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MATH WORKSHOP

REVISED EDITION

Robert W. Wirtz ▪ Morton Botel
Max Beberman ▪ W. W. Sawyer

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Teacher's Guide

MATH WORKSHOP

REVISED EDITION

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INTRODUCTION

I. A PERSPECTIVE

Loose talk about reform in the teaching of elementary mathematics has spread confusion and shattered the confidence of many teachers. Uneasiness and fear have developed. This mood has damaging effects in the classroom.

Teachers have been told they are teaching horse-and-buggy mathematics in a space age. New and esoteric language has begun to flood the literature and to be introduced into discussions. Some innovators have pointed the finger of scorn at traditional subject matter, pronouncing it "bad mathematics."

Relax! The mathematics our pupils are learning in traditional classrooms is good, sound, useful mathematics; it is good enough to play a central role in any moon shot. Traditional content in elementary school arithmetic is sound; it is at the core of man's advance, including his recent bursts into space.

Our real concern is with finding solutions for two interrelated problems. How can we help pupils (a) to learn *more* mathematics, and (b) to develop a *deep appreciation* for this increasingly important part of our culture? All proposals and discussions should be evaluated in terms of their contributions to the solution of these fundamental problems.

The traditional content taught to pupils in elementary schools is completely valid. Our elementary schools must teach pupils how to carry out the fundamental operations with whole numbers and fractions and how to use these skills in solving practical problems. But the truth of the matter is that pupils simply aren't mastering the traditional content. Many pupils have too weak a grasp on basic facts such as $7+2=9$ and $8\times 9=72$. (Most of these pupils will drop mathematics as soon as they can, losing contact with a tool that is essential in coping with the problems of the times.) The lack of mastery cannot be remedied just by giving more drill. Pupils will be able to master more of the traditional content if we give them freedom to *invent* ways of finding answers, if we take steps to make sure that they *understand* what they are doing, and if they have ample opportunity to *use* what they learn.

The job of teaching more mathematics will become easier to the extent that pupils develop an appreciation for the subject. This appreciation can develop only when mathematics is learned with understanding and through active participation—it will not take place when mathematics is learned by rote. Mathematics is a world of endless fascination. It is a world brimming with surprises, patterns, and enticing vistas. To see a pattern emerge from a conglomeration of facts, to organize seemingly disconnected ideas—these are some of the joys of learning mathematics. It is this sense of excitement in mathematics which pupils ought to be experiencing in elementary schools.

Giant steps are currently being taken to alter the traditional approach to teaching mathematics in order to get more mathematics into the elementary and secondary curricula. To this end, brilliant pioneering work is being done at the Universities of Illinois, Maryland, Minnesota, Syracuse, Stanford, and elsewhere. Generous financial support has been given by the National Science

Foundation, the Carnegie Corporation of New York, and the U. S. Office of Education for research projects, workshops, and institutes.

The authors of MATH WORKSHOP have participated in this pioneering work. Our program has evolved during a period of over ten years. Its ideas and techniques have been field-tested with thousands of teachers and tens of thousands of pupils.

II. CONTENT AND TEACHING APPROACH

The authors of MATH WORKSHOP believe that a sound mathematics program can be made ineffective by methods of teaching which rely mainly on rote learning. We also believe that even inspired teaching of isolated fragments of mathematics will fail to produce pupils who understand and can use what they have learned. The MATH WORKSHOP puts its *major subject matter emphasis* on basic principles and unifying ideas which tie together the facts and manipulative procedures of arithmetic. This emphasis simplifies the subject, makes it understandable, insures recall, and increases interest. Our *major pedagogical emphasis* is that of teaching for discovery. We present content to pupils so that they are challenged to think about relationships, to look for patterns and clues, to draw logical conclusions for themselves, and to invent their own ways of finding answers.

A. The *What* of Mathematics

The Spiral Curriculum

The *24th Yearbook* of the National Council of Teachers of Mathematics talks about "the basic mathematical understandings which should be continuously developed and extended throughout the entire mathematics curriculum, grades K-12 and beyond." This concept of the spiral curriculum has been used in designing MATH WORKSHOP. At each level of the MATH WORKSHOP the main ideas of mathematics have been introduced. Each successive year's activities deal with these unifying ideas with increasing sophistication.

However, pupils are not expected to master such ideas as, say, multiplication of fractions or addition of positive and negative numbers the first time they meet them. They will encounter such ideas again and again, building always more layers of understanding.

The spiral curriculum rejects the "building block" approach to arithmetic which has been used in the past. Rather, the spiral curriculum calls for the early introduction of fundamental and unifying ideas with repeated reconsideration at later times. It calls for continual nurturing of important concepts year after year as understanding gradually develops.

The hope for mastery must be limited to the basic facts of arithmetic. Here, mastery can and must be achieved. But where big ideas are involved, mastery does not come in this way. It develops slowly and is based completely on understanding.

The Unifying Ideas or Strands

MATH WORKSHOP emphasizes nine unifying ideas in mathematics:

1. *Structure*
2. *Sets*
3. *Number and counting*
4. *Numeration*
5. *Addition and subtraction*
6. *Multiplication and division*
7. *Functions and relations*
8. *Geometry*
9. *Measurement*

The coverage of these ideas on the pupil pages and on the guide pages is summarized in the chart on pages 36-37. Each teacher's guide page lists the unifying ideas considered on the corresponding pupil page and contains many suggestions for extending these ideas beyond what is called for on the pupil page.

Let us consider each of these ideas.

Structure. Suppose that you came upon some sentences from a Martian arithmetic book:

1) | ☆ □ = □ ☆ |
2) ⊓ ☆ ⧄ = ⧄ ☆ ⊓
3) (⊠ ☆ ┌) ☆ □ = ⊠ ☆ (┌ ☆ □)
4) (□ ☆ ⧄) ☆ | = □ ☆ (⧄ ☆ |)
5) □ ☆ ⊠ = □
6) ⧄ ☆ ⊠ = ⧄

Somehow you have found out that the symbols:

|, □, ⊓, ⧄, ⊠, ┌

represent different cardinal numbers. You guess that the symbol ☆ is an operation sign, and you wonder which operation it stands for. Sentences 1 and 2 convince you that ☆ stands for neither subtraction nor division, for these sentences suggest that ☆ stands for an operation for which order does not affect the results. Sentences 3 and 4 also support this conviction because they suggest that grouping does not affect the results. However, there is nothing about sentences 1-4 which would help you choose between addition and multiplication as the operation denoted by ☆. Perhaps the other sentences will help.

Sentences 5 and 6 turn out to be of little help. In fact, both of them suggest that ☆ could denote addition, multiplication, subtraction, or division. Fortunately, sentences 1-4 rule out both subtraction and division. If ☆ stands for addition, then ⊠ denotes the number 0; if ☆ stands for multiplication, then ⊠ denotes the number 1.

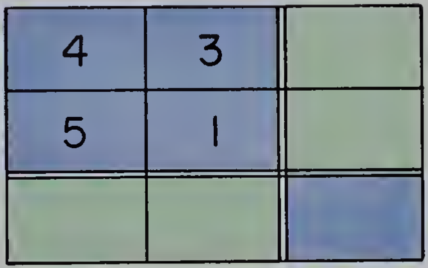
It turns out that, based on what can be learned from sentences 1-6, the operation sign ☆ could stand for addition or it could stand for multiplication.

The reasons we can't decide from sentences 1-6 if addition or multiplication is denoted by ☆ are that these operations have certain properties in common and these properties are reflected in sentences 1-6. The *additive structure* of the cardinal numbers is similar to the *multiplicative structure.* Each operation is commutative (sentences 1 and 2) and each is associative (sentences 3 and 4). Also, each operation has an identity element (sentences 5 and 6).

Whenever you provide exercises for pupils which bring any of these properties to the surface, you are paying attention to the structural properties of the number system. For example, exercises such as the ones which ask pupils to fill the frames to make the following into true sentences:

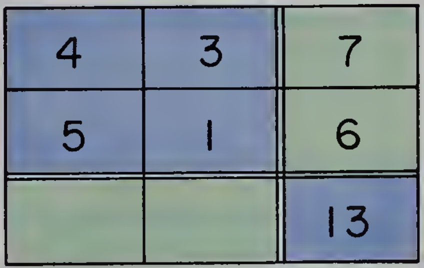
$$8 \times 7 = \square \times 8$$
$$7 + (3 + 5) = (7 + \triangle) + 5$$

are exercises which help pupils develop an awareness of commutativity and associativity. Another such exercise is the one which asks pupils to complete the addition *Cross Number Puzzle:*



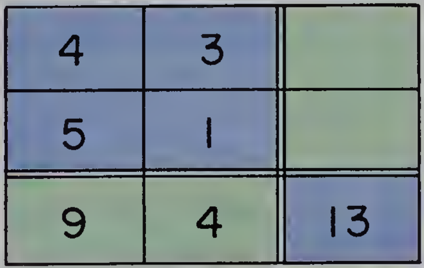
4	3	
5	1	

Notice that the cell in the lower right-hand corner gets filled with the same number whether the pupil adds the numbers in the rows and then adds the sums:



4	3	7
5	1	6
		13

or adds the numbers in the columns and then adds the sums:



4	3	
5	1	
9	4	13

This invariance of the total highlights the rearrangement properties of addition. (See pages 18-23 in the Introduction.)

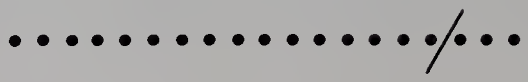
Of course, the advice that one should give pupils exercises which bring structural properties to the surface does not mean that pupils need to be told the names for these properties. A pupil who has discovered these properties by working with displays of physical objects has a sound understanding of the properties. He thinks of them almost as laws of nature. Any attempt to name these properties or to formalize them would be regarded by him as an attempt to glorify the obvious. A young pupil who tells you that

$$7+3 \text{ is the same number as } 3+7$$

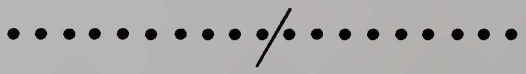
because "addition is commutative" has surely been coerced into using these words. He knows that $7+3=3+7$ because any display of objects which represents $7+3$ can be rearranged (without changing the total number of objects) into a display which represents $3+7$.

A pupil can be aware of and can use the structural properties without knowing their names. Exercises can be found which encourage pupils to make use of the properties. For example, a number line exercise which asks pupils to find the landing point when the starting point is 3 and the jump is 16 to the right could be solved by starting at 16 and jumping 3 to the right. The fact that this alternative procedure gives the correct answer strengthens a pupil's awareness of the commutative property of addition and helps to convince him of its usefulness.

If a pupil finds the sum of 16 and 3 by finding the sum of 6 and 3 and then finding the sum of 10 and 9, he has made use of the associative property of addition in a helpful way. (Of course, he has also made use of the fact that the numbers are expressed in decimal notation.) He does not become convinced of the correctness of his technique by being told that it is based on the fact that addition is an associative operation. Rather he becomes convinced by observing that this display of counters:

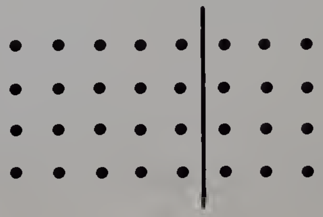


can be rearranged into this display:



In addition to the structural properties of commutativity, associativity, and the existence of an identity element for addition, the operation of adding a number has the property of being invertible—it can be "undone" by subtracting that number. Similarly, the operation of multiplying by a (nonzero) number can be "undone" by dividing by that number.

Pupils also learn that certain multiplication facts can be discovered by using other multiplication facts and addition. For example, pupils can find the product of 8 by 4 by adding the products of 5 by 4 and 3 by 4. That this procedure is valid is based on the "physical" fact that an 8 by 4 array of counters can be partitioned into a 5 by 4 array and a 3 by 4 array:

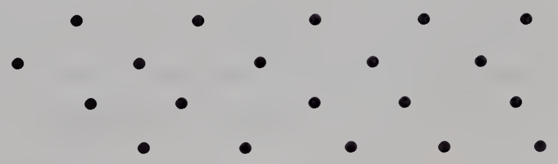


This experience helps pupils become aware of the fact that multiplication distributes over addition—a property of multiplication and addition. The term "distributive property" is not used on the pupil pages and need not be used in class. The important thing is that pupils use this technique in deriving certain products from others.

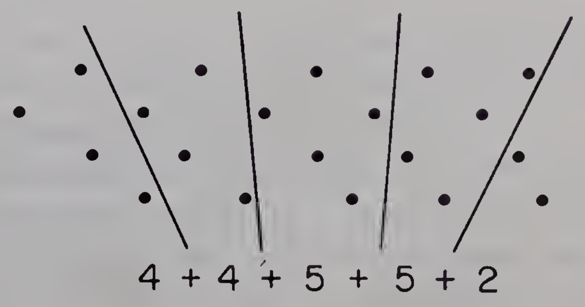
Sets. The principal contribution of sets to elementary arithmetic is that they can be used to clarify the notion of cardinal number. For a long time people thought that the word "two" in the expression "two apples" played the same kind of role as the word "green" in the expression 'green apples." This was the cause of much confusion about the nature of cardinal numbers. An individual apple can have the property of "green-ness" but it can't have the property of "two-ness." Only sets can have the property of number. A set consisting of two green apples is a set whose cardinal number is two. Also, any set whose members can be matched with those of a set of two apples is a set whose cardinal number is two. From here it is an easy step to say that the cardinal number two is the class of all such sets.

Now, a set is an abstract entity just as a number is abstract. You cannot see a set of two apples any more than you can see the number two or justice or liberty. For example, it is complete nonsense to ask the *set* of boys in the classroom to stand or to ask pupils to look at the *set* of counters on the table. There will come a time in the pupil's mathematical education when he will need to use the word "set" and we do not think it is a good idea for him to develop in the primary grades the notion that a set is a tangible object. Therefore, we have rarely used the word "set" in the pupil's text, and have used it sparingly in the teacher's guide pages. Instead we have used nontechnical terms such as "display," "group," and "array."

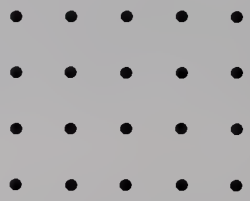
Number and Counting. In the primary grades the basic mathematical idea is that of counting. The operations of addition and multiplication are presented as shortcuts to counting. Given a display of objects:



and asked to tell how many objects are in the display, a pupil might proceed to count them one by one. This is the most primitive approach. A more sophisticated approach is to partition the display, obtain the count for each of the smaller displays, and add the counts:



An even more sophisticated approach is to rearrange the objects in the display into a rectangular array:



If one is familiar with the count for a 4 by 5 array, he knows immediately that the original display contained 20 objects.

Numeration. The idea of numeration grows out of the need of reporting a count, and the mode of reporting is culturally determined. For example, if the objects in the display shown above represent glasses of milk, we might report "1 gallon and 1 quart" in answer to the question: How many? If the objects represent eggs, we might report "1 dozen and 8 more" or "4 less than 2 dozen."

Each of these reports is a description of the result of some partitioning procedure. When decimal partitioning is used, the English language provides us with the words for giving brief reports. For example, the report "3 ten-groups and 7 one-groups" or "3 tens and 7 ones" can be abbreviated to "thirty-seven" and can be recorded in positional notation as 37.

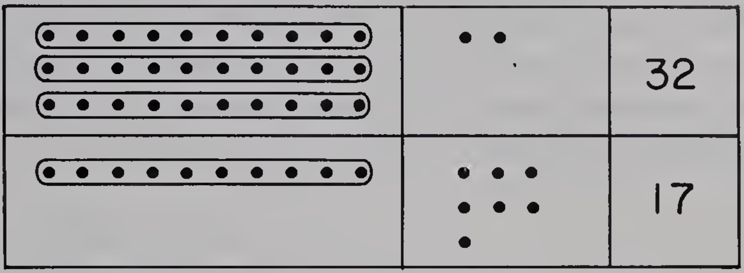
Addition and Subtraction. The operation of addition is developed as a shortcut to counting. Given enough opportunities to find the sum of 5 and 4 by counting, for example, a pupil soon commits to memory the fact that $5+4$ is 9. He makes use of this fact as well as of positional notation and associativity when finding the sum of 25 and 4: $25+4=(20+5)+4=20+(5+4)=20+9=29$. The addition facts from $0+0=0$ through $9+9=18$ should be memorized during the primary grades.

We use the word subtraction to denote the process used in undoing the result of adding. So, for example, to compute $8-3$ amounts to finding the number whose sum with 3 is 8. From this point of view, it is natural to learn subtraction facts at the same time one is learning addition facts. Thus, we have *fact teams*, groups of four sentences, such as:

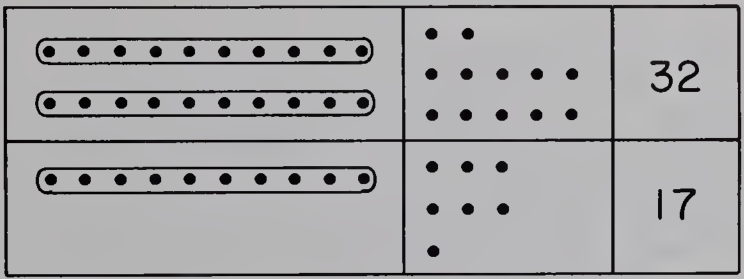
$$\begin{aligned}5+3&=8\\3+5&=8\\8-3&=5\\8-5&=3\end{aligned}$$

which are derived from one addition fact with the help of commutativity and the fact that subtracting a number is the *inverse* of adding that number.

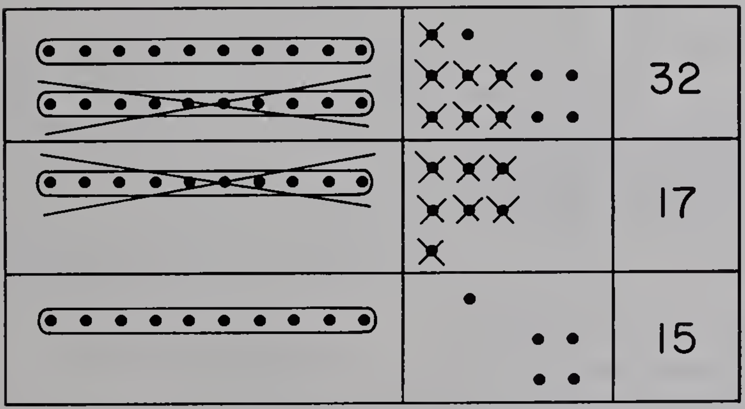
The addition and subtraction algorithms are developed as end products of activities in which pupils manipulate physical objects and keep records of the manipulations. The algorithm becomes a record-keeping device for imagined manipulations. Thus, the algorithm works because of laws which govern the manipulation of physical objects. For example, consider the algorithm used in subtracting 17 from 32. In terms of physical objects, one can find the difference by counting what remains when 17 objects are removed from a display of 32 objects. We start by giving ourselves decimally partitioned displays of 32 objects and 17 objects:



In order to carry out the take-away step, we need to break into one of the ten-groups. So, we rearrange the display for 32 by exchanging a ten-group for 10 one-groups:



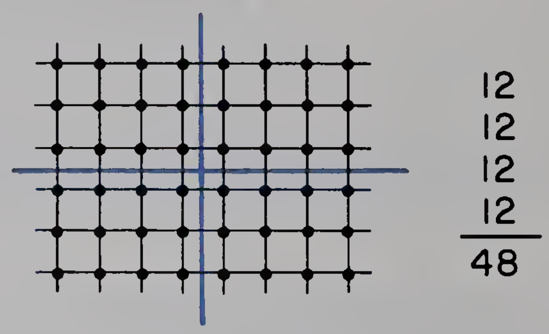
and then take away 1 ten-group and 7 one-groups:



These three stages are recorded in the steps of the algorithm:

$$\begin{array}{r}32\\-17\\\hline\end{array}\qquad \begin{array}{r}\overset{2}{\cancel{3}}{}^{1}2\\-17\\\hline\end{array}\qquad \begin{array}{r}\overset{2}{\cancel{3}}{}^{1}2\\-17\\\hline 15\end{array}$$

Multiplication and Division. As in the case of addition, multiplication is developed as a shortcut to counting. In this case the counting involves objects arranged in rectangular arrays or otherwise grouped to encourage counting by groups. For example, counting the spaces in an egg carton and telling the end point of a sequence of 5 number line jumps of 2 to the right starting at 0 are multiplicative situations. By continual practice in the primary years a pupil should commit to memory the multiplications facts from $0\times0=0$ to $9\times9=81$, and each of these facts should be discovered by the pupil. Some of the facts should be discovered by direct counting, but the discovery of others should involve the partitioning of arrays and adding the counts of the parts. For example, a pupil might learn that 3×4 is 12 by counting the spaces in a certain type of egg carton. He should use this fact to learn that 6×8 is 48 by drawing 6 horizontal lines and 8 vertical lines and then partitioning the resulting display of crossing points:



Division is the process used in undoing the result of multiplying. So, for example, to compute $24\div8$ amounts to finding the number whose product with 8 is 24. From this point of view, it is natural to learn division facts at the same time one is learning multiplication facts. Thus, as in the case of addition and subtraction, we have fact teams for multiplication and division such as:

$$\begin{aligned} 8\times3&=24\\ 3\times8&=24\\ 24\div3&=8\\ 24\div8&=3 \end{aligned}$$

which are derived from one multiplication fact with the help of commutativity and the fact that dividing by a (nonzero) number is the *inverse* of multiplying by that number.

Functions and Relations. An example of a function is the operation addition-of-2. This function assigns a unique number to each number. For example, it assigns 5 to 3, 19 to 17, and 202 to 200. A common way to exhibit a function is to show a table which lists some of the assignments. For example, to exhibit the function addition-of-2 we might show the table:

3	4	9	12	18	30
5	6	11	14	20	32

The numbers listed in the top row are the arguments of the function and those listed in the bottom row are the values of the function. The use of two colors helps to distinguish arguments from values. Of course, this same table shows another function—subtraction-of-2—which is the inverse of addition-of-

2. (See the teacher's guide Notes on Pupil Page 118 for more on functions.)

All functions are examples of mathematical objects known as relations. Numerical relations which are not functions are those which assign more than one number to given numbers. For example, the greater-than relation assigns 4, 5, 6, etc., to the number 3.

Geometry. Under this general heading one finds a variety of topics—classification of common figures, comparisons of area and perimeter, reflections, and similarity transformations (doubling and halving of line segments).

Measurement. Under this heading we find work with denominate numbers and the conversion of units.

B. How to Teach Mathematics

It's easy to ask a question so easy that almost anyone can answer it. And it's easy to dig up trick questions—there are books full of them.

But challenging questions that encourage independent research and lead to discovery must be neither too hard nor too easy, and must not rely on clever tricks. Rather, challenging questions are less than obvious but will yield answers if the pupil applies thoughtfully what he already knows.

Such questions are hard to come by. They are worth their weight in gold. Often they lurk in very simple situations.

Practically every page and activity in MATH WORKSHOP poses such less than obvious, challenging questions. MATH WORKSHOP presents these questions through a wide assortment of techniques, many of which were invented by the authors of the program. Challenging questions are posed through the game of *What's My Rule?*, *Cross Number Puzzles*, *All About* ____ pages, the game of *Watch The Signs!*, and through problem solving. Let us examine examples of these vehicles we use for asking challenging questions.

What's My Rule?

In first grade, Miss Atkins tells the children that she has a secret rule and she will help them read her mind.

"Call out a number and I will answer with a number."

"Five," a child calls out.

"Four," Miss Atkins answers. "Someone call out another number."

"One," a child calls out.

"Zero," the teacher replies.

"Seven," another child calls out.

"Six," the teacher answers. "Have you guessed my rule?"

"Yes," Bill is quick to reply.

"Don't give my secret away. Just call out a number and tell me what you think I will reply."

"If I say 9, you are going to say 8."

"That's right. I *think* that you know my secret." Thus the children have learned a way to demonstrate their discovery.

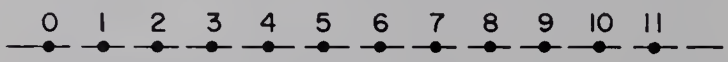
Many children are not as quick as Bill. So Miss Atkins may put on the board a record of the development thus far, first listing the number called out and then the reply:

5→4
1→0
7→6
9→8

The last item was Bill's contribution. Others might be:

6→5
2→1
4→3

When the game is new or the rule difficult, a number line on the board will help:



"Let's act out the game we have played. When someone called out 5, I answered 4. Point to 5 on the number line. Keep your finger on that point. I answered by saying 4. Point to 4 on the line."

Each step is similarly acted out.

"Now, if you call out 8, put your finger at that point on the number line."

8→_____

"What do you think I will reply?"

"Seven, of course!"

In some such way the idea can be made clear even to the slowest children. Care was taken not to talk about the teacher's rule itself—only about the results of applying it. Thus, every child must make the discovery for himself—some with minimum clues, others only after several different kinds of clues. There is a bit of artistry in giving clues only when needed and finding ways to reserve at least a little for even the slowest child to find out for himself.

Usually some child will note a curious fact about this rule.

"Suppose that I call out zero. What will you answer?" a child may ask.

"Point to zero on the number line," the teacher suggests. "You can see that I would have to ask you to pick some other number." (Unless the teacher would like to talk about positive and negative numbers and zero.)

A new game begins with another secret rule. Some child may want to take over the teacher's role and invent a rule of his own.

Rules can become more complicated.

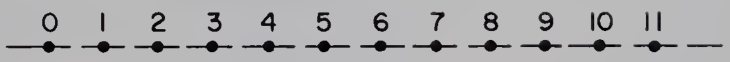
"I have a new rule. Call out a pair of numbers and I will reply by saying a number." Here is a record of a simple rule:

2, 3→5
4, 0→4
3, 1→4
1, 2→3
1, 1→2
3, 1→__

A more difficult rule is behind the following record:

2, 4→3
0, 2→1
4, 0→2
5, 9→7

These last two items may puzzle the children. Perhaps a number line can help shed light on the rule.



"If someone called out 2 and 4, I would reply with 3. Point to the numbers called out (2 and 4). I would reply with 3.

"If someone called out 4 and 0, I would reply with 2. Point to 4 and 0 on the number line. I would call out 2.

"Point to 5 and 9—and remember that I would answer 7. What would I answer if someone called out 1 and 7?"

1, 7→__

"Four," Bill might answer, and quickly ask: "But what would you do if I called out 5 and 6 or if I called out 3 and 0?"

"I would have to ask you to pick another pair of numbers." (Unless the teacher is ready to discuss fractions.)

An unlimited number of rules can be devised on any level of sophistication. Further, there are countless ways to suggest different rules, such as:

a. 1, 4, 7, 10, 13, __, __, . . .

b.

3	1	4	10	0	9	6
7	3	9	21	1		

c.

7	4	3	1
9	9	0	9
11	6	5	1
	8	6	
6			2

Mathematics is essentially a study of patterns. It has its own fascinating patterns and it is often employed to help reveal patterns in the physical world. When a scientist or any observer is searching for a better understanding of what he notices, he usually tries to set down a description of events. Whenever possible he will try to write down numbers that in some way help him record relationships. He might record the following observations:

	1st	2nd	3rd	4th	5th
minutes	1	3	4	5	10
miles	1	9	16		

In his first three observations he notes time and distance. A pattern is strongly suggested. He may check his guess by making further checks for 5 minutes and 10 minutes. If his results are 25 miles and 100 miles, respectively, he will feel that he can formulate a rule or law that will suggest a pattern in certain real events. ($1 \times 1 = 1, 3 \times 3 = 9, 4 \times 4 = 16, \ldots$)

In mathematics we study patterns because they suggest basic principles of the subject itself and because mathematics helps record and explore patterns of the world we live in.

Thus, this exercise or game of *What's My Rule?* has three important dimensions: (1) It can be used to motivate drill in the basic combinations of arithmetic. (2) It poses problems that children can think through to a solution. (3) It uses a method fundamental to the experimental sciences.

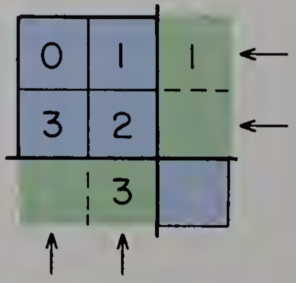
All About _____

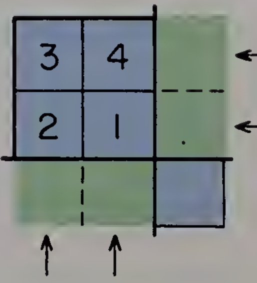
This activity can best be discussed in terms of specific examples. (See pupil page 35.) On the *All About*_____ pages we show pupils various ways in which counts are reported. These ideas are fundamental to understanding the decimal numeration system. Furthermore, the *All About* _____ pages show pupils that numeration systems arise from the needs of the real world.

Cross Number Puzzles

Too few of us realize that even six- and seven-year-olds can carry out research problems in arithmetic. Do you doubt it? Let's leave speculation and get to work.

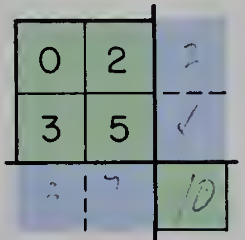
In the spaces indicated by arrows, write the sum of the two numbers listed in each row and in each column:

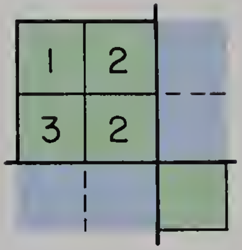




Do you have a desire to fill in the lower right-hand corner of each diagram above? Notice the two numbers listed above the corner and the two numbers listed to the left of the corner: 1 and 5, 3 and 3 in the first example; 7 and 3, 5 and 5 in the second example.

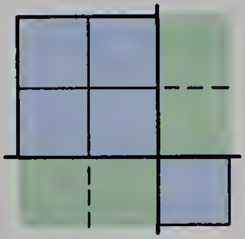
Most people would feel an urge to write 6 in the lower right-hand corner of the first example, and 10 in the corresponding corner of the second example. Try two more:

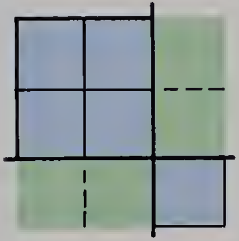




Again, a similar result. This is the kind of surprise that is the spice of arithmetic!

Can you select four numbers to list in the four blue-shaded cells so that some surprise happens in the lower right-hand corner? Try your hand at two examples:

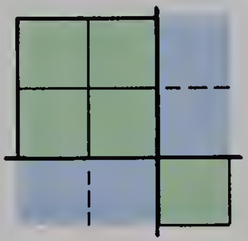


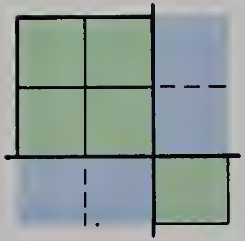


Did you succeed?

This problem is relatively simple. Even first-graders can find examples that work. They are putting their arithmetic to use, practicing the basic facts of addition.

Now try another problem. Find four numbers to list in the four green-shaded cells at the upper left—numbers that will not work in the same way. Try two examples:





Do you find that this problem is easier? Or is it more difficult?

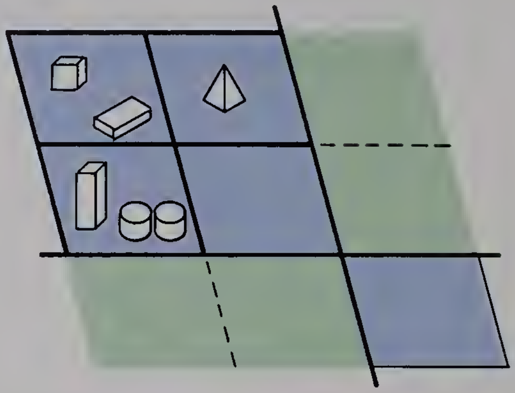
When children tackle this problem they sometimes claim their numbers won't work. But when the arithmetic involved is checked, mistakes turn up—and the examples work after all.

Some children jump to the conclusion that it's impossible to find exceptions. Most will extend the search, trying the largest numbers they can handle. But the search will not yield an exception.

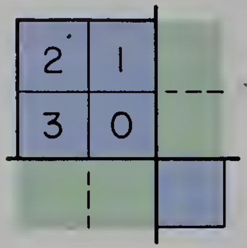
This certainly will be a surprising result to most children (and adults). Yet all children who have at least a few facts of addition at their command can discover this principle.

An understanding of the principle can be illustrated in an activity designed for children who are just learning to count and to record their results.

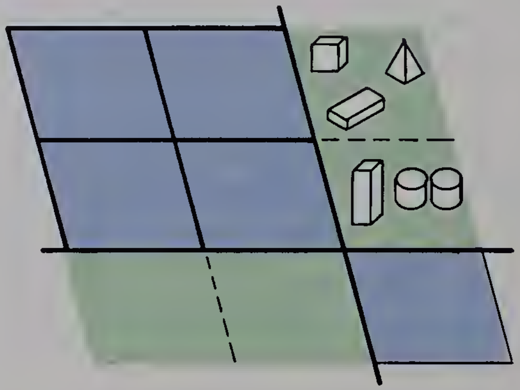
Suppose that we create a small model on a table with masking tape—or on the floor of the classroom. We place objects in three of the four boxes:



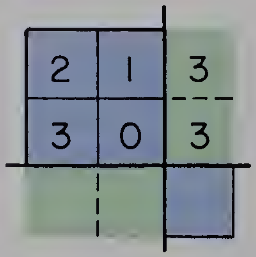
On the chalkboard a record is made for later use:



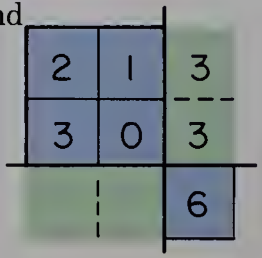
We now sweep the objects in the top row of boxes to the right and count them. There are 3 (2 plus 1). Next we sweep the objects in the second row to the right and count them. There are 3 (3 plus 0).



Include these results in the record on the board:

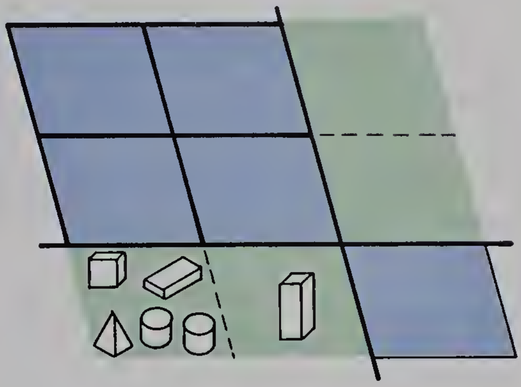


Now sweep the objects into the box in the lower right-hand corner and count them. There are 6 (3 plus 3). Include this in the record:

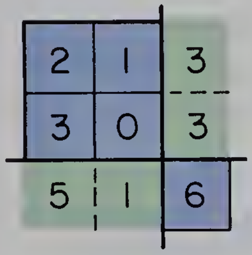


Can we put the objects back into the original four boxes so that each has the same number of objects as it had in the beginning? We have an adequate record on the board. It does not tell us which objects were in each box, but it does tell us the number of objects—and we make it plain that we are interested only in the number of objects.

After the objects have been replaced so that they conform with the original record, we sweep the objects in the boxes on the left down to the bottom and count them. There are 5 (2 plus 3). Similarly, we sweep the other object down and count it. There is 1 (1 plus 0).



Include these results in the record on the board:

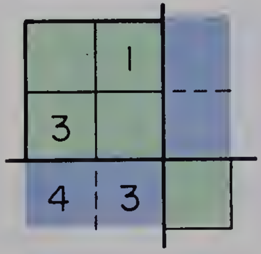


2	1	3
3	0	3
5	1	6

We sweep all the objects into the bottom right-hand corner and count them. There are 6 (5 plus 1).

We find that we do not need to record this result, since we already noted that 6 objects were in that corner when we swept them there by a different route. Are you surprised? Certainly not. In fact, if any other result had occurred, you would be understandably disturbed.

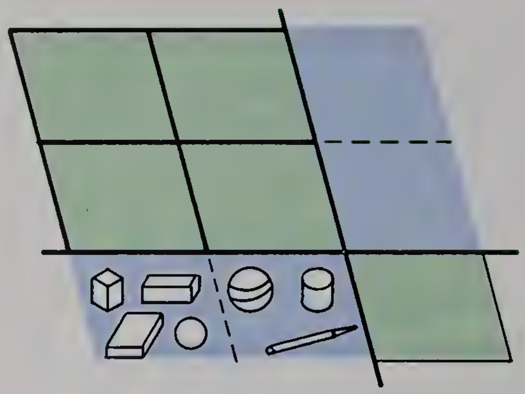
Reconstructing the puzzle. Now suppose we start this activity a second time by putting a partially completed record on the board:



	1	
3		
4	3	

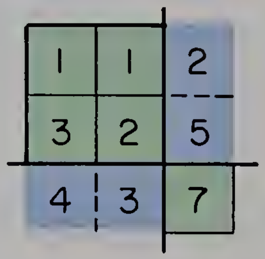
Can we find out the number of objects that were originally placed in each of the four main squares?

If we place four objects in the space beneath the two boxes on the left, and three objects in the space beneath the other two boxes, the result might look like this:

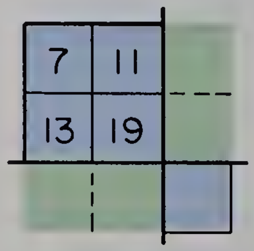


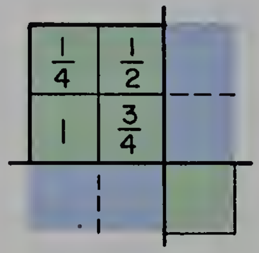
Since the record tells us that three of the four objects on the left were in one of the boxes, place three of them there—and there is one left for the other box on the left.

In a similar fashion, we can think and work our way through to a completion of the record. It would look like this when completed:

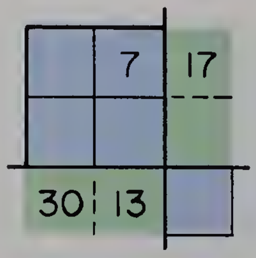


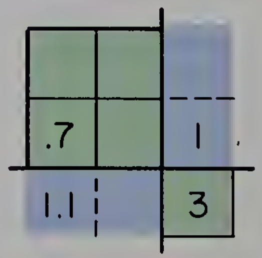
Although the simple activity just described is for beginners, the underlying principle is the same when the initiated are faced with such *Cross Number Puzzles* as:





Similar but more sophisticated reasoning is required for reconstructing the full record from such fragments as:





The *Cross Number Puzzle* requires considerable practice of the basic facts of arithmetic; it produces surprises; it helps to build a conviction that the outcome in addition is not influenced by the order or the grouping of the addends.

The *Cross Number Puzzle* is used with great effectiveness to get pupils to discover the algorithms for addition and subtraction. (See pupil pages 58-65.)

Watch The Signs! These are exercises very much like *Cross Number Puzzles*. They are self-checking and force pupils to shift their thinking between addition and subtraction. Some of these exercises lead directly to the subtraction algorithm. (See D on pupil page 27.)

Problem Solving and Applications

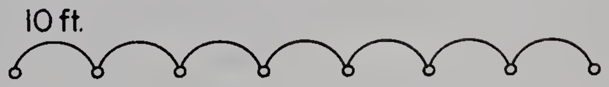
Story Problems. Over the years, "story problems" or "word problems" have been a source of vexation to many teachers of arithmetic, to many children—and to many parents. This activity, which properly seeks to relate the abstractions of mathematics to the world of events, often ends without a sense of accomplishment for anyone.

Story or word problems are usually reserved for the end of a development, as a sort of test or exercise of the pupil's ability to apply an acquired knowledge of arithmetic.

Examples vs. problems. A critical analysis of the questions usually asked in traditional arithmetic programs reveals that they constitute primarily a test of a child's reading ability, and that if the child can read, the questions are trivial and constitute *examples* rather than *problems*.

Example: Mr. Jones bought 8 fence posts. He paid $10 apiece. What was his total expense? (If you can read, this constitutes an example of the fact that $8 \times 10 = 80$.)

Problem: Mr. Jones bought 8 fence posts. He set them out in a straight line, 10 feet apart. What is the distance from the center of one end post to the center of the other end post? (If you can read, this is a problem that requires careful consideration of the situation presented; perhaps a sketch to show that there are 7 spaces between posts.)



The first situation is an example; the second is more of a problem. If a child clears the reading hurdle then the example taxes little more than his memory, while the problem depends on a less than obvious question that makes greater demands on the mind.

Examples and problems play different roles in helping children understand mathematics—and both are important.

When we notice a pattern repeating itself in the limited world around us—the world we can manipulate—we often use an expression in the language of arithmetic that points to that pattern, such as

$$2 + 3 = 5$$

At this point we ought to direct children's attention to an application of this idea to the wide world of fact and fancy that they live in.

"Suppose that this is a headline."

$$\boxed{2 + 3 = 5}$$

"Tell a story that might fit such a title or headline," the teacher urges.

"There are five children in our family, two boys and three girls."

"Fine. Who has another story that would fit?" the teacher asks.

"I have three cookies out of the five I started with. I ate two of them."

"It's the second of December and my birthday is on the fifth. My birthday is in three days."

"On my hand I have a thumb and a little finger. There are three fingers in between."

"Out of my allowance of five cents I have two cents left."

"Three hours from now will be five o'clock." (Check with a clock to determine whether or not this is a true sentence.)

Each of these situations has within it the conditions we point to in the headline. Each of the story-tellers reveals that he is beginning to grasp the relationship between arithmetic and the real world.

Whenever we abstract a bit of mathematics out of the world at hand and write it down in the language of arithmetic, good classroom technique should include the next logical step—to find examples from the world that reflect this idea.

The authors of MATH WORKSHOP believe strongly that basic mathematics should be derived from the world around us. For this reason we continually refer to practical situations on both the pupil pages and in the guide pages.

Each guide page contains a section entitled Headline for the Day. This might be copied on the chalkboard at the opening of the school day, thus giving pupils a chance to ponder it. For example, you might write the headline $2 + 3 = 5$ given above, and then let pupils tell various stories. Or, you might put a story on the chalkboard and ask for headlines. For example, if you wrote the story:

Bill has 2 toy cars. His father gave him 3 more. You should expect headlines such as:

$$2 + 3 = 5$$
$$5 - 3 = 2$$
$$3 > 2$$
$$5 = 3 + 2$$
$$5 - 2 = 3$$
$$5 > 2$$
$$2 + 3 = \square$$
$$\square - 3 = 2$$
$$2 < 5$$

Please allow the pupils plenty of freedom in composing headlines.

Starting with the teacher's Notes on Pupil Page 45 we have included a pair of stories which you may wish to use on a drill sheet or put on the chalkboard. At least part of the discussion in each lesson should remind pupils that mankind has developed the language of mathematics to help report events that happen in the world of things.

Teaching for Discovery

Each page in MATH WORKSHOP and the activities suggested in the guide pages are designed so that there are several levels on which children can participate—including opportunities for success for even the slowest learners.

The activities we present are truly open ended, with challenge for even the most advanced and able children in the class. While some questions will have been settled, more will have been left open. Some children will want to explore these open questions under their own steam. Such children ought to learn to work independently, sparing the teacher's time for those who need it more. Actually, what the more able students need most is encouragement and opportunity to develop a taste for independent study.

The highest level of learning has been reached when a child begins to pose questions of his own and to work toward their solution.

There is in MATH WORKSHOP provision for introducing or practicing the basic facts of arithmetic. Usually such facts must be recalled and organized so that a pattern emerges. As patterns appear, they provide answers to less than obvious questions.

Discovering patterns is one of the most rewarding experiences in mathematics. It is called the "Aha!" or the "Gee-Whiz!" experience—a gasp; the light dawns! "I see it!" These are events teachers bend their best efforts to help bring about. One of the most disheartening facts teachers must face is that some children simply "don't see it." Yet this is also a challenge—it calls for exercising skill in teaching. Slow learners need more clues—more obvious clues; but the finest teacher will reserve a wee bit for each child to "see" for himself. If the teacher's best efforts fail, he may have to give away part of the secret and try again next time. This is an art that does not come easily, but the eventual payoff is deeply rewarding.

When the results of work in arithmetic exhibit an unusual regularity, the inquiring mind looks for an explanation. One problem may be solved but others are suggested. This is an exciting characteristic of mathematics.

When children discover a relationship, they are excited and eager to stay with that discovery. They will readily put this new idea to work, showing more examples of *it*. They come alive, often with an "Oo-o-o! I've got it!" or with a burst of industry using a newly-discovered relationship to reel off answers with speed and confidence.

What is the *it* they have glimpsed? Well, if we distract their attention with a request for them to *tell about it* or *put it in words*, we have selected a method that could not be better calculated to turn children against discovery.

That a child can move ahead in putting *it* to work successfully is certainly all the evidence we need in order to know that he *has it*. Our only justification for veering off in the direction of a word-pulling exercise is a feeling that *it* is a generalization that ought to be verbalized in order to be usable. But research does not support this as a valid reason.

Transfer to future experiences is not increased by attempting to *verbalize a generalization*. Subvocal or nonverbal awareness, the new-found ability to move ahead more speedily, more directly, more confidently, is our main goal.

When children experience an insight—when they leap ahead because they have discovered a pattern—we ought to be very careful about letting words into the act too soon. The children probably do not have the language needed to express their discovery in words. If they try, we must, of course, help them. But too often such help and explanation degenerate into hairsplitting, an activity far removed from the act of discovery.

Gertrude Hendrix of the University of Illinois Committee on School Mathematics, who has done much research on discovery in the teaching of mathematics, warns that imprecise verbalization may "mutilate" discovery.

On the other hand, when some children (and teachers) make discoveries, they want others to know about it. But herein lies a danger. Whenever a person reveals his secret to one who has not been as fortunate, the latter is deprived of the delight of discovering it for himself.

So we invent ways in which children can share their discovery without revealing the secret too widely. We have them whisper their new idea in the teacher's ear; we encourage them to make up additional examples which will provide further clues for those who haven't found a pattern that underlies several examples. We name the new idea after its discoverer—Bill's Rule or Mary's Law or John's Pattern. But we keep urging children to keep the secret—for a while at least.

Once in a while we find children who have not invented shortcuts for solving certain kinds of problems. For example, suppose that children use counting by one in counting problems which involve fairly large totals. Hopefully, such children will become discouraged and begin to look for easier ways of determining the total, such as counting by five. If the easier ways are discovered, the children should be highly praised and urged to demonstrate this technique to the entire class. If this example is not sufficient to shake all children loose from counting by one, then you must step in and urge the conservative ones to drop the slower technique. Please temper the discovery method with common sense.

III. REPRESENTATIONAL MATERIALS

When mathematics loses contact with the world of things and with the events out of which it came, the subject becomes mysterious to most children. If they do not understand that we are discussing the way things happen as we manipulate objects, they must resort to rote memory.

MATH WORKSHOP activities stress the fact that arithmetic provides a means of recording or reporting what happens in the physical world.

Many kinds of manipulative or representational materials are suggested and discussed in this Teacher's Guide. These include various types of counting materials, geometric shapes, Cuisenaire® rods,* number lines, etc. (See the reference in section G on page 32 of the Introduction.)

Perhaps the most accessible source of representational materials is the pupil pages themselves. We believe that all of elementary mathematics can be abstracted out of experiences with the physical world. The very word WORKSHOP in the title of our program indicates our belief in the value of learning mathematics by working with physical objects. In some cases, the pupil pages are the workshop—pupils work with the two-dimensional pictorial representations of physical objects and apparatus (*Cross Number Puzzles*, number lines, colored strips) and use the forms provided to keep records. Frequently the teacher will decide that pupils need to work with three-dimensional equipment. In those cases, the teacher may provide equipment just like that pictured on the pupil pages or like that suggested in the guide pages.

IV. MASTERY OF BASIC FACTS AND ALGORITHMS

Mathematics provides a language to help report events that happen in the world of things. Counting is at its core. Because this process becomes tedious when many things must be counted, shortcuts to counting have been invented and improved. These shortcuts depend on two characteristics —memory and patterns. Thus, elementary school mathematics is essentially concerned with counting and with shortcuts to counting.

At first glance, MATH WORKSHOP activities may appear to be highly motivated exercises to provide practice in the basic facts and combinations and processes of arithmetic. In the vernacular, they wear the garb of "sugar-coated drill." Actually, they are valid educational devices, because they help establish that automatic recall of "fundamentals" without which children are mathematical stutterers.

Every teacher knows that many children are mathematical stutterers because they can't easily recall the basic facts. They hesitate and flounder; the discussion bogs down; they lose the argument; they miss the feeling of movement toward a solution; they cannot enjoy mathematics.

This area of trouble is so universally known that it is surprising so many are only casually concerned! There is much hue and cry to bring new and unusual content into the elementary school curriculum—numeration in bases other than ten, sets, properties of numbers, and other concepts—yet there seems to be silence about such a crucial matter as helping children master the fundamental combinations without which mastery their progress in understanding mathematical ideas is very painful.

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*Cuisenaire—Registered trademark of Cuisenaire Company of America, Inc.

While we have emphasized the scope of our program in terms of nine unifying ideas, *the basic facts of arithmetic and algorithms are taught and retaught throughout the program.*

To provide you with help toward the goal of mastery, we have included on practically every guide page (in a section entitled DRILL) a sample work sheet which can be easily converted into a Dittoed®* sheet. We have felt that it would be too expensive and perhaps a little silly to use any of the pupil's pages for strictly drill purposes. (You may find it useful to send a Dittoed drill sheet home with a pupil rather than have him take home his text.)

V. THREE WAYS TO USE PUPIL PAGES

There are at least three generally successful ways in which individual pages or groups of pages can be used in the classroom:

1) Use the page to conclude a development.
2) Use the page as the basis for discussion.
3) Let children work independently.

1) In those areas in which the teacher has many ideas of his own, there is no substitute for direct teaching—with liberal use of manipulative material, the blackboard, and class discussion. The page is then used late in the development to tie the idea down, or simply to record the progress that has been made toward deeper understanding.

There are limitless ways to initiate lively activities around important and useful concepts. The MATH WORKSHOP provides many such activities for you to choose from.

After handling manipulative materials, playing number games, and other such preliminary work, pupils are ready to move through the page in the final few minutes of the arithmetic period.

2) In contrast, some pages might be gone through very carefully with an entire class. Such a page or sequence of pages offers many opportunities to dig deeply into: What's going on here?

3) A third approach might be called *Do-it-all-by-yourself*. Begin by saying, "Open your book to (name the page). Look it over. Think about it. See if you can decide for yourself what is to be done. Wait a while before you ask for help. And let's make today a 'quiet day.' "

Often, the first question the page poses to children is: What am I supposed to do? There are almost always sufficient clues provided so that with a bit of study children can discover the intent of the exercises. Traditional materials have provided too little incentive for children to figure out for themselves the nature of the business at hand.

Now you feign reluctance to help but are, of course, watching carefully for those who haven't sufficient self-confidence to get started on their own. If several children are unable to get started, suggest that they work in pairs, helping each other quietly. Enlarge these helper combinations if necessary. Grouping like this will usually get the activity under way for everyone. Let time run out on the session if you wish, suggesting that those who haven't finished may want to take the work home with them to finish.

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*The name DITTO is a registered trademark of Ditto Incorporated, Chicago, Illinois.

VI. TEACHING AIDS IN PROVIDING FOR INDIVIDUAL DIFFERENCES

A. In the Teacher's Guide

The Introduction presents a point of view and a rationale for our approach to arithmetic. You will find this useful in interpreting many of the things we have included in the pupil pages. You will also find it useful in explaining the program to parents.

Each pupil page is reproduced in the Teacher's Guide in full size and in full color with answers printed in red.

Each pupil page is accompanied by guide pages which are divided into several sections. Let's consider some of these sections:

Tasks. This section contains a very brief description of what the pupil will be doing as he works on the pupil page. We suggest that you study the pupil page in a pupil's book and work on it before you read our description and the answers we give.

Purpose. Here we describe what we think the pupil will learn as he works on the pupil page or as he carries out some of the activities described in The Lesson. (Naturally, our description contains a conservative estimate of what might happen. Mathematics is full of surprises and so are pupils.) If you are aware of our purpose, you will find it easier to help pupils achieve it.

Unifying Ideas. In this section we list those of the nine unifying ideas mentioned on page 8 to which the exercises on the pupil page and the recommendations in the guide can be expected to contribute. Please regard the list as a minimum statement, and be prepared for pupils who may wish to go beyond what we have described for the page.

Materials for Teacher. This is a description of the equipment you would need if you followed all of the suggestions made in The Lesson. Some teachers will be able to look at this description and then plan their own lessons without reading further.

Materials for Pupils. This section describes the equipment needed by individual pupils in carrying out the tasks or in pursuing the activities suggested in the lesson.

The Lesson. In this section we describe one or more of the lessons we might teach after studying the pupil page.

Practically all of our lessons would involve representational materials. We might decide to use a workshop and laboratory approach, using the equipment directly suggested by the pupil page (see section III on page 26).

In that case, the pupil page would be used as review material.

We might use equipment quite unlike that suggested by the pupil page, but in keeping with the purpose and unifying ideas. In some cases, the pupil page might be worked on at the beginning of the lesson, so that the laboratory would then provide an application of what was learned on the pupil page. In other cases, we might reverse the order and make the pupil page the application of what was learned in the laboratory.

We would assign some pupil pages to be worked on with very little explanation from us (see section V on page 28). For other pages, we would ask lots of questions. Hopefully, we would find some means of projecting such pages on the chalkboard so that we could talk about them easily (see section B, below).

No teacher could possibly find the time to use all our suggestions in any school year.

Choose the ones which best suit your style of teaching, and vary your choices from year to year.

B. "Math Workshop Supplementary Activities"

A package of supplementary activity worksheets is available at every level, A through F. Each package includes 32 copies of 64 different practice pages. These can be used independently by pupils for seatwork and for homework. Able pupils should be given supplementary activity material at advanced levels. Each set of supplementary activities includes a Teacher's Manual with brief suggestions for each page and an answer key.

C. "Work Forms"

Many of the interesting and versatile page-forms developed in MATH WORKSHOP texts have been extracted and made available as Ditto and Multilith masters called *Work Forms*. In a few minutes, a teacher can convert one of these forms with pen or pencil into a rich supplementary activity for the whole class or for a group within the class. Providing, as they do, significant independent learning activities for creative seatwork and meaningful homework, and taking into consideration, as they do, the wide range of individual abilities within any class, the *Work Forms* should prove very useful to busy teachers.

The Supplementary Work Forms Manual itself suggests many ways in which these forms can be used to help develop important ideas (and, at the same time, provide more drill in the fundamental facts). Teachers can create similar activities by adding whatever numbers they want in a matter of minutes.

Countless other uses for the *Work Forms* can be found in the pupil's texts and the teacher's guides to the other levels of the MATH WORKSHOP. The ideas in the particular adaptations found there are easily applied to any level of development.

The *Work Forms* provide the teacher with a flexibility and versatility never before available in an organized and integrated program. While they are not essential to the use of the MATH WORKSHOP texts, they are an invaluable supplement, an effective means of meeting individual needs, and a help to every teacher in expressing his own individual approach to helping children understand arithmetic.

D. "Mastery Tests of the 390 Basic Facts of Arithmetic"

These tests are contained in a booklet including a battery of six tests, each of which presents all the basic facts of addition, subtraction, multiplication, and division. An answer sheet is provided so that children can check their own or their neighbor's work. In addition, an individual record sheet enables each child to chart his own progress in terms of accuracy and time required to recall all the basic facts.

E. "Discovery in Elementary School Mathematics"

This is a professional development textbook which suggests to teachers many ideas for teaching mathematics at every grade level. This book is divided into two major sections—Multilevel Research Activities and Multilevel Worksheet Activities.

Multilevel Research Activities present 27 exploratory experiences which can be introduced on the chalkboard and with simple, teacher-made manipulative materials at any grade level. Each activity starts simply and spirals upward with extensions and elaborations which will appeal to the more mature and sophisticated mind. Each activity focuses on mathematical structure in a learning context of discovery.

Multilevel Worksheet Activities present 24 units, each of which explores some of the unifying ideas of mathematics. Each unit begins with a brief introductory statement regarding mathematical structure and discovery. Each unit includes six activity worksheets, scaled in difficulty so that the teacher can present the ideas discussed in the unit to pupils at any level. These pages may be copied or modified by teachers for classroom use.

F. "Helping Children Discover Arithmetic"

This is a brief guide to help parents understand the ideas behind MATH WORKSHOP. Specifically, it helps parents see:

1. How we can get children to think creatively in arithmetic by posing challenging questions.
2. How we help children master the basic facts of arithmetic.
3. How parents can help children learn arithmetic.
4. How multiplication, one of the central ideas of arithmetic, is taught from the first through sixth grades in a spiral curriculum.

G. "Games and Enrichment Activities"

We have prepared a manual on the use of representational materials, such as Cuisenaire rods and Geo-boards. These materials can be used as the basis for games and other enrichment activities—many of which are mentioned or introduced on the guide pages. They contain many unusual applications of the concepts considered in the pupil pages.

H. Tests in the Pupil Book

Scattered throughout MATH WORKSHOP, LEVEL B are pages which we suggest you use as tests. These are pages 28-30, 68, 79, 94, 105, and 117. The guide pages suggest point distributions for these tests.

As with any evaluation techniques for young children, care must be taken not to put more faith in tests than they deserve. Your own judgment of a child based on your face-to-face work with him is more valid and probably more reliable than scores on tests.

This brings up the general problem of marking or grouping pupils as they work on all of the pages of the text. The pages should be considered opportunities for investigation and exploration on the part of the pupil rather than as pages to mark for a grade. There are many pages on which pupils will not be expected to get 100% on the first try. Nothing would be more damaging to their regard for mathematical inquiry than to be pummelled with red marks and low grades. By all means, keep a light touch rather than a heavy hand when exploring these pages.

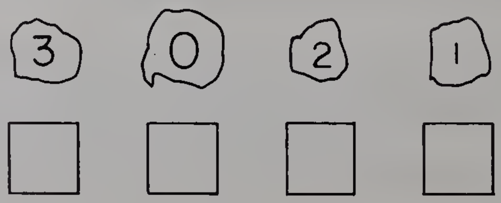
VII. SOME FINAL THOUGHTS

A. Number or Numeral—What Distinctions Shall We Make?

Suppose that you were conducting the following activity with your class and the class was being observed by a visitor who knows nothing about numbers:

"Let's cut out numbers and paste them on a piece of paper in some sensible order. In what order shall we paste them?

"Suppose that these are my cut-out numbers: (Put a sketch on the board.)



How should I paste them down?"

A discussion may ensue. "Let's put the smallest number on the left. Which one is that? Zero, of course. Then, the one, the two, and the three." The pupils then work alone or in groups, cutting out the various shapes and pasting them on a sheet of paper with the smallest number on the left.

The visitor would be completely mystified by the pupils' paste-ups. Surely, the pupils must be mistaken, because the smallest cut-out is the 2, not the 0.

The children are not confused. When you tell these children to draw a tree, they know that you expect them to draw a *picture* of a tree. When you tell them to cut out numbers, they know that you expect them to cut out *symbols* for numbers. They know this because they have learned from activities involving displays of objects and one-to-one correspondence that numbers are abstract entities. They know that they can see 3 stars, they can eat 3 candy cars, and they can lift 3 books, but they can't see, feel, hear, smell, or taste the number 3. They can write a name for this number—the simplest name looks a little like a turned-around capital E. They can say this name.

If children have a rich background of experience in handling physical objects and comparing displays of such objects, they will not confuse numbers with the symbols (called *numerals*) which are the names of the numbers. The reformers of secondary school mathematics education in the 1950's found it necessary to make students aware of the distinction between numbers and numerals because many of these students came out of elementary school with impoverished backgrounds in arithmetic. Such students were taught a brand of arithmetic which consisted largely of symbol-pushing. Big mathematical ideas for such students were rules about turning fractions upside down and moving decimal points. Whatever understanding they may have developed about the abstract existence of numbers through manipulation of physical objects in the first grade diminished severely in the later grades under an avalanche of symbol-pushing activities.

The authors of MATH WORKSHOP have decided to abide by common usage in which the word "number" has two meanings—the abstraction and the symbol. To help pupils build the kind of understanding which will enable them to pick the correct meaning from context, the authors urge a rich and varied series of activities which involve physical objects. (Of course, we also abide by common usage when we speak of Roman numerals and Arabic numerals.)

B. Achievement and Intelligence

Makers of I.Q. tests search for measures of the child's ability to think. They try to rule out questions that merely assess achievement in subject-matter areas. Therefore, they would not ask children to complete the following familiar examples from arithmetic. These questions would more likely appear in an achievement test.

$$\begin{array}{r} 3 \\ +7 \\ \hline \end{array} \qquad \begin{array}{r} 41 \\ -16 \\ \hline \end{array} \qquad \begin{array}{r} 346 \\ \times\ 27 \\ \hline \end{array} \qquad 5\overline{)135} \qquad 8 \times 3 =$$

$$\frac{1}{2} + \frac{2}{3} = \qquad\qquad 1.19 = 17\% \times \underline{\quad}$$

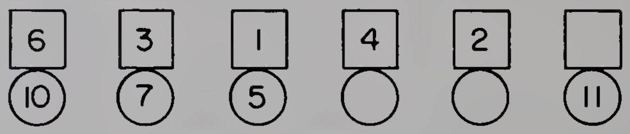
For measurement of an I.Q. the questions from mathematics would be more like those that follow:

I. Extend each of the following sequences:

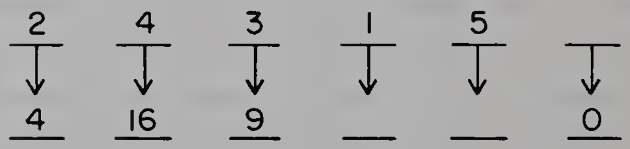
a. 2 , 5 , 8 , 11 , 14 , ___ , ___ , ___ .....
b. 1 , 2 , 4 , 8 , 16 , ___ , ___ , ___ .....
c. 3 , 4 , 6 , 9 , 13 , ___ , ___ , ___ .....
d. 1 , 1 , 2 , 3 , 5 , ___ , ___ , ___ .....

II. Fill in the blanks in the following examples:

a.



b.



c.

1 , 3 → 8   3 , 2 → 10
5 , 0 → 10   6 , 3 → ___
2 , 5 → 14   ___ , 4 → 12

III. Circle the number in each row that seems most out of place:

a. 6 , 10 , 2 , 7 , 8 , 4
b. 25 , 70 , 43 , 16 , 31 , 52
c. 13 , 15 , 19 , 3 , 11 , 7

Such problems requiring a familiarity with arithmetic are held to be more appropriate as measures of the ability to think. They are designed to test intelligence rather than achievement.

Mathematics is a fertile source for I.Q. test questions because mathematics is fundamentally a study of patterns and relationships. Many of the questions selected are similar to the examples given above. Behind that choice is the test-maker's estimate that children have not already explored and investigated such questions.

What an indictment of the elementary school curriculum in mathematics!

If arithmetic is limited to memory of number combinations and rules for manipulation, then it is a dead, dull, dreary task against which an active, eager, creative mind must surely rebel. However, if we broaden our thinking and look everywhere for patterns and relationships, the subject stimulates the mind to inquire and search out and discover!

Every page in MATH WORKSHOP probes beneath the basic facts and processes. We are always exploring the question: Why does arithmetic work? Each activity invites the child to ask: What's going on here?—and urges him to find out.

C. A Final Word

The MATH WORKSHOP has not invented a new mathematics. Nor have its authors spun novel theories of education. We have tried to bring into a whole the most universally accepted ideas of both educators and mathematicians.

Robert W. Wirtz
Morton Botel
Max Beberman
W. W. Sawyer

SCOPE AND SEQUENCE CHART

Level B Revised

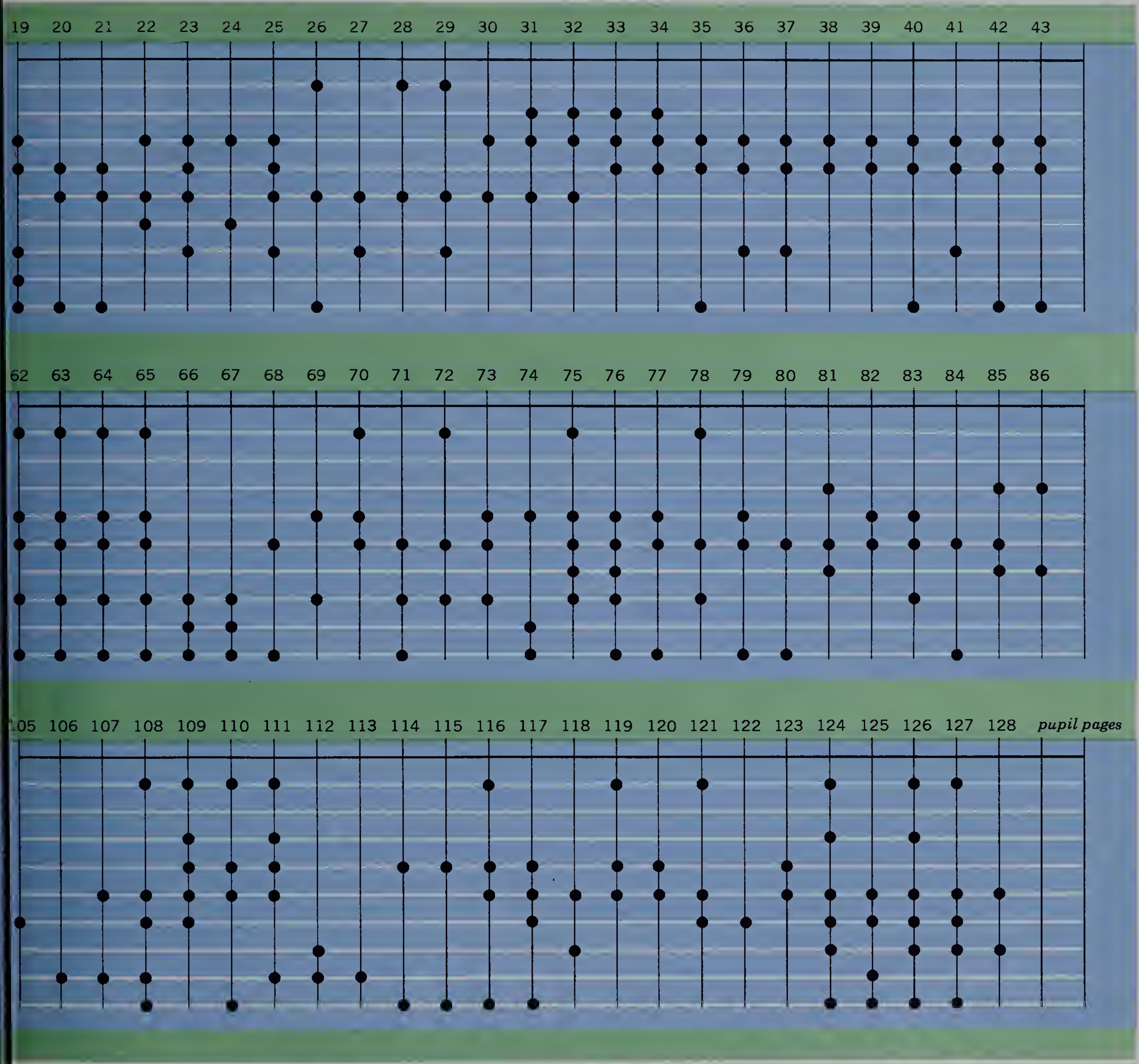
pupil pages	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Unifying Ideas																		
1. Structure					●	●	●						●					●
2. Sets			●	●	●													
3. Number and counting	●	●	●	●	●	●	●	●			●	●						●
4. Numeration															●	●		
5. Addition and subtraction	●	●		●	●	●	●	●		●	●	●	●	●	●	●	●	●
6. Multiplication and division											●	●						
7. Functions and relations			●		●	●		●	●		●			●			●	
8. Geometry	●	●							●									
9. Measurement		●					●		●	●			●	●			●	●

	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61
Unifying Ideas																		
1. Structure															●	●		
2. Sets								●	●									
3. Number and counting			●	●	●	●	●	●	●	●		●		●	●	●	●	●
4. Numeration			●	●	●	●	●	●	●	●	●	●		●	●	●	●	●
5. Addition and subtraction			●	●					●			●	●	●	●	●	●	●
6. Multiplication and division				●									●					
7. Functions and relations			●					●		●								
8. Geometry	●	●																
9. Measurement	●	●											●					

	87	88	89	90	91	92	93	94	95	96	97	98	99	100	101	102	103	104
Unifying Ideas																		
1. Structure						●												
2. Sets																		
3. Number and counting	●	●	●	●	●	●	●		●	●	●							
4. Numeration								●										
5. Addition and subtraction								●		●							●	
6. Multiplication and division	●	●	●	●	●	●	●		●	●	●	●	●	●	●	●	●	●
7. Functions and relations																	●	
8. Geometry								●										
9. Measurement								●							●	●		

Consider each unifying idea . . . on how many pages will you find it?

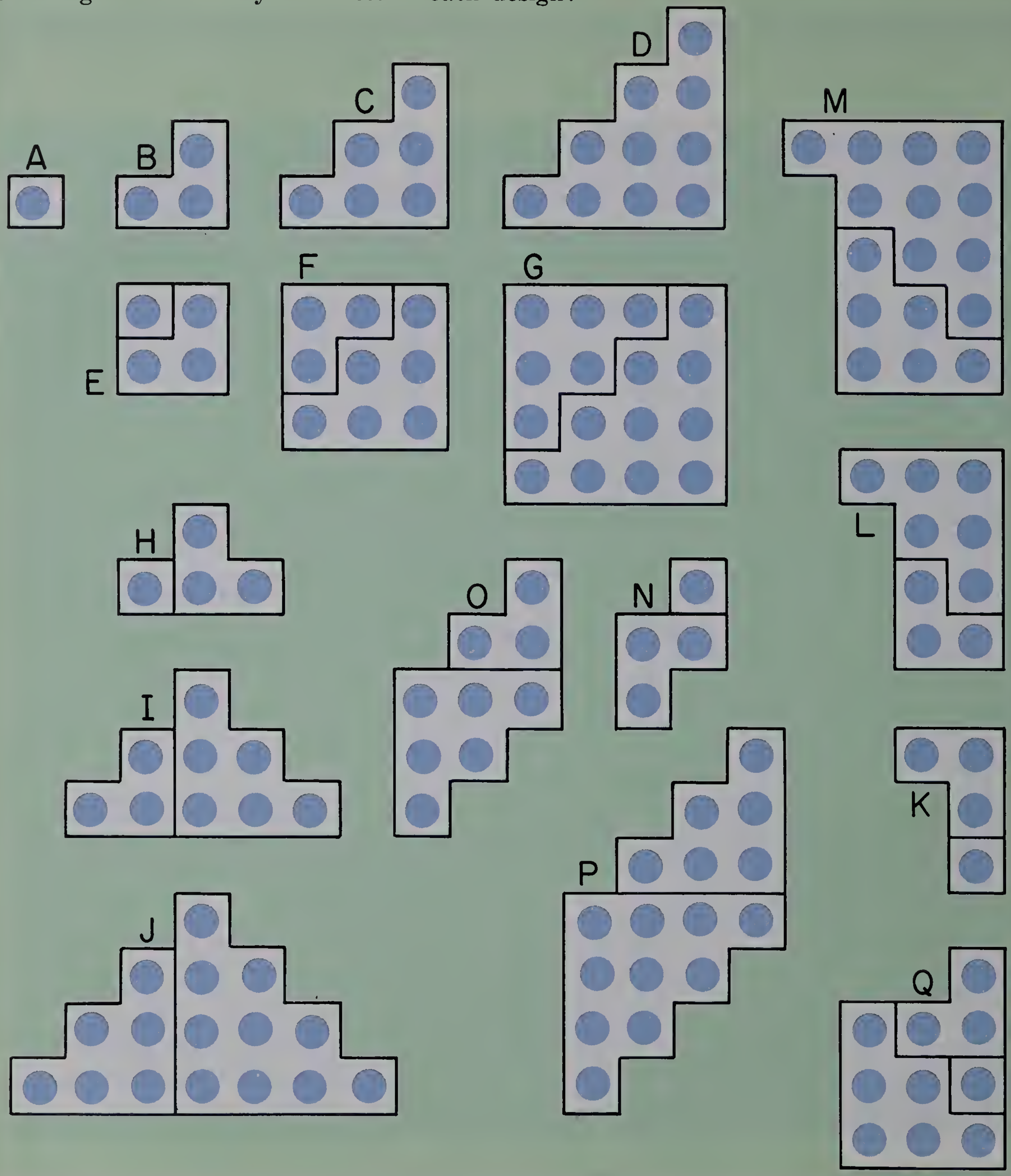
1. Structure	29 pages
2. Sets	9 pages
3. Number and counting	61 pages
4. Numeration	58 pages
5. Addition and subtraction	78 pages
6. Multiplication and division	38 pages
7. Functions and relations	60 pages
8. Geometry	17 pages
9. Measurement	42 pages



Consider each page . . . how many unifying ideas will you find on it?

9 pages contain just one unifying idea,
38 pages contain two unifying ideas,
49 pages contain three unifying ideas,
18 pages contain four unifying ideas,
12 pages contain five unifying ideas,
2 pages contain six unifying ideas.

Counting — How many blue dots in each design?


A
B
C
D
M
E
F
G
H
L
O
N
I
K
P
J
Q

A	B	C	D
1	3	6	10

E	F	G
4	9	16

H	I	J
4	9	16

K	L	M
4	9	16

N	O	P
4	9	16

Q
10

Notes on Pupil Page 1

Tasks: *Pupils find the number of dots in each display of dots and record this number in the appropriate blank at the bottom of the page.*

Purpose: (a) To stimulate the use of previously learned addition and subtraction facts; (b) to encourage the recognition of patterns in representations of the *triangular numbers* and the *square numbers;* (c) to remind pupils of the fact that the number of objects in a display of objects is not affected by rearrangements of the display; (d) to provide practice in associating physical representations with addition and subtraction facts.

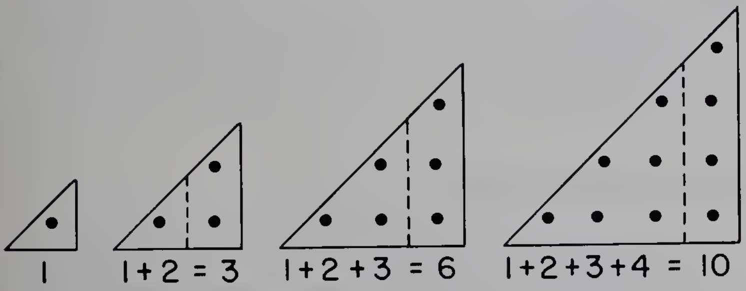
Unifying Ideas: Number and counting; addition and subtraction; geometry.

Materials for Teacher: Three sets of demonstration models of the displays A, B, C, and D shown on the pupil page; several demonstration-size disks or squares to be used as counters.

Materials for Pupils: Cuisenaire rods or similar materials; desk-size counters (disks or squares).

The Lesson:

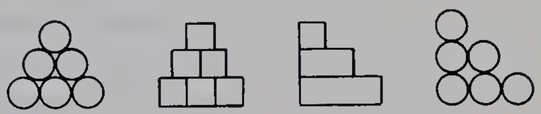
Let's use displays of dots to represent sums of consecutive counting numbers starting with 1: 1, 1 + 2, 1 + 2 + 3, 1 + 2 + 3 + 4, etc. We discover that these displays can be shaped like triangles. The diagram below shows the displays for the first four sums:



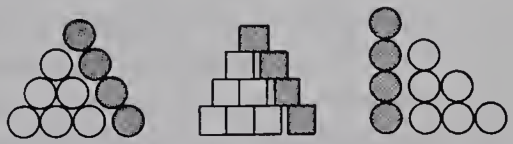
Compare these with displays A, B, C, and D on the pupil page.

These sums of consecutive counting numbers starting with 1 are called *triangular numbers.* The dashed lines in the diagram above suggest a method for computing triangular numbers. For example, to find the second triangular number, add 2 to the first triangular number. To find the fourth triangular number, add 4 to the third triangular number. In terms of the displays of dots, to find the fifth triangular number, adjoin a column of 5 dots to the display of dots for the fourth triangular number and count the dots in the new display. The fifth triangular number is 1 + 2 + 3 + 4 + 5 = 10 + 5 = 15. The first 10 triangular numbers in order are 1, 3, 6, 10, 15, 21, 28, 36, 45, and 55.

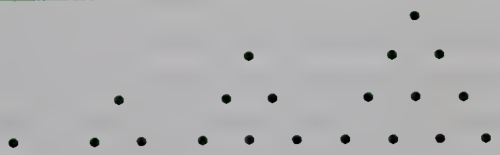
Before pupils begin work on this page, you may wish to do the following to develop an awareness of the pattern of the triangular numbers. Ask the pupils to use their counters or rods to build triangular displays:

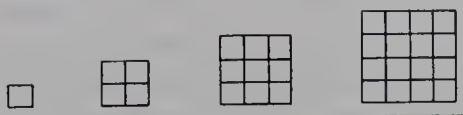


How many in the top row? (Write a 1 on the board.) How many in the next row? (Write + 2.) How many in the bottom row? (Write + 3.) How many in all? (Write = 6.) Ask how many counters, or how long a rod, would be needed to make a display one row taller but with the same shape. Have your pupils do this. They can check their constructions with your demonstration models. Do any pupils see that, if they are using separate round or square counters, they can adjoin a new side instead of a new base in order to make a display for the next triangular number?



Headline for the Day: Continue the sequences:





Write the sentence $1 + 2 + 3 + \underline{\qquad} = \underline{\quad}$. Ask for volunteers to complete the sentence.

Triangular Numbers and Square Numbers

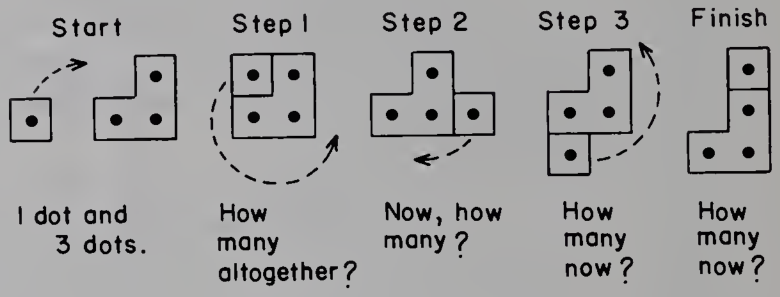
If you play a bit with the demonstration models, you will readily discover that the sum of any two consecutive triangular numbers is a square number. This property is clearly illustrated in displays E, F, and G on pupil page 1. Display E is composed of displays A and B ($1 + 3 = 2 \times 2$); F is composed of B and C ($3 + 6 = 3 \times 3$); G is composed of C and D ($6 + 10 = 4 \times 4$).

We hope that pupils will determine the number of dots in each of displays E, F, and G by adding the component triangular numbers instead of by straightforward counting. You can pave the way for this by using your demonstration models. Hold display A in your right hand and display B in your left hand and face the class. How many dots do you see on this card? (Answer: 1) How many dots do you see on this card? (Answer: 3) Write a 1 and a 3 on the board leaving space for a plus sign. Then, resuming your original position in front of the class, move display B toward display A in such a way that display E is formed. How many dots altogether are there in this *square?* Let pupils count if they wish to. Then complete the sentence you started ($1 + 3 = 4$) and read it aloud (1 plus 3 is—or equals—4). What is the *sum* of 1 and 3? Repeat this activity for displays B and C, and for displays C and D.

You might find a pupil who is aware that the total number of dots in, say, G can be found by adding 6 and 10, but who has forgotten that 6 + 10 is 16. He should be praised for understanding that addition can be used to find the total, and then encouraged to rediscover the addition fact by counting. This illustrates a general teaching procedure to be used whenever a pupil has forgotten an addition combination or fact—let him use physical representations of the addends and then determine the sum by counting. *When in doubt, count!*

We have noticed that the sum of each pair of consecutive triangular numbers is a square number. Displays H, I, and J show that these square numbers can be represented by displays which are not square shaped. How do we know that H, I, and J represent square numbers? Because H, I, and J are each composed of displays for two consecutive triangular numbers, and the sum of two consecutive triangular numbers is a square number. Display H is merely a rearrangement of display E. Since rearranging the dots in a display does not affect the number of dots, both H and E must contain the same number of dots. Since E contains 4 dots, so does H.

To help pupils become aware of this principle of rearrangement, you might use your demonstration models in the following sequence:



With this preparation, it is likely that pupils will do the work on page 1 with an eye open for patterns and shortcuts. We hope they discover that displays K, L, and M are representations of square numbers and that so are displays N, O, and P.

Triangular numbers are considered again on pupil page 124 and square numbers on pupil pages 126 and 103. There is an effective tie-up between the last three pictures in the first row on pupil page 103 and displays H, I, and J on pupil page 1. Both groups of displays do a good job of suggesting that the square numbers are sums of consecutive odd numbers starting with 1. Triangular numbers are the sums of consecutive numbers starting with 1: 1, 1 + 2, 1 + 2 + 3, . . . Square numbers are the sums of consecutive odd numbers starting with 1: 1 + 3, 1 + 3 + 5, . . . It makes one wonder what the pentagonal numbers are.

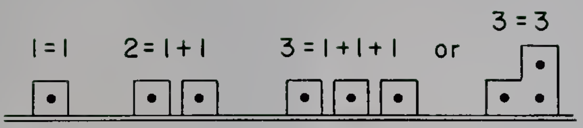
Representing Numbers

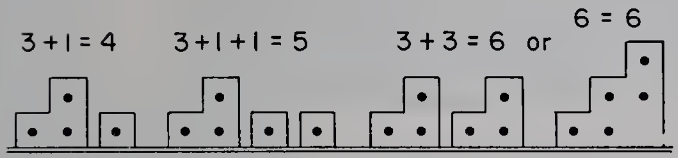
As follow-up activities to the work on pupil page 1 or as further preparatory activities, consider the following problems.

Problem 1

Using not more than 3 out of the 12 cards (3 sets of cards showing displays A, B, C, and D), can you arrange cards along the chalkrail to represent each of the numbers 1, 2, 3, 4, 5, etc.? See how high you can go without skipping a number.

As pupils arrange the cards, write the addition facts above their arrangements:





If you try this problem yourself, you will see at once that it entails a great deal of drill on addition and on counting.

Let pupils keep a record of results as they carry out the experiments:

1	2	3	4	5
1	1 + 1	1 + 1 + 1	1 + 3	1 + 1 + 3

6	7	8
3 + 3 6	1 + 3 + 3 1 + 6	1 + 1 + 6

9	10	11	12
3 + 3 + 3 3 + 6	1 + 3 + 6 10	1 + 10	1 + 1 + 10 3 + 3 + 6 6 + 6

13	14	15	16
1 + 6 + 6 3 + 10	1 + 3 + 10	3 + 6 + 6	3 + 3 + 10 6 + 10

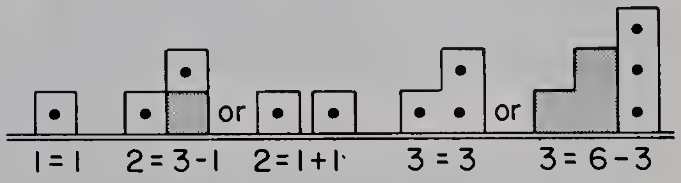
17	18	19	20
1 + 6 + 10	6 + 6 + 6	3 + 6 + 10	10 + 10

21	22	23
1 + 10 + 10	6 + 6 + 10	3 + 10 + 10

The number 24 cannot be represented under the given conditions.

Problem 2

Let's agree that placing one card face down on another suggests subtraction. Now try to represent each of the numbers from 1 through 10 with not more than two cards:



4 = 3 + 1 = 10 − 6   5 = 6 − 1   6 = 6 = 3 + 3   etc.

The displays on the pupil page suggest additional facts to be recorded. For example, display E suggests:

$1 + 3 = 4$ and $4 = 1 + 3$
$3 + 1 = 4$ $4 = 3 + 1$
$4 - 1 = 3$ $3 = 4 - 1$
$4 - 3 = 1$ $1 = 4 - 3$

Display J suggests:

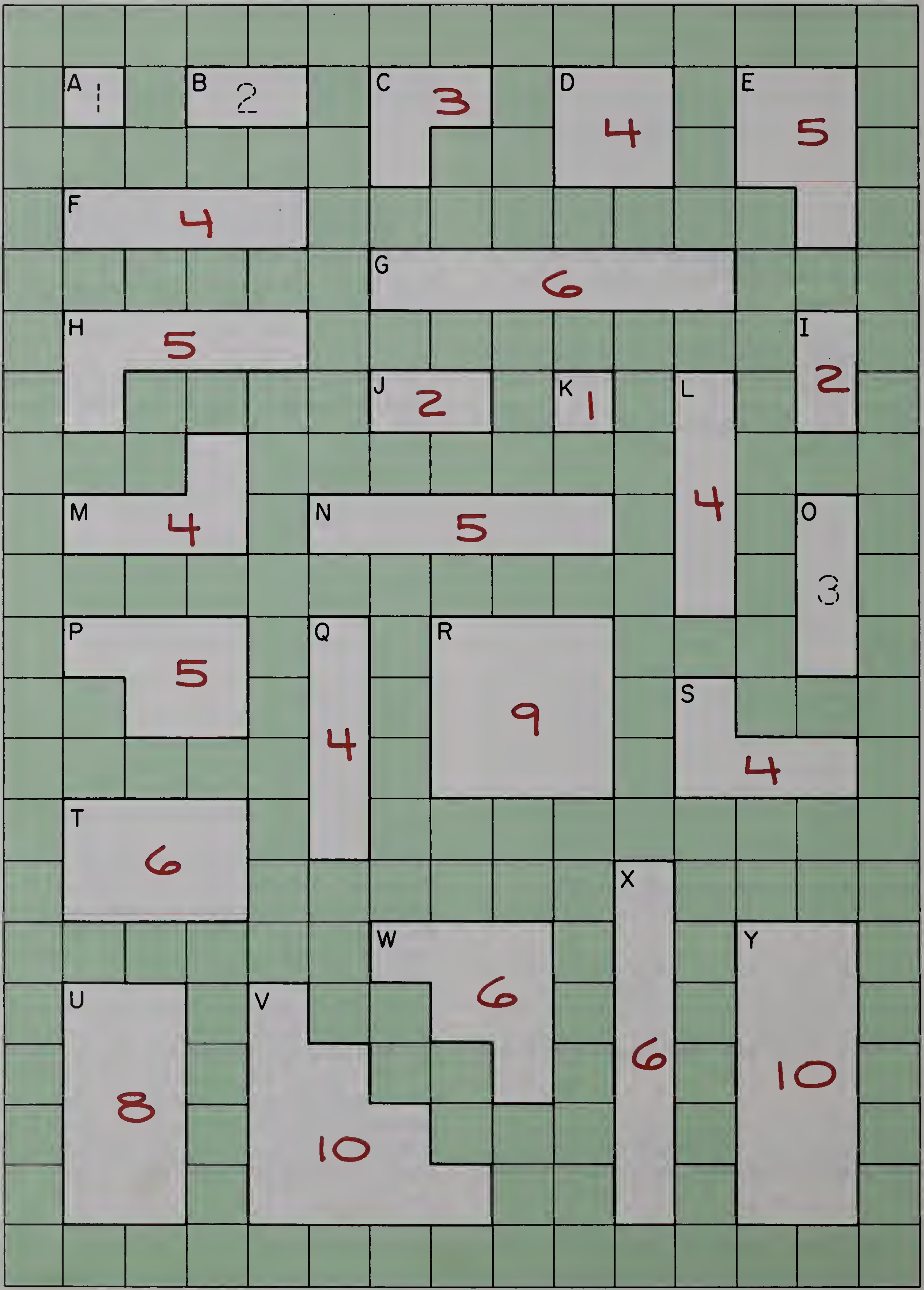
$$16 = 6 + 10 = 1 + 3 + 5 + 7$$

Display P suggests:

$$16 = 6 + 10 = 4 + 4 + 4 + 4$$
$$= 1 + 2 + 3 + 4 + 3 + 2 + 1$$

Let pupils study the displays and state additional facts.

We call your attention to the booklet *Games and Enrichment Activities* which accompanies this Teacher's Guide. This booklet contains many practical suggestions for supplementary and enrichment work. For example, if you find pupils who need help with addition, you will find useful suggestions under "Finger Play—A Thinking Game" on pages 11-12 of the booklet and under "Domino Cards" on pages 15-17. The booklet contains a large section on Cuisenaire rods (pages 31-79) which will prove very helpful. From time to time we shall call your attention to the *Games and Enrichment Activities* booklet. But don't wait for us. Look it over right now and familiarize yourself with its contents.


A 1
B 2
C 3
D 4
E 5
F 4
G 6
H 5
I 2
J 2
K 1
L 4
M 4
N 5
O 3
P 5
Q 4
R 9
S 4
T 6
U 8
V 10
W 6
X 6
Y 10

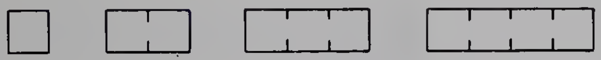
Notes on Pupil Page 2

Tasks; *Pupils determine the number of "tiles" that are missing in each of the "holes in the floor" and record this number in the "hole."*

Purpose: To provide practice in (a) using shortcuts to counting, (b) recognizing rectangular shapes, (c) finding areas of geometric figures, and (d) working with odd numbers and even numbers.

Unifying Ideas: Number and counting; addition and subtraction; geometry; measurement.

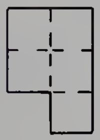
Materials for Teacher: Demonstration-size rectangular strips of cardboard with short separation marks:



Prepare three each of the four sizes of cardboard strips. (Coat the back of each strip with rubber cement so that it will readily adhere to a flannel board.) The smallest of the strips should be square in shape.

Materials for Pupils: Desk-size versions of the teacher's materials, or Cuisenaire rods.

The Lesson:

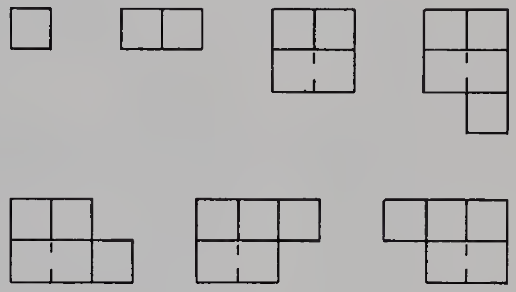
Some pupils may find the number of missing tiles in a hole such as E by drawing a vertical line and two horizontal lines  and then counting the 5 tiles. This is a good but primitive way of proceeding. We hope that such pupils will gradually come to use a more sophisticated technique. For example, after determining that D requires 4 tiles, they should see that, since E is 1 tile larger, it requires 5 tiles. Similarily, upon determining that X requires 6 tiles, they should note that Y is 5 tiles high ($6 - 1 = 5$) and, since Y is 2 tiles wide and $5 + 5 = 10$, Y requires 10 tiles. Or, a pupil might recognize that Y is 2 tiles wide and then count by 2's to determine that Y requires 10 tiles.

The ability to recognize a rectangular shape and the ability to use this recognition to eliminate counting will help a pupil learn multiplication. You can help pupils develop these abilities and get the most out of pupil page 2 by carrying out some of the following preliminary activities.

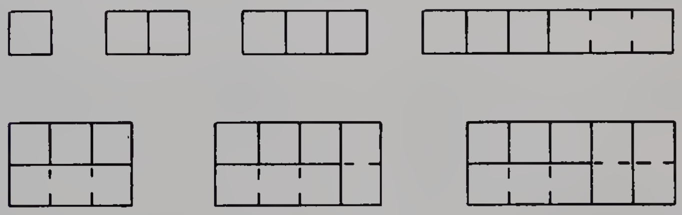
Activity 1

Display on the flannel board each of the following designs, one at a time. For each design ask: How many tiles? The second design in any of the following sequences is made by modifying the first, the third by modifying the second, etc.

Sequence I

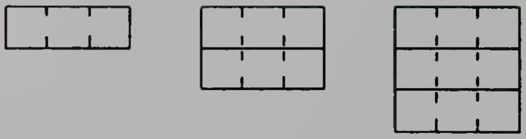


Sequence II

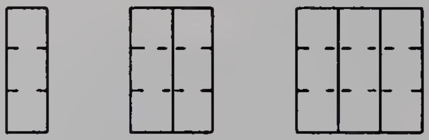


Headline for the Day: Complete the headline $3 + 5 = \square$ and tell a money (or time) story to fit it.

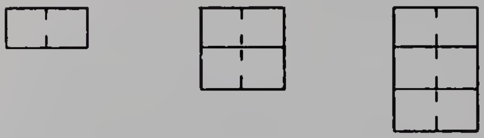
Sequence III



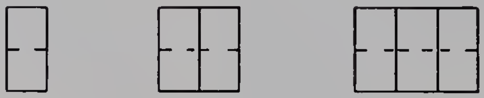
Sequence IV



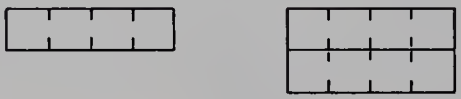
Sequence V



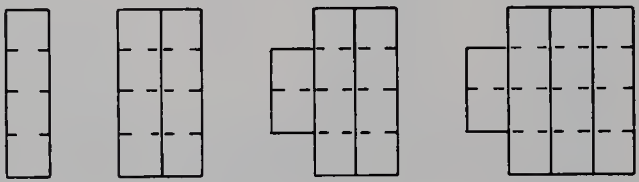
Sequence VI



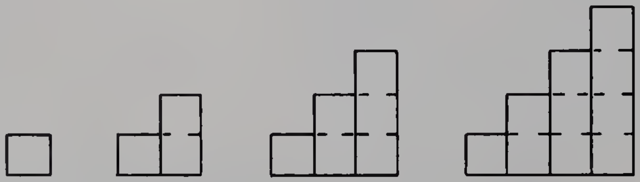
Sequence VII



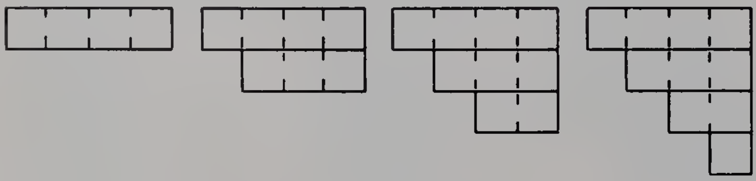
Sequence VIII



Sequence IX



Sequence X



Activity 2

Ask pupils to use *exactly two* of the rectangular strips to make designs to represent the whole numbers from 2 through 8. Have them record the appropriate addition facts as they proceed. Examples: 2 = 1 + 1, 3 = 1 + 2, 4 = 2 + 2, etc.

Activity 3

Repeat Activity 2 but use *exactly three* strips to represent the whole numbers from 3 through 12. Examples: 3 = 1 + 1 + 1, 4 = 1 + 2 + 1, . . . 9 = 2 + 3 + 4, etc.

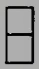
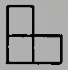
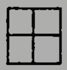
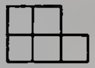
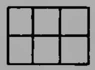
When pupils have completed page 2, you may wish to pursue the following activities.

Activity 4

Using exactly two strips of the same size, which of the numbers from 1 through 8 can be represented? Answers: 2, 4, 6, 8, because 2 = 1 + 1, 4 = 2 + 2, 6 = 3 + 3, and 8= 4 + 4.

Activity 5

Put the following table on the chalkboard and invite helpers to extend it:

	I	II
1	1	
2		 2
3	 3	
4		 4
5	 5	
6		 6
7		
8		

What are the numbers in Column I called? (Odd numbers.) What are the numbers in Column II called? (Even numbers.) There are two kinds of designs here, one kind for even numbers and one kind for odd numbers. Can you combine the designs for a pair of even numbers to get a design for an even number? (Yes, you can combine them to make a rectangle which is 2 tiles high, and this is a design for even numbers.) Can you combine the designs for two odd numbers to get the design for an odd number? (No, you can combine these designs to make a rectangle which is 2 tiles high, and this represents an even number.)

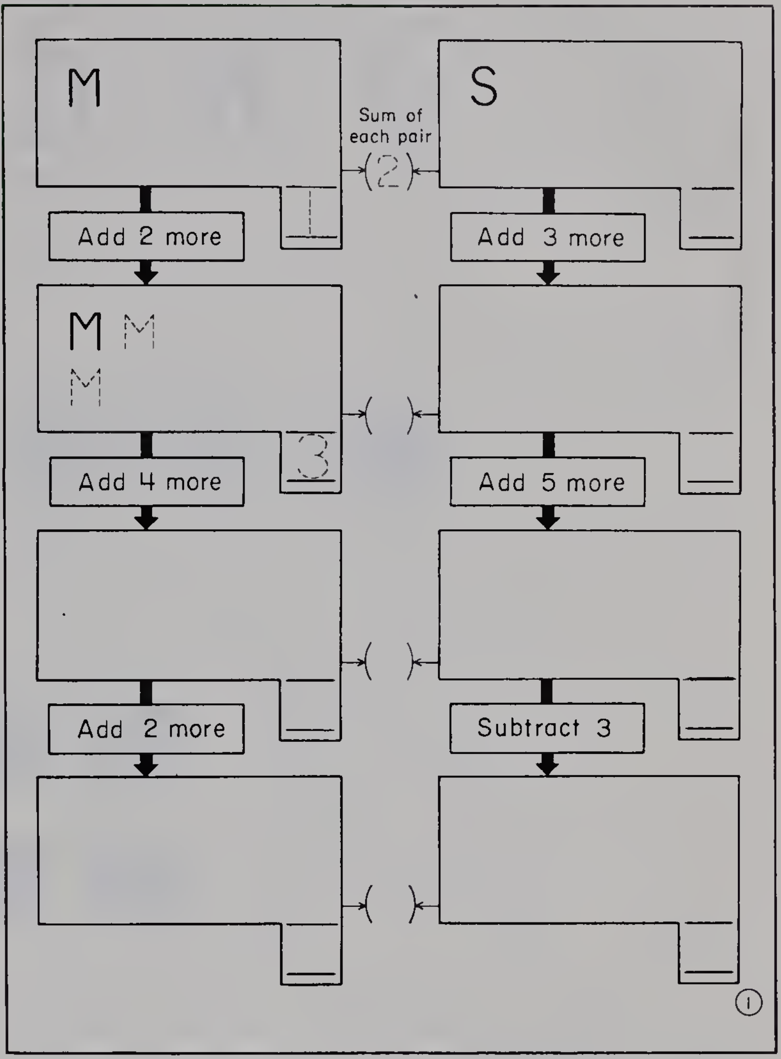
Activity 6

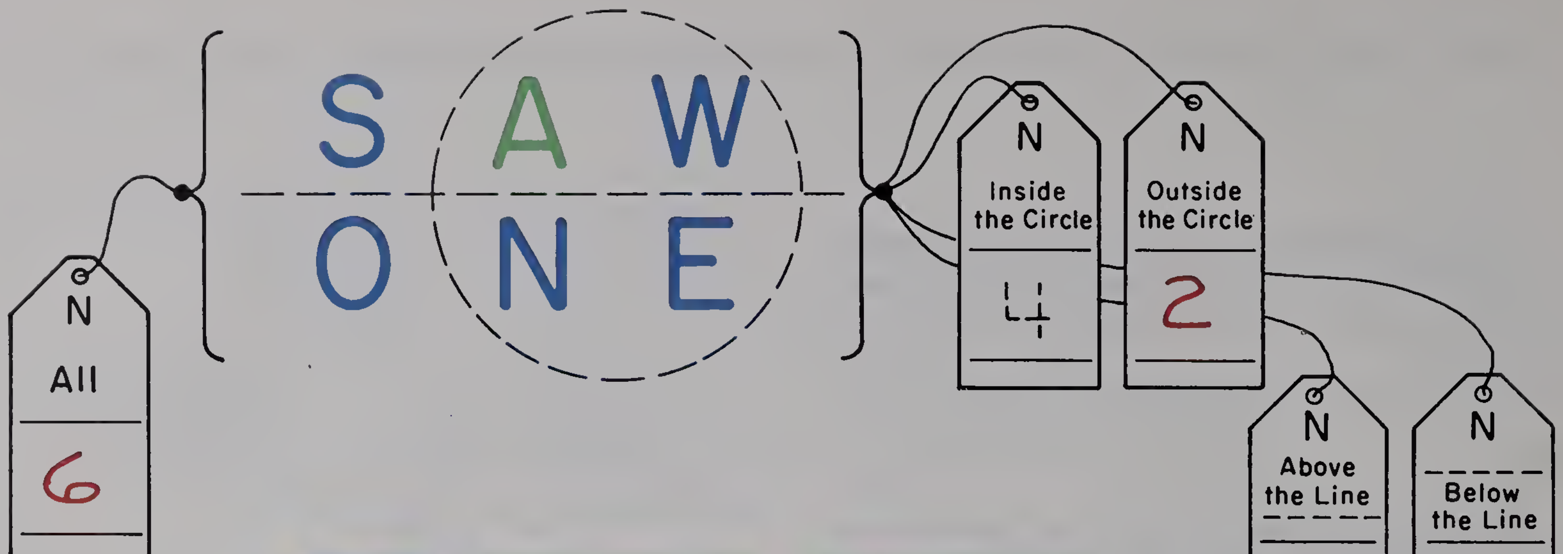
Ask pupils to classify the "holes" on pupil page 2 according to whether they represent even numbers or odd numbers. They might write an O or an E on each hole.

Drill

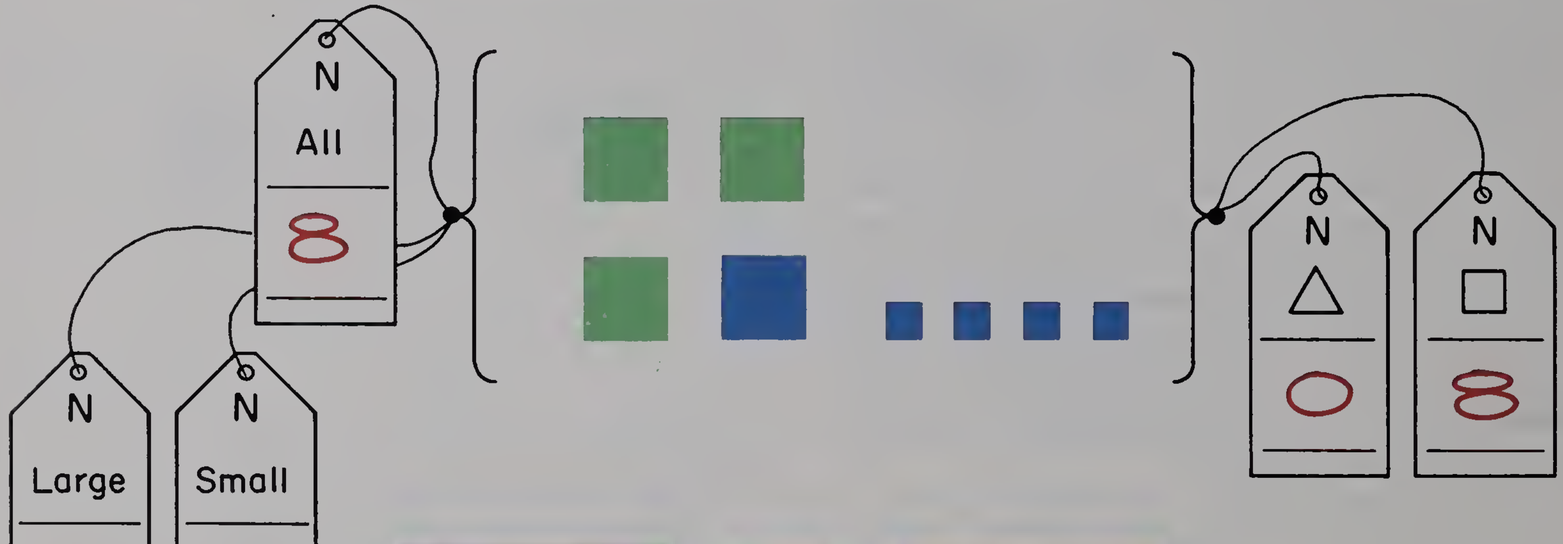
The authors of Math Workshop have put together supplementary materials which can be used for drill work or for remedial work. From time to time we shall call your attention to work sheets from Math Workshop Supplementary Activities, Level B. There are 64 work sheets in the set and these are accompanied by a teacher's manual. For example, you might wish to use page 1, shown in the next column, with your class at this time.

If many of your pupils need extensive remedial work, you might consider using work sheets from Math Workshop Supplementary Activities, Level A. On the other hand, for the "front-runners" you might try some work sheets from Math Workshop Supplementary Activities, Level C. The teacher's manual for each level contains answer-marked copies of the 64 work sheets for that level plus suggestions for their use.




N
All
6
S A W
O N E
N
Inside the Circle
4
N
Outside the Circle
2
N
Above the Line
3
N
Below the Line
3

6 = 4 + 2 = 3 + 3


N
All
8
N
Large
4
N
Small
4
N
△
0
N
□
8

4 + 4 = 8 = 0 + 8

What's My Rule?

A.

x	2	5	1	7	9	14	16	18	19	29
x + 1	3	6	2	8	10	15	17	19	20	30

B.

x	6	2	4	9	1	10	13	21	20	30
x − 1	5	1	3	8	0	9	12	20	19	29

Notes on Pupil Page 3

Tasks: *For the top and middle thirds of the page, pupils count and record the objects (letters, squares, triangles) in each display as described on the number tags and then complete the equations. For the bottom third, pupils discover the rules and demonstrate their discoveries by filling in the blank cells in the table.*

Purpose: (a) To relate the partitioning of sets and addition; (b) to remind pupils that sets of two or more members can be partitioned in more than one way; (c) to give practice in observing patterns; (d) to give practice in adding and subtracting 1.

Unifying Ideas: Sets; number and counting; functions and relations.

Materials for Teacher: Colored chalk; colored Ditto masters; green and blue construction paper.

Materials for Pupils: Crayons; Dittoed sheets; Cuisenaire rods.

The Lesson:

The top third of the page shows the results of two partitionings of the given display. The given display consists of 6 letters arranged in 2 rows and 3 columns. The braces serve as a boundary for the display and as something from which to hang the number tags; the braces are not part of the display.

One of the partitionings was made when someone enclosed 4 of the letters in a circle. The circle separates the display into two smaller displays, one consisting of 2 letters and the other consisting of 4 letters.

The other partitioning was made when someone drew the horizontal dashed line. It separates the given display into two smaller displays, each consisting of 3 letters.

The fundamental notion that for any partitioning of a display the sum of the numbers for the parts is the number for the given display, is indicated by the equations $6 = 4 + 2$ and $6 = 3 + 3$. The equation $4 + 2 = 3 + 3$ illustrates the general principle that no matter how you partition a display, the sum of the numbers for the parts is fixed. This general principle is referred to again and again throughout arithmetic. It provides us with a basis for numeration systems and for the algorithms for addition, subtraction, multiplication, and division.

The exercise in the middle third of the page asks pupils to classify the objects in a display according to criteria which, in contrast to the preceding exercise, do not depend upon the positions of the objects in the display. As in the case of the previous partitionings, if the classes are disjoint (no object in two classes) and exhaustive (each object in some class) then the sum of the numbers for the classes is the number of objects in the given display.

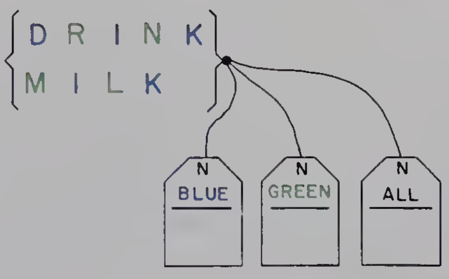
The use of the triangle-square classification may present a conceptual difficulty. Pupils can be helped with this by considering questions like:

> How many elephants are there in one classroom?
>
> If Bill spent all of his money, how many pennies does he have left?

The answer to each question is 0.

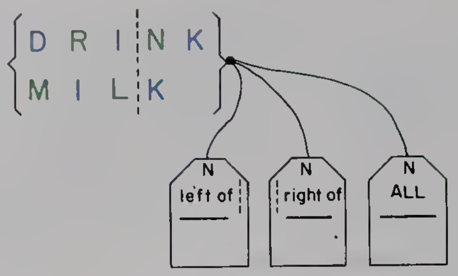
Prepare pupils for the page by constructing, in stages, similar exercises at the chalkboard, letting volunteers fill in the tags and the equations. Make sure that pupils know how to read an equation ($5 + 4 = 9$ is read as five plus four is—or equals—nine).

Exercise 1

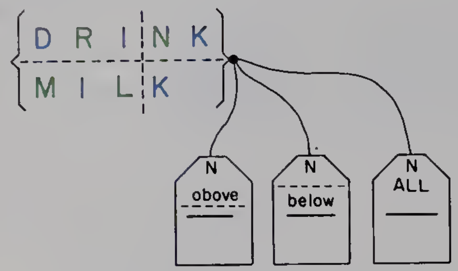


Equation: ________ + ________ = ________

Headline for the Day: I have 10 cents. How many coins do I have?

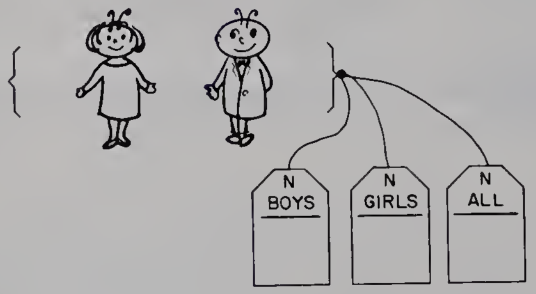


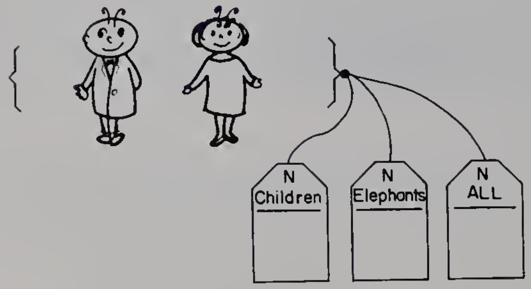
Equation: ____ + ____ = ____



Equation: ____ + ____ = ____

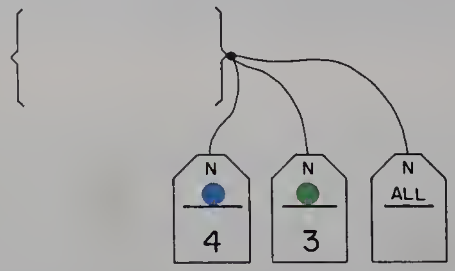
Exercise 2





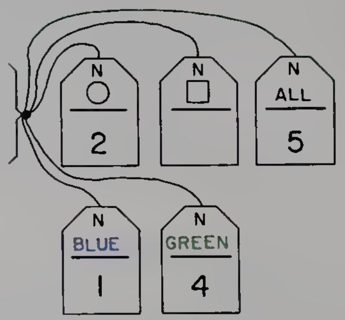
Some highly provocative discussions can result from presenting pupils with reconstruction exercises like the following, which could be represented on Dittoed sheets.

Exercise 3



Equation: 4 + 3 = ____

Exercise 4



Equations: 2 + ____ = 5

1 + 4 = ____

You may wish to prepare pupils for the middle third of the page as follows. Cut out from green and blue construction paper some large green squares, some large blue squares, and some small blue squares. Each pupil should receive 3 large green squares, 1 large blue square, and 4 small blue squares. The pupils are asked to separate their squares into two piles so that squares of one kind are in one pile and those of a different kind are in the other. Any pupil who doesn't construct either of the two situations shown below will probably have difficulty with the problem on the pupil page.

Situation A — green squares | blue squares

Situation B — large squares | small squares

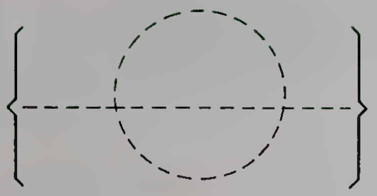
Questions similar to the following may be of some help to these pupils:

Are all the squares large or are some large and some small?

Separate your squares into two piles so that one pile is the large-squares pile and the other is the small-squares pile. Does the large-squares pile have any blue squares in it? Does the small-squares pile have any green squares in it? In what other way could we have chosen squares for two piles?

Make two piles, one for each color. Does the blue pile have any large squares in it? Does the green pile have any small squares in it?

Another useful preliminary activity is the following. Distribute Dittoed sheets with the picture that is shown on the next page on them.



Have the pupils place some red and light green Cuisenaire rods on their papers so that some are inside the circle, some outside the circle, some above the line, some below the line. Then ask questions such as the following:

> Are all your rods in the circle? Outside the circle? Are some inside and some outside? Are all the rods inside your circle the same color? Are all those outside the circle the same color? Are the rods all above the line? Below the line? Some above and some below? Are the rods above the line the same color? Below the line the same color?
>
> How many red rods are inside your circle, John? How many outside? How many green rods inside your circle, Jane? How many outside? Bill, how many red rods are above your line? How many below your line? Joan, how many green rods are above your line? How many below?

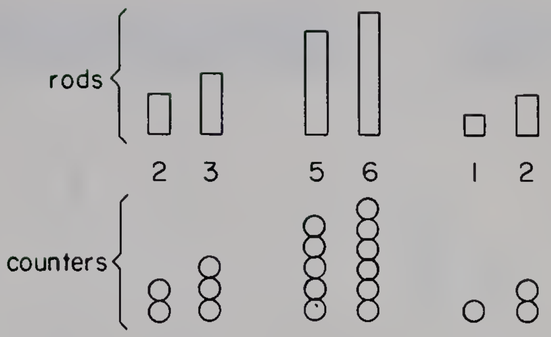
For more easy work of this kind, see "Spatial Relations Activities" on pages 5-6 of the booklet *Games and Enrichment Activities*. For some very challenging work like this, see "And-Or Games" on pages 28-29 of the same booklet.

What's My Rule?

The *What's My Rule?* exercises are a recurrent feature of MATH WORKSHOP. To familiarize pupils with these exercises, we suggest the following preliminary activity. Ask volunteers to call out their "favorite number." As each is called out, you reply according to some secret yet simple rule. After a few rounds, record the results on the chalkboard. The record might look like this:

Called out	Reply
2	3
5	6
1	2

Pupils can record these results visually at their desks with rods or counters, representing the number called out on the left and the reply on the right; for example:



As some pupils show signs of having guessed your rule, ask them if they can tell what your reply will be to some number they call out. You said your favorite number is 4. What do you think I will reply? In this way a pupil can demonstrate his discovery of the rule without giving away the secret. Each example is another clue to those who are still looking for the rule.

Such rules can be further explored as real-life situations. For example, all we know is that Bill has one more pencil than Jim has. Now the class can complete such statements as:

If Jim has 5 pencils then Bill has ___ pencils.
If Jim has no pencils then Bill has ___ pencils.
If Bill has 4 pencils then Jim has ___ pencils.

A record of the questions and appropriate answers might be:

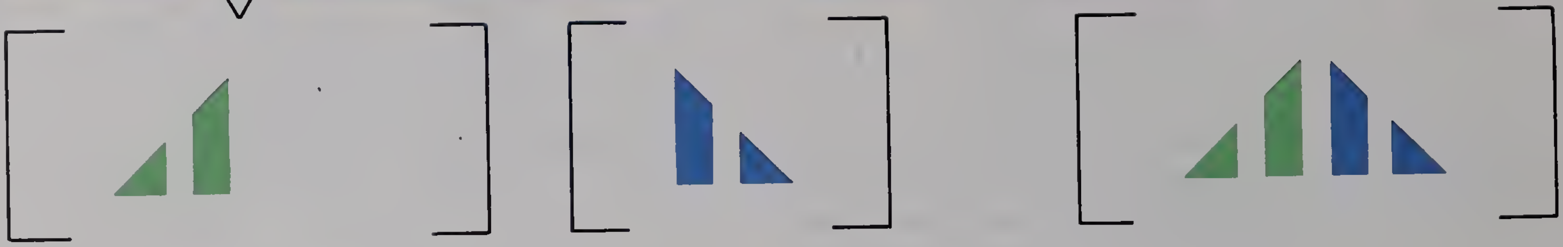
Number of pencils Jim has	Number of pencils Bill has
5	6
0	1
3	4

Many pupils will wish to continue playing *What's My Rule?* as a game, making up their own rules for the others to discover. Also, you may wish to put some difficult examples on the board and leave them for several days. The first pupil to reveal the "secret" becomes the "expert" for that example.

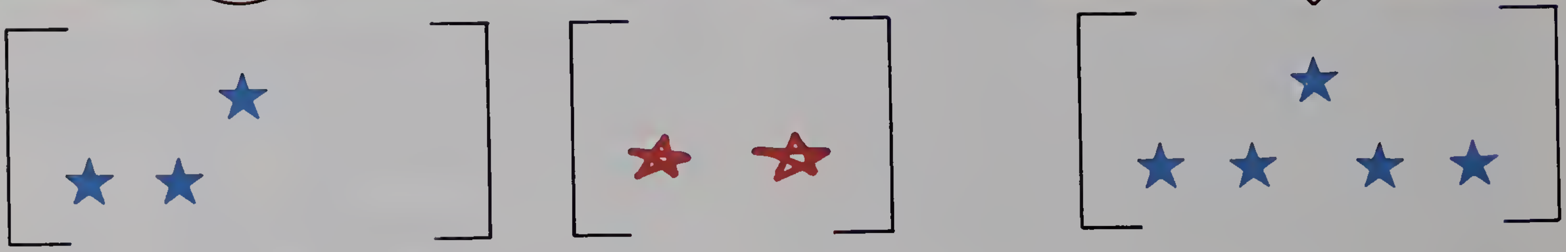
Pattern Indicators

Note the expressions x and $x + 1$ at the left ends of the rows in the first *What's My Rule?* exercise. Avoid discussion of the "meanings" of these expressions. We are using them as pattern indicators. This use will dawn on pupils as they proceed through the course. If you refer to them at all, use them as row labels—if 7 is in row x then 8 should be in row $x + 1$.

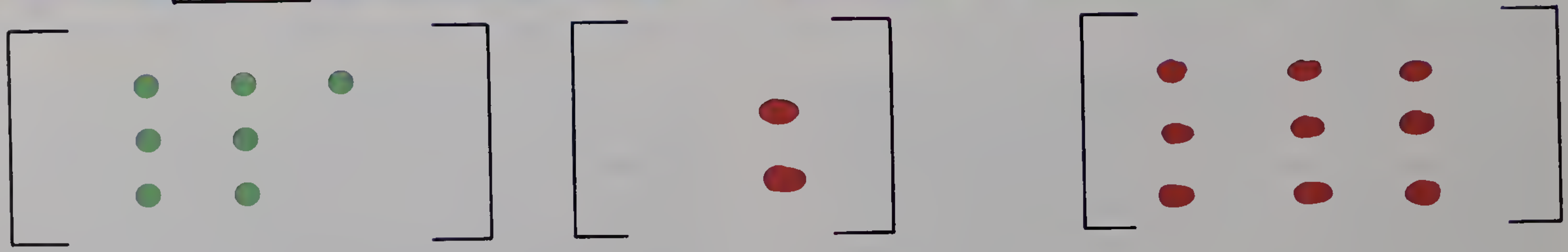
I 2 + 2 = 4



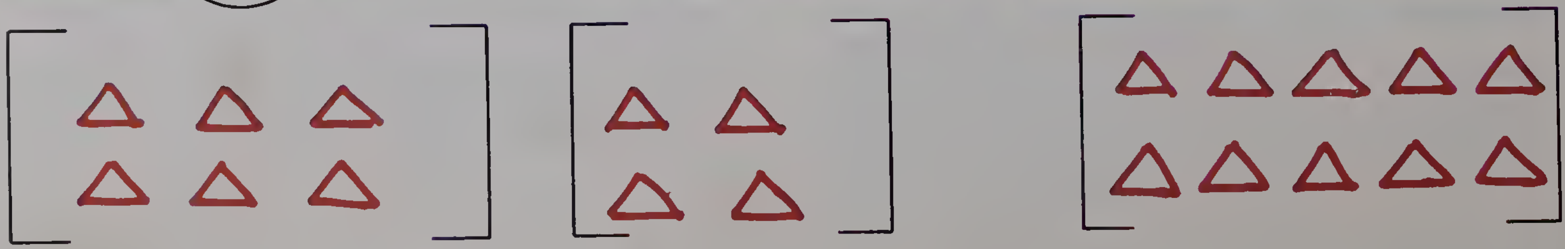
II 3 + 2 = 5



III 7 + 2 = 9



IV 6 + 4 = 10



Tasks: Pupils fill the frames and draw displays so that each completed addition sentence is represented by the corresponding pictures.

Purpose: To review addition in terms of manipulating objects.

Unifying Ideas: Sets; number and counting; addition and subtraction.

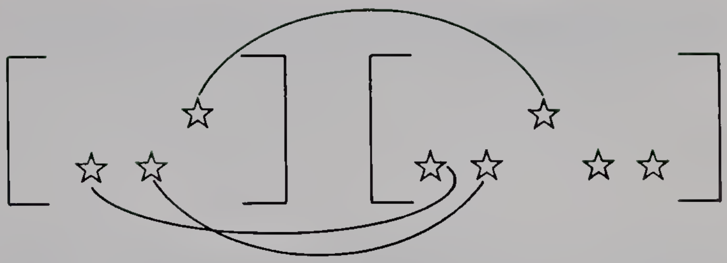
Materials for Teacher: Ditto masters; cards described below.

Materials for Pupils: Counters; Dittoed sheets; Cuisenaire rods for supplementary activity.

The Lesson:

In Exercise I the pupil sees three displays, the first two displays having no objects in common, and the third display consisting of the objects in the first two displays. This physical situation is a faithful reflection of the formal definition of addition in terms of sets. To find the sum of 2 and 2, one considers a pair of 2-sets which are disjoint from each other, and then considers their union. The number of members in the union is, by definition, the sum of the numbers of members in the disjoint 2-sets. The pupil expresses this fact by counting the members of each of the three sets and writing the equation $2 + 2 = 4$.

In Exercise II the pupil is faced with the problem of finding the missing addend in the equation $3 + \triangle = 5$. This may be solved by working with displays. Choose a representation of the first addend (3) and a representation of the sum (5):



Then match each object in the first display with an object in the second display. The number of unmatched objects in the second display is the missing addend.

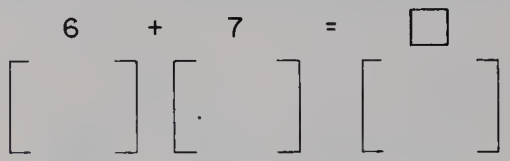
Exercise III presents a variation of exercise I. Here the problem is to find the sum of 7 and 2. We are already given a representation of 7. We choose a representation of 2 which does not overlap the representation of 7, that is, the representations have no objects in common. Then we combine the representations into a single display. Since the single display is a representation of the sum, we can find the sum by counting the objects in the single display. Thus, $7 + 2 = 9$.

Exercise IV presents the basic problem of finding the sum of two numbers. One does this by selecting disjoint (nonoverlapping) displays for the addends, combining these into a single display, and counting the objects in the single display.

Naturally, we do not expect pupils to do all their addition problems by working with displays. Our purpose in this lesson is to make sure that pupils realize that the addition operation for numbers is based on combining displays. Any time a pupil is faced with an addition problem which he cannot solve because he cannot recall or doesn't know the relevant fact, he may find the answer by working with displays.

Headline for the Day: $7 + 3 = 3 + 7$

To prepare for this page, give them problems at the chalkboard like the following:

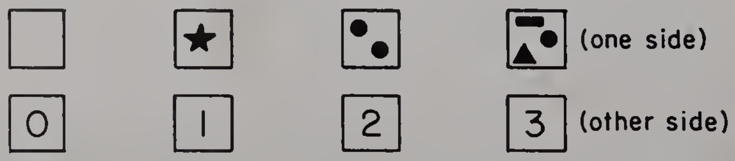


A volunteer comes to the board and makes six cross marks under the 6. Another makes seven cross marks under the 7. A third volunteer erases each of the cross marks, one by one, and makes a new cross mark for it under the □. A fourth volunteer counts the cross marks under the □ and writes the total in the □. Pupils should do the same problem on a Dittoed sheet at their desks. They should use counters to represent 6 and 7, move the counters with a sweep of the hand into the space under the □, and count to find the sum.

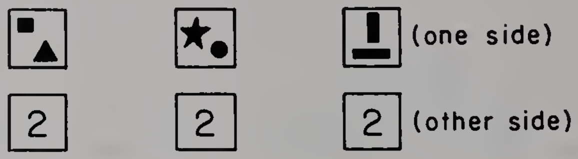
It is important that pupils realize that the same sum is obtained no matter what choice is made of physical representations for the addends. Square counters give the same result as round counters, and round counters give the same result as cross marks or apples or rabbits.

Another issue of importance is that the representations or displays used for the addends must be disjoint. You can make this point by trying to find the sum of 3 and 4 in the following *incorrect* way. Call Mary, Ezra, and Sue to the front of the room and point out that these are 3 children. Ask them to take their seats. Then announce that you will ask 4 children to come to the front of the room, and call upon Fred, Beth, Ezra, and Alice. The class will agree that these are 4 children. After the 4 children have taken their seats, ask all the children you have called to come to the front of the room. Ask the class to tell you how many children are at the front. Does this mean that 3 + 4 = 6? Let the children decide what is wrong. (See "And-Or Games" on pages 28 and 29 of the booklet *Games and Enrichment Activities.*)

Lest pupils get the idea that number sentences *must* be represented in the manner shown on this page, present the following activity. Make a deck of cards each with a number listed on one side and pictures of objects on the other side:



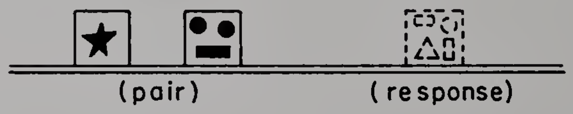
Preferably, you should make several cards for each number with different objects pictured on the other side:



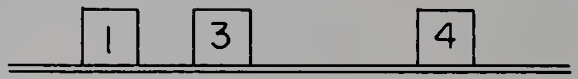
One use for these cards is to place several on the chalkrail in random order with the objects-sides showing. A pupil selects a card, counts the objects shown—1, 2, 3—and turns it over, showing the 3. Each card is used similarly.

Or you may ask: How many pictures on the back of this card (pointing, perhaps, to a 2-card)? As soon as any response is given, turn the card over and count. This will help initiate the idea that ought to grow—*When in doubt, count.*

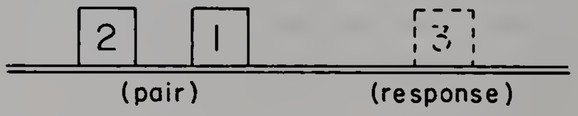
Later, these cards can be used in a way that anticipates addition. Display a pair of cards. Can you find a single card that has the same number of pictures as the two together have?



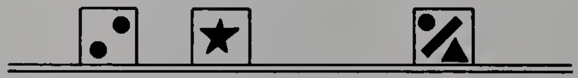
Now turn the cards over. The pupils see:



Later, begin with the numbers:



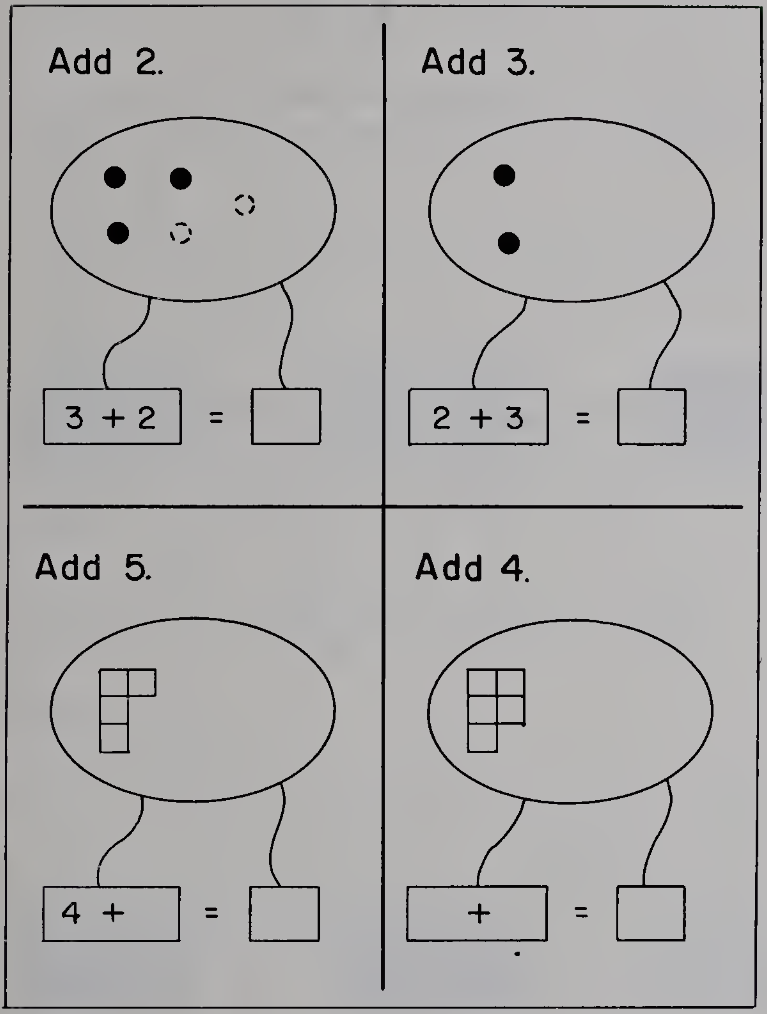
When the cards are turned over, the total number of objects for the pair taken together is to be the same as the total number of objects on the card selected:



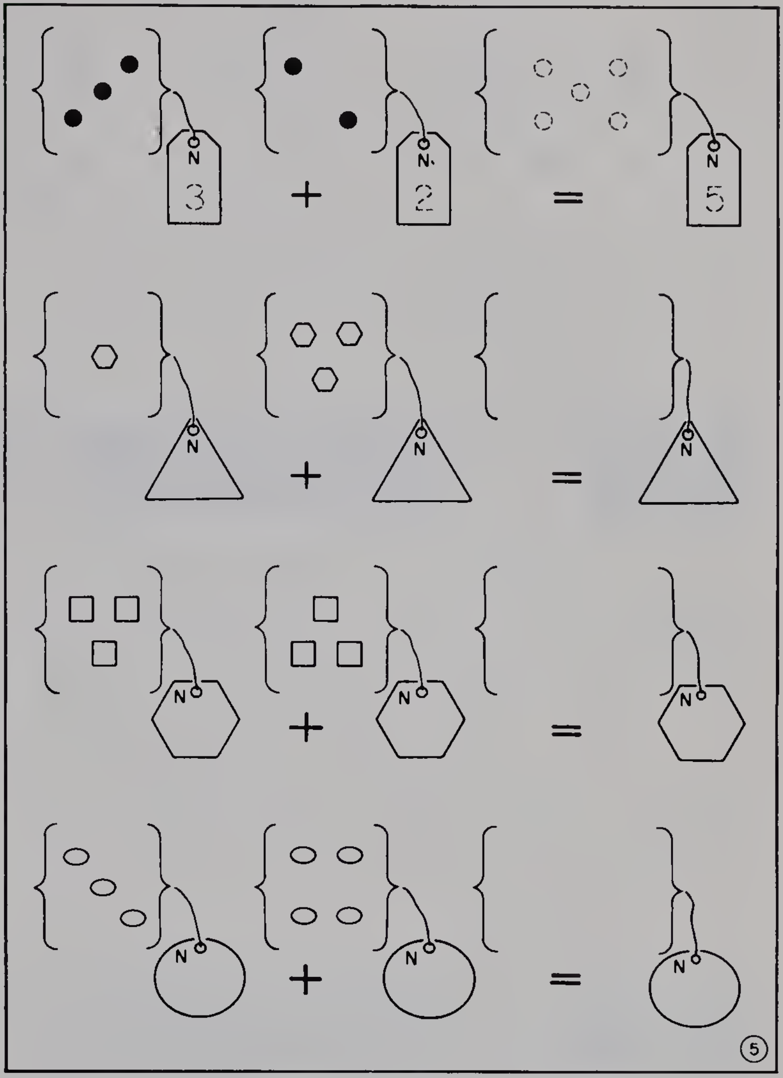
If you have Cuisenaire rods, you will be interested in "Number Sentences," "The Missing Sum Game," "The Missing Addend Game," and "The Missing Sum or Missing Addend Game" on pages 41-44 of the booklet *Games and Enrichment Activities.*

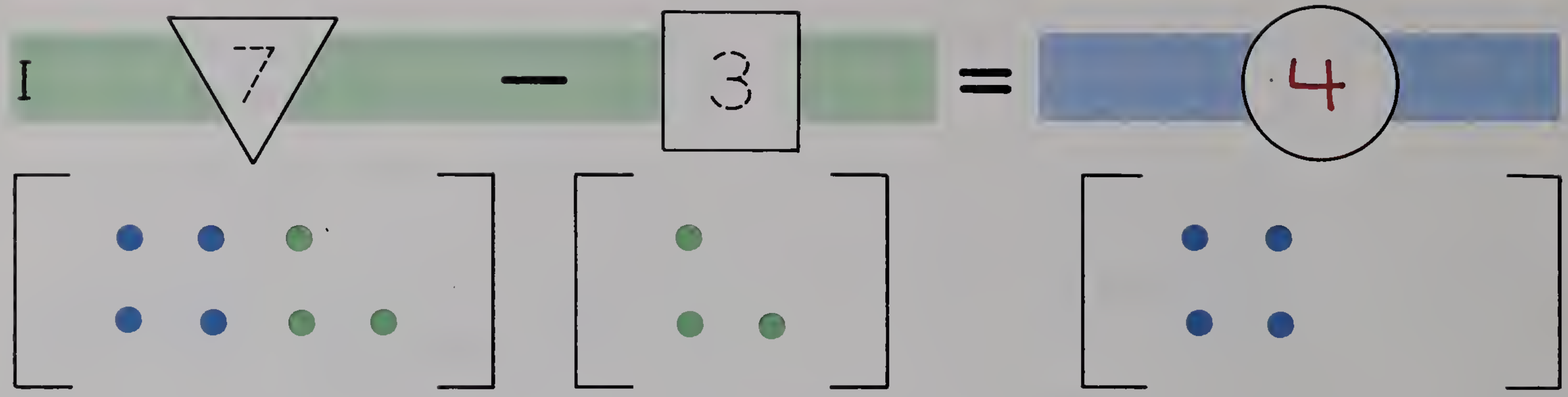
Drill

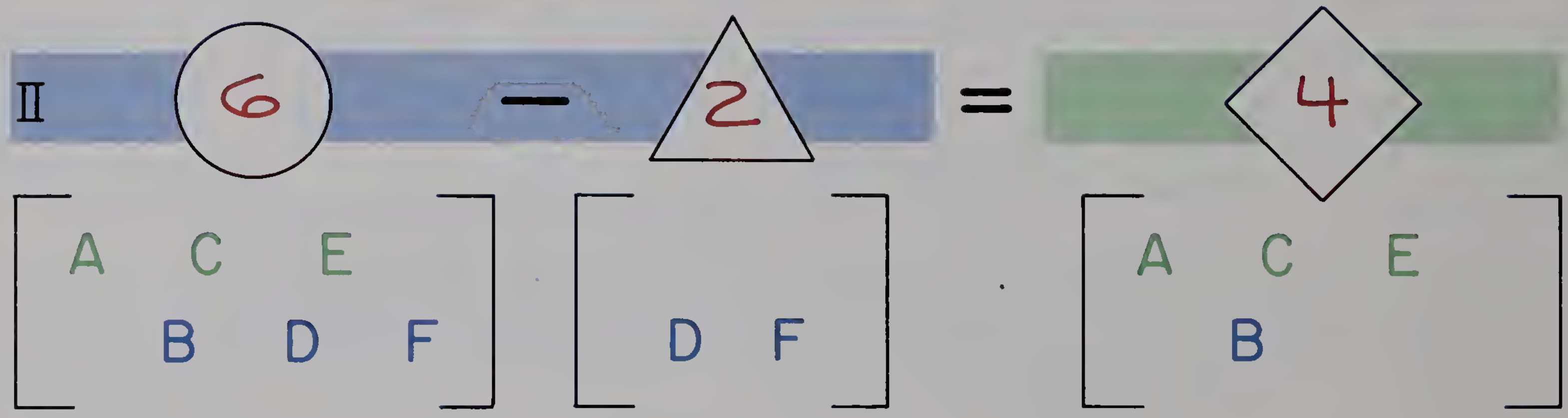
Here is a suggested drill sheet.

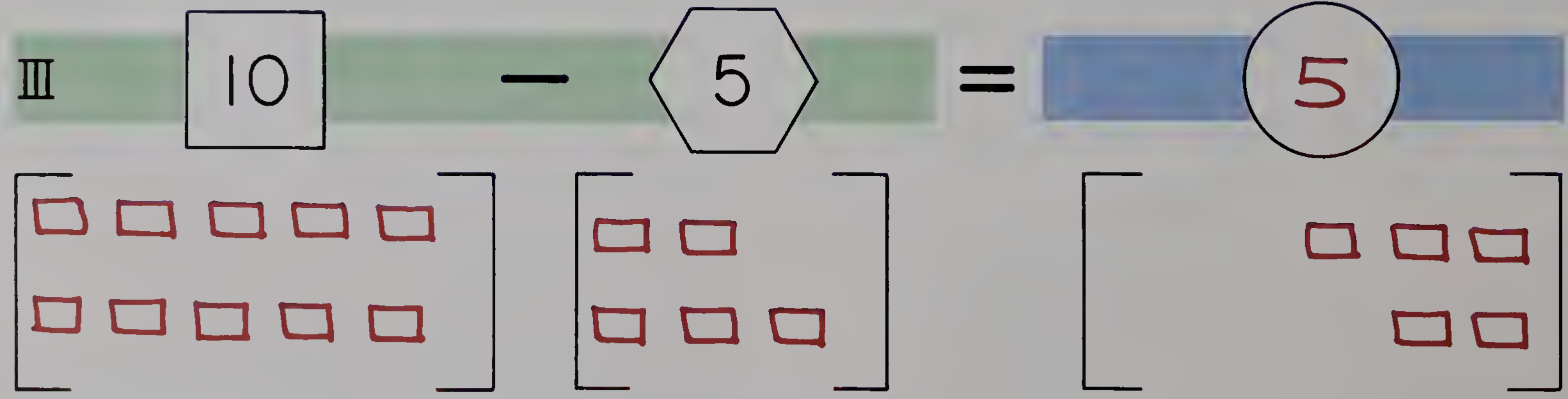

Add 2.
3 + 2
=
Add 3.
2 + 3
=
Add 5.
4 +
=
Add 4.
+
=

Also consider using as a drill sheet page 5 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:


N
3
+
N
2
=
N
5
N
+
N
=
N
N
+
N
=
N
N
+
N
=
N
⑤




II
6
−
2
=
4
A C E
B D F
D F
A C E
B



What's My Rule?

A. 15, 16, 17, 18, 19, 20, 21, 22, 23, 24 . . .

B. 16, 15, 14, 13, 12, 11, 10, 9, 8, 7 . . .

C. 0, 2, 4, 6, 8, 10, 12, 14, 16, 18 . . .

Notes on Pupil Page 5

Tasks: *Pupils fill the frames and draw displays so that each completed subtraction sentence is represented by the corresponding pictures. In the bottom fourth of the page, pupils propose the rules of formation for the sequences and use the rules to fill in the blanks.*

Purpose: (a) To review subtraction of numbers in terms of partitioning a display; (b) to provide practice in observing patterns.

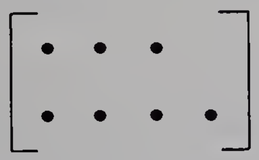
Unifying Ideas: Structure; sets; number and counting; addition and subtraction; functions and relations.

Materials for Teacher: Three boxes, one with a triangular lid, one with a square lid, and one with a circular lid; counters or other small objects; Ditto masters.

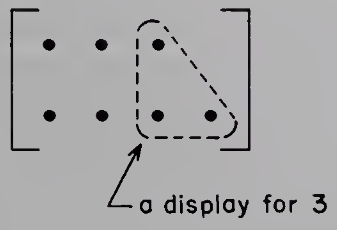
Materials for Pupils: Counters; Dittoed sheets.

The Lesson:

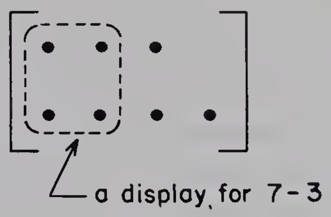
The main purpose of this lesson is to show pupils that a subtraction problem can be solved by partitioning a display. In exercise I we are faced with the problem of subtracting 3 from 7. Start with a display for, or a representation of, 7:



Select as a representation of 3 some part of the display for 7:



The number of remaining objects in the display for 7 is the difference between 7 and 3:

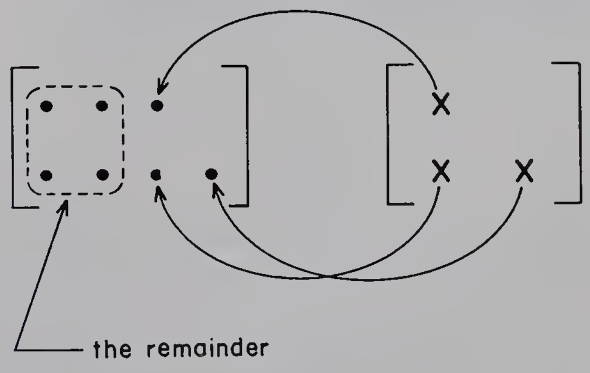


Thus, $7-3=4$.

The physical procedure of partitioning a display is a faithful reflection of the definition of subtraction in terms of relative complementation of sets. For example, to subtract 3 from 7, one considers a 7-set and any subset of the 7-set which has 3 members. The number of members in the complement of the 3-set relative to the 7-set is, by definition, the difference $7-3$.

The foregoing definition of subtraction in terms of sets corresponds with what is sometimes called the "take-away" concept of subtraction illustrated by the problem: *Bill had 7 marbles and lost 3 of them. How many does he have left?*

The so-called "comparison" concept of subtraction is also readily defined in terms of sets and complements of sets. The comparison concept is illustrated by the problem: *Bill has 7 marbles and Tom has 3 marbles. How many more marbles does Bill have than Tom?* In solving this problem in terms of displays, we first match a display for 3 with some part of a display for 7. Then we count the remainder of the display for 7:



Headline for the Day: $12-3+3=\square$

Thus, any problem involving the comparison concept is reducible to a problem involving the take-away concept.

Each of the three physical situations on pupil page 5 can represent a take-away problem or a comparison problem. For example, in exercise I we can ask:

> If you have 7 marbles and you lose 3, how many do you have left?

Or we can ask:

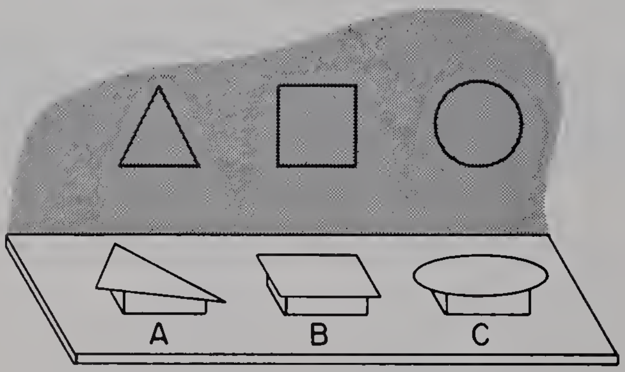
> If you have 7 marbles and your friend has 3, how many more do you have than your friend?

Of course, exercise I also suggests addition problems:

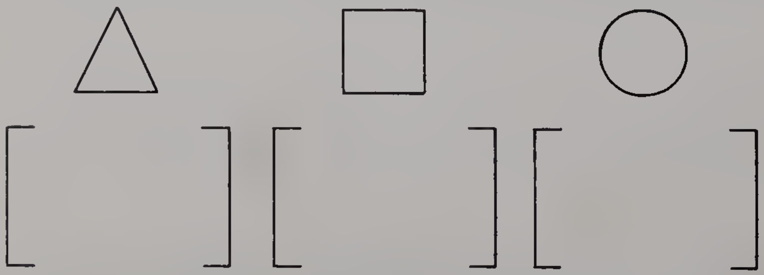
> If you have 4 marbles left after losing 3, how many did you have before you lost any?
>
> If your friend has 3 marbles and you have 4 more than he does, how many do you have?

The Game of Three Boxes

Pupils should have an opportunity to work with concrete objects in preparation for page 5. Place a table in front of the chalkboard, display three empty boxes on the table, and draw on the chalkboard pictures of the lids of the boxes:



(For ease in the present discussion, we refer to the boxes by letter.) Put 8 objects in box A. Have a pupil count them and announce the total to the class. Write an 8 in the triangle on the chalkboard. Each pupil should have a Ditto sheet with the following drawn on it:

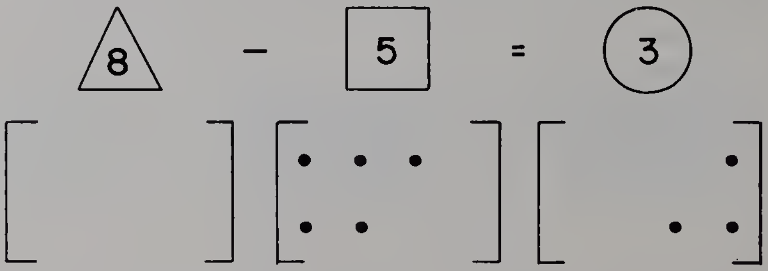


A pupil keeps track of your actions by putting counters on the space between the brackets under the △ and writes an 8 in the △.

Now reach into box A, take out 5 objects, and put them in box B. Have a pupil count the objects in B and announce the result. Write a 5 in the □. Each pupil moves 5 of his 8 counters into the space between the brackets under the □ and writes a 5 in the □.

Summarize by saying that you started with 8 objects in box A (point to the 8 on the chalkboard) and took away (you and the pupils write a minus sign between the △ and the □) 5 objects and put them in box B (point to the 5). Ask pupils to figure out how many objects are now in box A. Move these objects to box C. A volunteer should count them for you. Write the total in the ○. Summarize again by saying that if you start with 8 objects in box A and take away 5 of them which are put in box B, and the objects left in box A are put in box C, then what was left in box A must be (you and the pupils write an equality sign between the □ and the ○) 3 objects.

At this point, each pupil has the following on his Dittoed sheet:



Empty the contents of boxes B and C into box A. The pupils sweep the counters into the space between the brackets under the △. Ask them to write a number sentence for which the action is an interpretation. (Any of the equaltions: $8 = 5 + 3$, $5 + 3 = 8$, $3 + 5 = 8$.) This shows that if you subtract 5 from 8 and then add 5 to the result, you get back to 8. *Adding a number is the inverse of subtracting that number.* This is an important aspect of the structure of the number system.

The next problem might start again with putting 8 objects in box A, but this time taking 3 of the 8 and putting them in box B.

Follow-up Activities

As a follow-up activity for pupil page 5, give pupils the following problems on a Dittoed sheet:

1) 6 − 3 = □
2) 8 − 6 = □
3) 7 − □ = 3
4) □ − 1 = 5
5) △ − □ = 2 (many correct answers)
6) △ − □ = ○ (many correct answers)
7) (8 − 2) + 2 = □
8) (17 − 9) + 9 = □

Another follow-up activity might consist of oral questions such as:

Put 10 counters in an empty box. Take out 2 counters and then put back 1. How many are now in the box?

Bill put some pennies in an empty box. Susan took out 3 and Ed took out 4. Then there were no more pennies in the box. How many did Bill put in the box?

Sarah put 25 pennies in an empty box. Fred took out 8 of them and Alice put in 8. How many are now in the box?

Order of Operations

Let's write a headline for the following problem:

Bill had 10 cents. His father gave him 4 cents, then he spent 5 cents, and then his brother gave him 3 cents. How much did Bill have altogether?

Here is a fitting headline:

a) 10 + 4 − 5 + 3 = △

Now consider another problem: Tony had 10 cents. His mother gave him 4 cents. Then he went to the corner store and bought a candy bar for 5 cents and a balloon for 3 cents. How much did Tony have altogether?

Here is a headline that fits this last problem:

b) 10 + 4 − (5 + 3) = △

Headlines (a) and (b) differ in an important way. When we look at the left side of (a), we see a chain of adding and subtracting operations which are to be performed in succession from left to right:

$$10+4-5+3=14-5+3=9+3=12$$

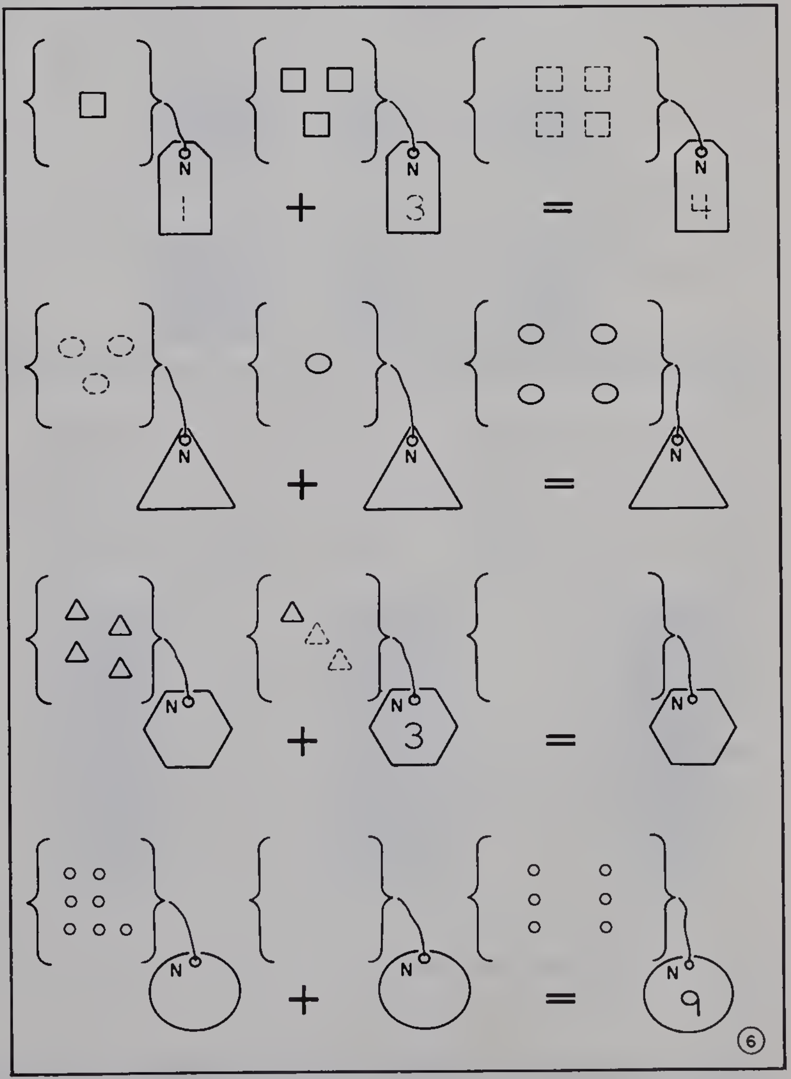
On the other hand, when we look at the left side of (b), the parentheses tell us that we should first add 3 to 5:

$$10+4-(5+3)=10+4-8=14-8=6$$

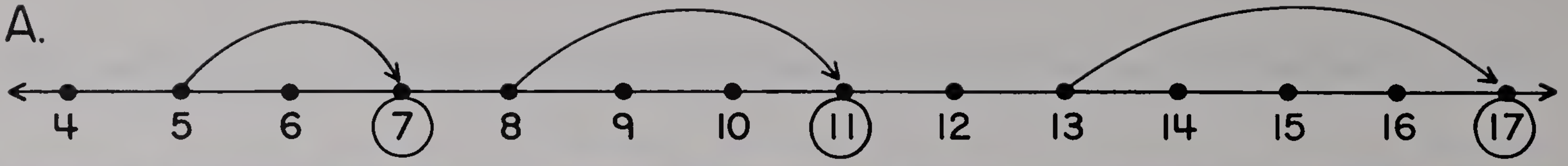
These interpretations of the left sides of (a) and (b) illustrate some important agreements that we follow in this textbook. If we have a string of additions and subtractions and no parentheses, we work from left to right. If we have parentheses, we work on what is inside the parentheses and then go from left to right.

Drill

Consider the following for a drill sheet. It is page 6 of Math Workshop Supplementary Activities, Level B.

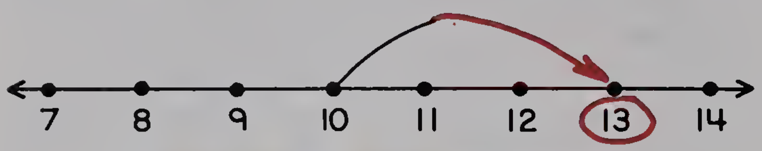


A.



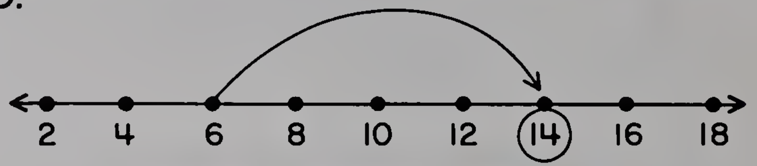
5 + 2 = 7   8 + 3 = 11   13 + 4 = 17

B.



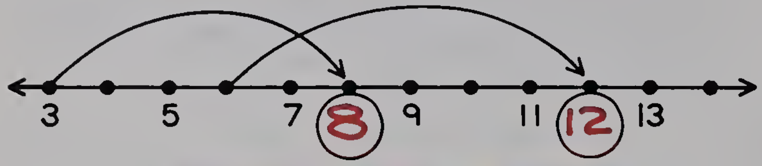
10 + 3 = 13

C.



6 + 8 = 14

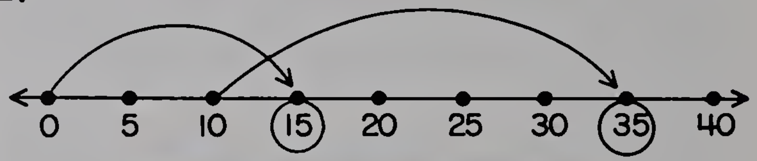
D.



3 + 5 = 8

6 + 6 = 12

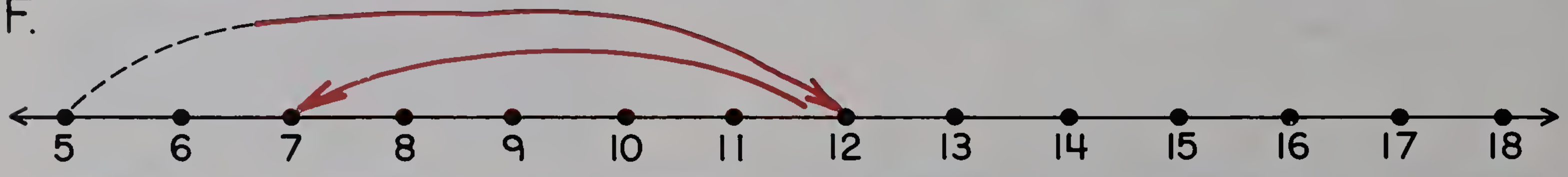
E.



0 + 15 = 15

10 + 25 = 35

F.



$$\begin{array}{r} 5 \\ +\ 7 \\ \hline 12 \end{array} \qquad \begin{array}{r} 7 \\ +\ 5 \\ \hline 12 \end{array} \qquad \begin{array}{r} 8 \\ +\ 6 \\ \hline 14 \end{array} \qquad \begin{array}{r} 6 \\ +\ 8 \\ \hline 14 \end{array} \qquad \begin{array}{r} 7 \\ +\ 9 \\ \hline 16 \end{array} \qquad \begin{array}{r} 9 \\ +\ 7 \\ \hline 16 \end{array}$$

G. What's My Rule?

n	0	3	6	1	8	9	14	18	27	29
n + 2	2	5	8	3	10	11	16	20	29	31

Notes on Pupil Page 6

Tasks: *In exercise A—E pupils draw arrows to show number line jumps and they fill the blanks in the equations. In exercise F they draw arrows and complete the addition problems. In exercise G they discover a rule and use it to complete the table.*

Purpose: (a) To show pupils that adding a number (other than zero) can be represented as a "jump to the right" on a number line; (b) to provide practice in observing patterns; (c) to provide practice in addition and subtraction and in adding and subtracting 2.

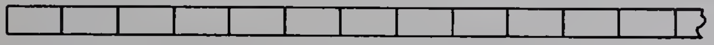
Unifying Ideas: Structure; number and counting; addition and subtraction; functions and relations.

Materials for Teacher: Colored chalk; Ditto masters; masking tape.

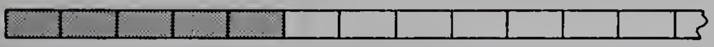
Materials for Pupils: Dittoed sheets.

The Lesson:

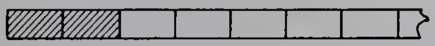
Adding whole numbers by combining displays is our basic approach to addition. We can derive from this another procedure for finding sums. Let us imagine that we have an inexhaustible supply of sticks which are lined up end to end (for convenience, let them all be of the same length):



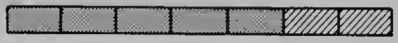
We shall now use these sticks to help us find the sum of 5 and 2 in accordance with our basic approach. The first step is to choose a display for 5. Instead of just any display for 5, choose one whose objects are the first five sticks in the lineup:



Put this display to one side, and then select a display for 2—not just any display, but one whose objects are the first two sticks in the remaining lineup:



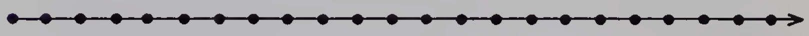
Combine the two displays by lining them up. The new display is a representation of 5 + 2:



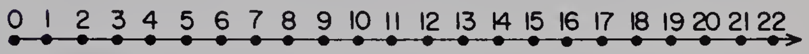
Count the objects in the new display and thereby learn that 5 + 2 = 7.

(If one uses sticks of the same length, it is easy to appeal to a pupil's ability to compare lengths by showing him that 5 + 2 is the same number as 2 + 5. A "train" built by lining up 5 "cars" and then "hooking on" 2 cars is just as long as a train built by lining up 2 cars and then hooking on 5 cars. In fact, if you have in front of you an already completed train, you really can't tell how it was put together. A representation of 5 + 2 can serve as a representation of 2 + 5, and vice versa.)

It is clear that every addition problem can be solved this way. Now, instead of using sticks, we can accomplish the same end with displays that, although somewhat more abstract, are much easier to handle. We represent the inexhaustible supply of sticks by drawing a line and marking points on it to represent the ends of the sticks:



To facilitate both the selection of the display for the first addend and the counting of the display for the sum, we assign numbers to the marked points in such a way that, for example, the points 9 and 0 are the end points of a train made up of 9 cars. Thus, to find the sum of 9 and 3, one places his finger on the 9-point, counts 3 cars to the right and reaches the 12-point. He has discovered that when one hooks a train of 3 cars onto a train of 9 cars he has produced a train of 12 cars. Thus, 9 + 3 = 12, and the only counting required was that involved in picking the display for 3. The numbered line reduces a problem requiring the counting of three displays to the counting of just one display. Pictures of a numbered line are called *number lines.*



Headline for the Day: Have pupils tell or write a time story illustrating one of the number sentences on pupil pages 5 or 6.

Advantages of Number Lines

The use of a number line in doing addition problems has several advantages other than convenience in counting.

For one thing, it keeps uppermost in the pupil's mind the order of the whole numbers. For example, since the 7-point comes after the 3-point, $7 > 3$.

It shows nicely the relationship between adding a number and subtracting that number. If adding 5 is represented by a jump of 5 cars to the right then subtracting 5 is a jump of 5 cars to the left. For example, it is clear that $(17 + 5) - 5$ must be 17.

It shows other structural properties of addition. It shows that addition is *commutative*—if you start at 8 and jump 2 to the right, you will reach the same point as if you had started at 2 and jumped 8 to the right. It shows that addition is *associative*—if you start at 8 and then jump 2 to the right and then 3 to the right, you will reach the same point as if you had started at 8 and had taken one big jump of $2 + 3$ to the right; in other words, $(8 + 2) + 3 = 8 + (2 + 3)$. It shows that you can *switch* two addition operations without affecting the final result: $(8+2)+3=(8+3)+2$. In general, it shows that you can *rearrange* addition and subtraction questions without affecting the final result:

$$\begin{aligned} 8-2+3-5 &= 8-2-5+3 \\ &= 8+3-2-5 \\ &= 8+3-5-2 \\ &= 8-5-2+3 \\ &= 8-5+3-2 \end{aligned}$$

(See "Order of Operations" in the Notes on Pupil Page 5.) Of course, such rearrangements are "legal" only if they do not involve subtracting a larger number from a smaller one. For example, it makes no sense to rearrange $8+7-10$ into $8-10+7$ because $8-10$ makes no sense when 8 and 10 are whole numbers.

If you space the marked points uniformly then you can use geometric intuition to discover numerical facts. Since a jump from 29 to 33 is of the same size and in the same direction as a jump from 69 to 73, it follows that $33 - 29 = 73 - 69$.

Finally, points can be marked on a number line to represent fractions, and addition and subtraction of fractions can be represented with jumps on a number line in precisely the same way as addition and subtraction of whole numbers.

Preparatory Activities

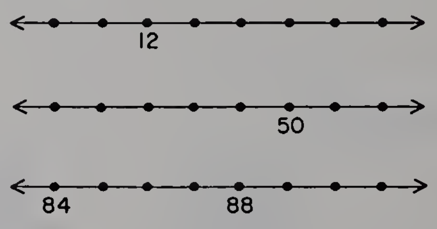
In designing pupil page 6, we have assumed that pupils are already familiar with number lines from their work in an earlier grade. If this is not the case for your pupils, you will need to do some preliminary work. Use a cardboard strip to mark the 0-point and the 1-point and two cardboard strips (of the same length) to mark the 2-point. It is essential that pupils understand the significance of the 2-point—the distance between the 0-point and the 2-point represents the length of a 2-car train, etc. Make up some exercises on a Dittoed sheet which are simpler than those on pupil page 6. For example, be sure to show the 0-point on the number lines and label each marked point. (See pages 23, 24, and 25 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B.)

Play some counting forward and counting backward games similar to the following:

From the number 10, count backward 2 numbers and count forward 3 numbers.

From 60, count backward 5 numbers and forward 5 numbers.

What are the numbers between 30 and 40? Draw some number line chunks on the board similar to the following and discuss naming the points.

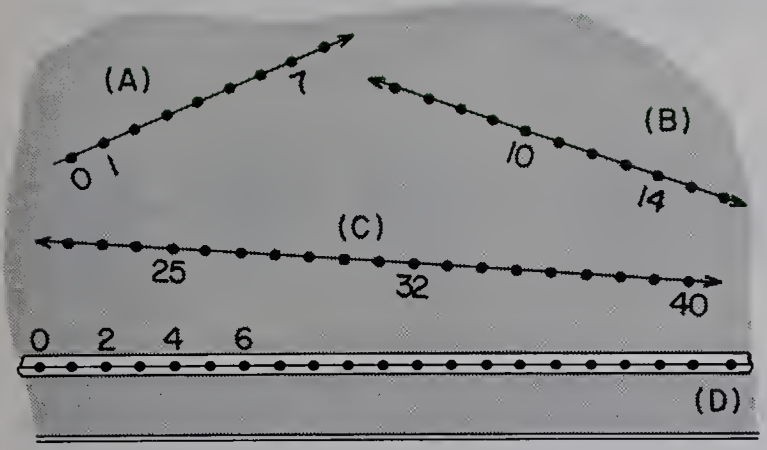


For some extremely simple number line activities, see pages 1-4 and page 13 of the booklet *Games and Enrichment Activities*. Be sure to modify the activities suggested there by including a marked point for 0 on each number line drawn.

When the foregoing preliminary activities have been completed (or if your pupils are already familiar with number lines), you may wish to carry out the following additional preparatory activities.

Activity 1

Display several number line chunks (shown at the top of the next page) on the chalkboard—A, B, and C in chalk, and D on masking tape.

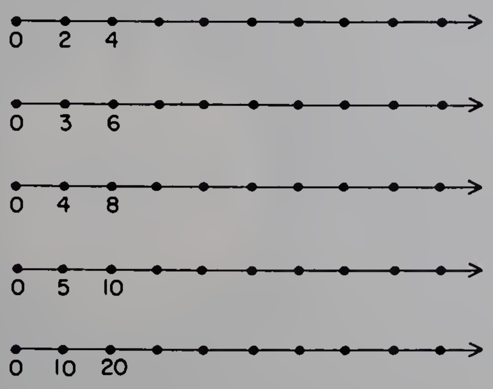


Select volunteers to complete the labeling of number lines A, B, and C. For number line D, proceed as follows. Start at the 0-point and label consecutively in red chalk the first few multiples of 2—0, 2, 4, and 6. Stop and offer the red chalk to a volunteer who should label the next few multiples of 2. Have him offer the red chalk to another classmate to continue the labeling a few more steps, etc. What numbers did we use? (The even numbers.) Now, using green chalk, label points 1, 3, and 5 in that order. Stop and offer your green chalk to a volunteer to label the next of the unlabeled points. Have him pass his green chalk on to another classmate to continue the game. What numbers did we use this time? (The odd numbers.)

Erase the board. Who remembers all the numbers we wrote in red chalk? List them on your paper. Billy, read your list. Make a list of the numbers we wrote in green. Jane, read your list to the class.

Activity 2

Distribute Dittoed sheets showing the following and ask pupils to label the marked points.



Activity 3

Draw a number line on the chalkboard, label the points for just the even numbers, and show the connection between a jump and an addition sentence. Start with jumps of 2 or 4 and with even numbers as starting points. For example, draw a jump from 4 to 6 and write the sentence $4 + 2 = 6$. Then write the sentence $6 + 4 = 10$ and ask a pupil to draw the jump. (Circle the labels for the landing points.) Move to jumps from odd-numbered points to even-numbered points and from odd to odd. This activity will prepare pupils for exercises A through E on pupil page 6.

Activity 4

Draw a number line on the chalkboard and use it to prepare pupils for exercise F. Present the compound sentence:

$$9 + 5 = \square \quad \text{and} \quad \triangle + 9 = 14$$

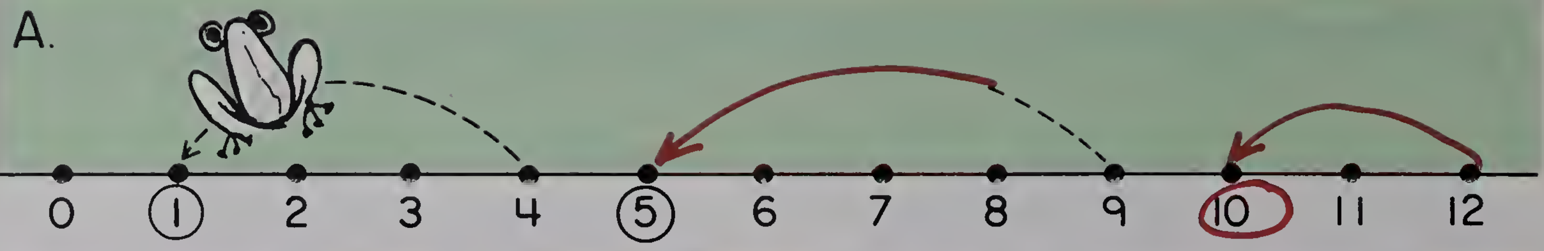
To find the missing sum, one starts at 9, jumps 5 to the right, and lands on 14. Then, to find the missing addend (in the second part of the sentence), one starts at 14 and jumps 9 to the left. A pupil who is aware of the fact that addition is commutative should be able to predict the landing point for the second jump. Continue with other compound sentences such as:

$3 + 7 = \square$ and $\triangle + 3 = 10$
$4 + 11 = \square$ and $\triangle + 4 = 15$
$13 + 5 = \square$ and $5 + \triangle = 18$

Pupils should now be ready for page 6. A word of caution is in order with respect to the way pupils react to the exercises on the page. Although pupils could handle exercises B and F without looking at the number lines provided, this is not so for exercises A, C, D, and E. In these exercises, the number lines have marked on them the information needed to fill the blanks in the equations. Most pupils will be able to discover the connection between the equations and the data marked on the number lines. Be on the lookout for those who might become frustrated because "teacher hasn't told them what to do."

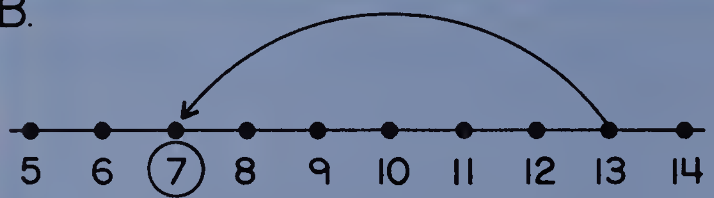
Note that exercise G describes a "jump 2" rule.

A.


0
1
2
3
4
5
6
7
8
9
10
11
12

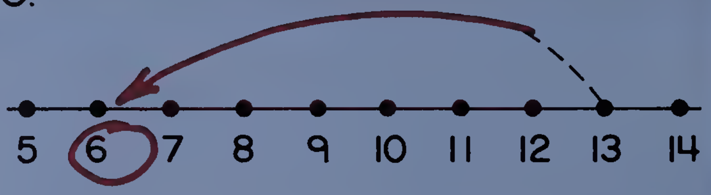
4 - 3 = 1 9 - 4 = 5 12 - 2 = 10

B.


5
6
7
8
9
10
11
12
13
14

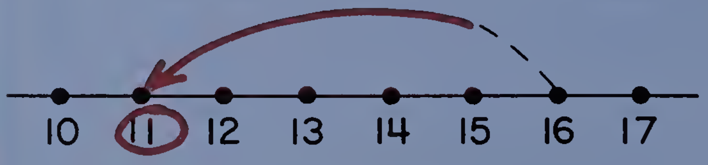
13 - 6 = 7

C.


5
6
7
8
9
10
11
12
13
14

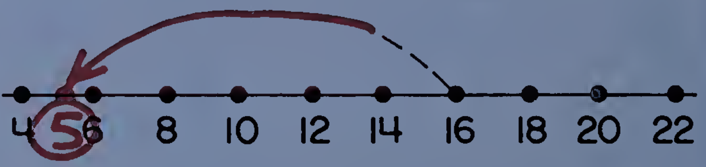
13 - 7 = 6

D.


10
11
12
13
14
15
16
17

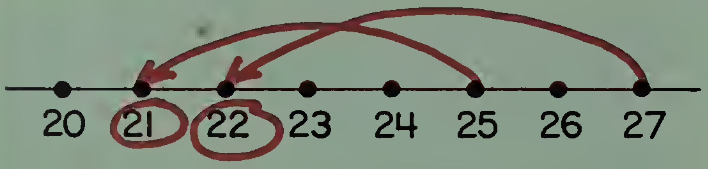
16 - 5 = 11

E.


4
5
6
8
10
12
14
16
18
20
22

16 - 11 = 5

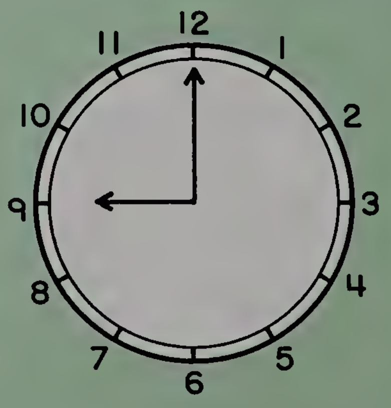
F.


20
21
22
23
24
25
26
27

25 - 4 = 21

27 - 5 = 22

G.


12
1
2
3
4
5
6
7
8
9
10
11

9 o'clock

Notes on Pupil Page 7

***Tasks:** In exercises A–F pupils draw number line jumps and fill blanks in equations. In exercise G pupils tell time.*

Purpose: (a) To show pupils that subtracting a number (other than zero) can be represented as a "jump to the left" on a number line; (b) to make pupils aware of the fact that subtraction facts come in pairs; (c) to provide practice in subtraction; (d) to alert pupils to the presence of scales (number lines) on common measuring instruments.

Unifying Ideas: Structure; number and counting; addition and subtraction; measurement.

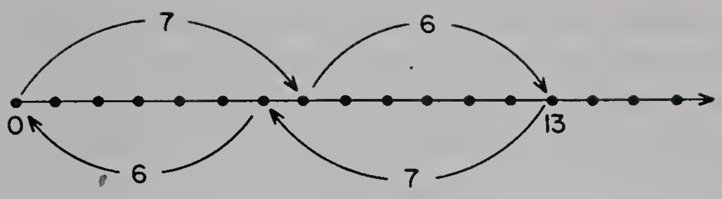
Materials for Teacher: Ditto masters; measuring instruments.

Materials for Pupils: Dittoed sheets.

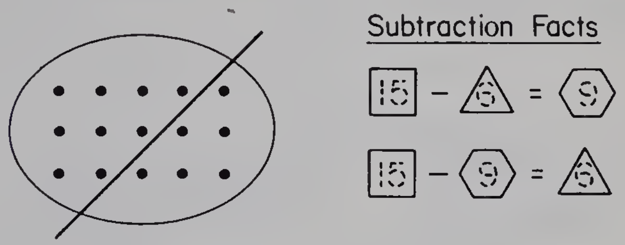
The Lesson:

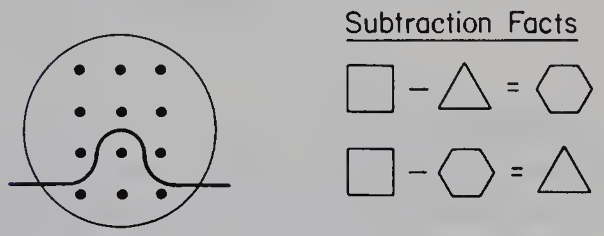
If pupils are successful with the work on pupil page 6, it will not be necessary to do much in the way of general preliminary activities for pupil page 7. However, it is helpful to prepare pupils so that they do not miss the point of exercises B-C and D-E. The fact that subtraction facts come in pairs (13—6=7 and 13—7=6) follows from the facts that addition is a commutative operation and that subtracting a number is the inverse of adding it. For example, we know that 13 — 6 = 7 because 13 = 7 + 6 and (7 + 6) — 6 = 7. But 7 + 6 = 6 + 7 and (6 + 7) — 7 = 6. So we also know that 13 — 7 = 6.

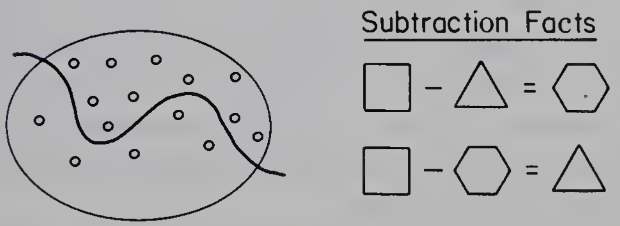
These facts can be shown nicely on a number line. If one starts at 0, jumps 7 to the right and then 6 to the right, he can return to his starting point by jumping 7 to the left and then 6 to the left:



These facts can also be shown by partitioning a display and deriving the pair of subtraction sentences from the partitioning. For example, if you have 10 pennies and give 3 to Billie, how many do you have left? If you start with 10 pennies and give 7 to Billie, how many do you have left? (10—3=7 *and* 10—7=3.) You may wish to prepare Dittoed sheets with exercises such as the following:



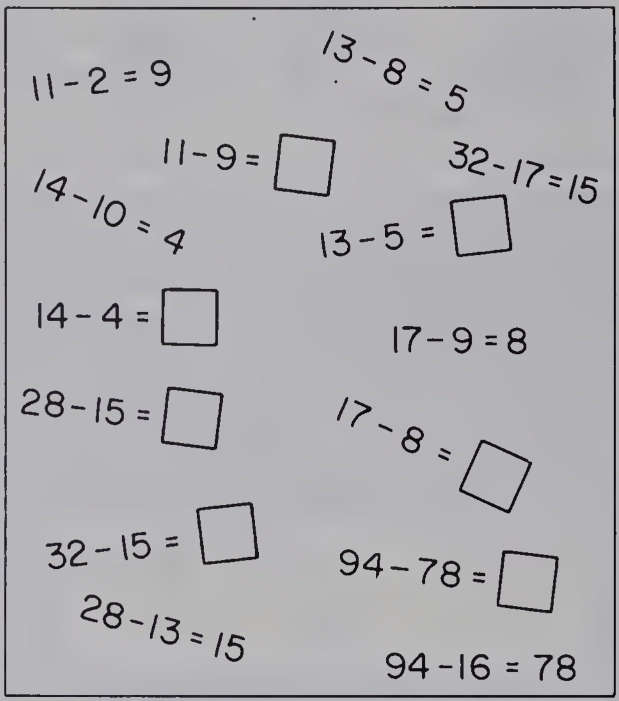




As a follow-up activity for exercises B-E, you may wish to prepare a Dittoed sheet of exercises which allow a pupil to capitalize on his awareness of the

Headline for the Day: It's now _____ o'clock. What time was it 2 hours ago? Make a chart.

principles illustrated in B-E. Instruct the pupils to fill the blanks and tell them that if they look carefully, they might see an easy way to solve the problems. Here is one possible arrangement of exercises:



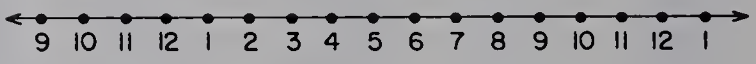
Measuring Instruments

Exercise G on telling time suggests an application lesson. Ask pupils to collect instruments used in measuring, or pictures of measuring devices. These may include: thermometer, light meter, scales (weight), sundial, ruler, micrometer, yardstick (in inches), measuring cup, meterstick (in centimeters), tape measure, egg timer, clock. Pupils should observe that the scale on a measuring instrument is very much like a number line.

The scale on a clockface is a kind of number line but jumps on this scale do not always illustrate ordinary addition and subtraction. If it is 10 o'clock now, what time was it 3 hours ago? What time will it be 3 hours from now? Put three headings on the chalkboard like those below. Record the time in any column. Have pupils complete the row. Let pupils use a clockface to help them in their computations.

3 hours ago	now	3 hours from now
7	10	1
8	11	2
9	12	3
10	1	4
11	2	5
12	3	6

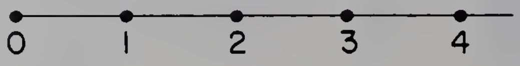
Ask your most able pupils if they could draw a kind of number line that would be useful in answering questions about "jumps" in time. What we have in mind is a scale such as the following:



With this scale it is easy to see that if it is 11 o'clock now, it will be 3 o'clock 4 hours from now and that it was 10 o'clock 13 hours ago.

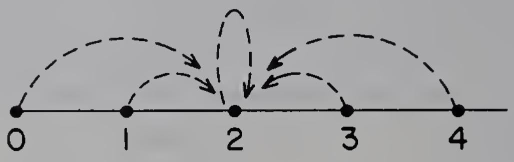
Number Line Jumps

Here is an interesting supplementary activity which will lead pupils into an extended investigation and which will encourage an active search for patterns. Consider this part of a number line:



In how many ways can we land at 2 on the number line if we start at one of the labeled points and take a single jump?

There are five ways—including starting at 2, jumping straight up in the air, and landing back at the starting place:



These jumps suggest some addition and subtraction number sentences:

$$0 + 2 = 2$$
$$1 + 1 = 2$$
$$2 + 0 = 2$$
$$3 - 1 = 2$$
$$4 - 2 = 2$$

(As descriptions of jumps, the sentences $2 + 0 = 2$ and $2 - 0 = 2$ suggest the same jump, so we include only one of them.)

How many different jumps could be taken on this portion of the number line, starting at one marked point and taking a jump to another marked point?

A convenient way to keep track of the jumps is to list the addition and subtraction combinations they suggest:

Starting Point	Direction ↶ or ↷	Size of Jump	Landing Point	Arithmetic Shorthand
1	↷	2	3	1 + 2 = 3
4	↶	4	0	4 - 4 = 0
0	↷	3	3	0 + 3 = 3

How would you know when you had found all possible jumps as we described them? One way would be to organize the jumps in terms of landing points, and proceed in some order until you had exhausted the possibilities. The shorthand of arithmetic could be used to record the fact that you had considered a particular jump:

$0 + 4 = 4$
$1 + 3 = 4$
$2 + 2 = 4$
$3 + 1 = 4$

$0 + 3 = 3$
$1 + 2 = 3$
$2 + 1 = 3$
$4 - 1 = 3$

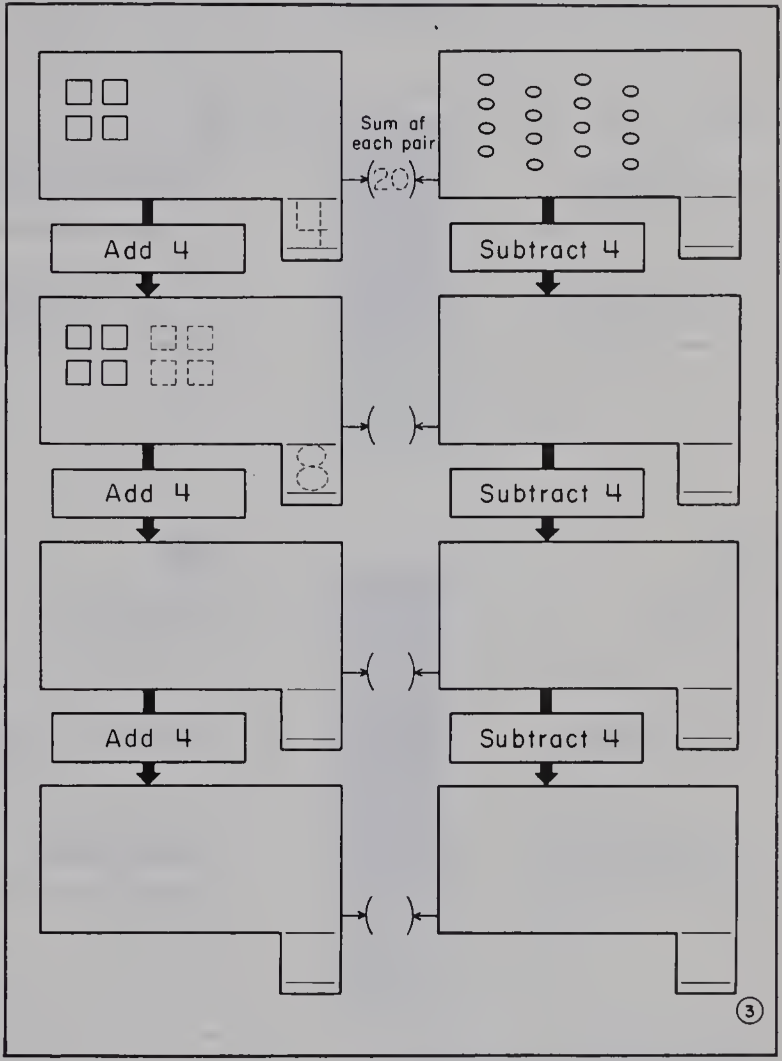
(Notice that we did not include $4 + 0 = 4$ or $3 + 0 = 3$ because our question asked for ways to jump from one point to *another*, which precludes landing at the starting point.)

Our results will show that there are four different ways in this situation that you could land at any one of the five points labeled—or twenty different jumps starting from one point and landing at another.

Such extended investigations of a simple situation provide opportunities to nurture important mathematical ideas. As a record is kept, patterns begin to emerge. Without noticing these patterns, we would have no way to be quite sure we had exhausted the possibilities.

Drill

Here is a suggestion for a drill sheet. This is page 3 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B.



HOW MANY LETTERS?

I

add	3
subtract	8
total number of letters	11

II

Monday	6
Tuesday	7
total	13

III

January	7
February	8
March	5
total number of letters	20

IV

seven	5
o'clock	6
total	11

V

Wednesday	9
Thursday	8
total	17

VI

April	5
May	3
June	4
total	12

VII

plus	4
minus	5
equals	6
total	15

VIII

Friday	6
Saturday	8
Sunday	6
total	20

IX

July	4
August	6
September	9
total	19

X

October	7
November	8
December	8
total	23

What's My Rule?

XI

n	3	7	8	11	20	31
n - 2	1	5	6	9	18	29

XII

x	3	0	5	10	7	15
x + 10	13	10	15	20	17	25

Notes on Pupil Page 8

Tasks: In exercises I-X pupils count the letters in each word, record the counts in the blue boxes, and record the totals in the green boxes. In exercises XI and XII pupils discover rules and apply them.

Purpose: (a) To review addition of two or three numbers with sums in the second or third decades and with problems displayed in vertical form; (b) to provide practice in observing patterns, in adding or subtracting 2, and in adding or subtracting 10.

Unifying Ideas: Number and counting; addition and subtraction; functions and relations.

Materials for Teacher: None.

Materials for Pupils: Counters or Cuisenaire rods.

The Lesson:

While it is not important that pupils be able to read the words that appear in the exercises, you may wish to find out which pupils can recognize them.

Prepare pupils for this page by giving them similar problems on the chalkboard:

one	3
two	3
total	

three	5
four	4
five	4
total	

Mary	4
Alice	5
James	5
total	

Pupils find sums by jumps on the number line, by counting, or by recall.

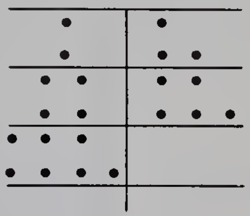
Classify the names of the pupils according to the number of letters in their names. Ask each pupil for the number of letters in his name and write his name in the appropriate column:

3	4	5	6	7	8
Ray	Paul	Helen	Sheila	Richard	Virginia
	John	Susan	Milton		
		Lynne			

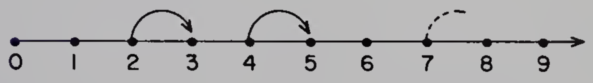
When all the pupils have been listed, draw lines at the bottoms of the columns and have pupils determine for each column the total number of entries and the total number of letters.

Prepare for the *What's My Rule?* exercises by starting games in which pairs of numbers are represented by counters, number line jumps, or rods.

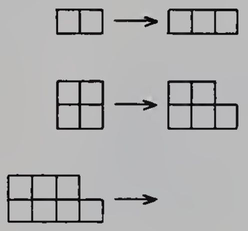
2	3
4	5
7	



2	4	7				
3	5					



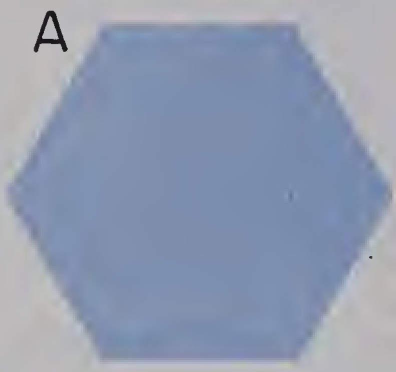
2	3
4	5
7	

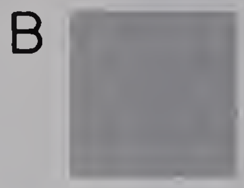


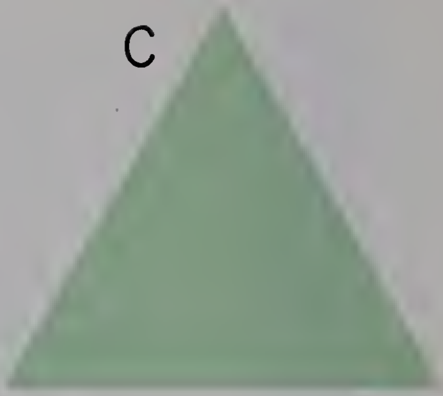
Invite pupils to extend the pattern. They should be encouraged to keep the secret, allowing others to discover it for themselves. Extending the pattern and whispering the secret to you are two ways in which quick pupils can be credited for their early discoveries.

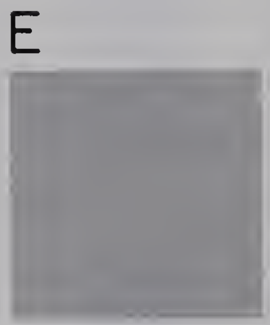
Headline for the Day: Tell stories suggested by the addition sentences on pupil page 8. Have the ones suggested by the first column (3 + 8 = 11, 5 + 6 = 11, 4 + 5 + 6 = 15) deal with time, the second column deal with money, and the third column deal with people.

LARGEST — IN-BETWEEN — SMALLEST

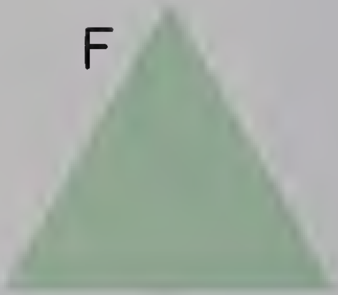


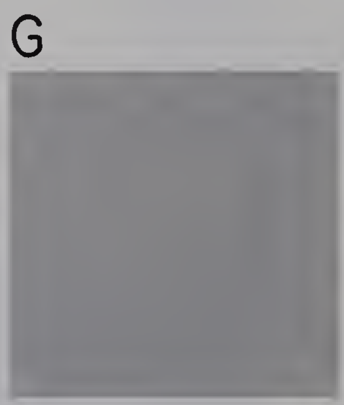


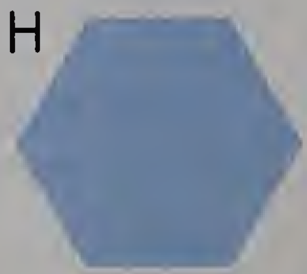


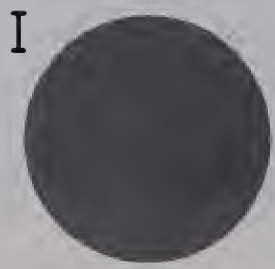


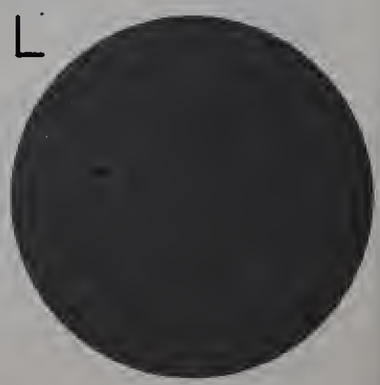
	LARGEST	IN-BETWEEN	SMALLEST
SQUARES	G	E	B
CIRCLES	L	I	D
TRIANGLES	C	F	K
HEXAGONS	A	H	J

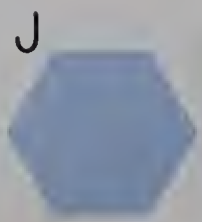


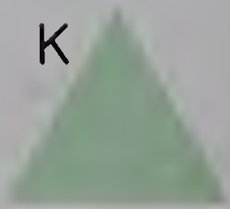












Tasks: *Pupils classify the given regions according to shape and classify regions of the same shape according to size.*

Purpose: (a) To review pupils' knowledge of terminology for geometric figures; (b) to give pupils an opportunity to judge relative sizes of similarly shaped regions.

Unifying Ideas: Geometry; measurement; functions and relations.

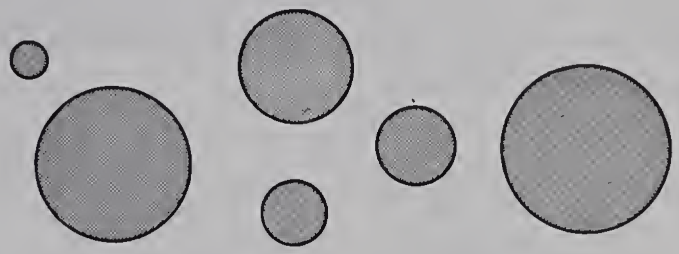
Materials for Teacher: Demonstration models of the figures on pupil page 9 and others mentioned in the lesson; small magnets for mounting figures on a magnetized board.

Materials for Pupils: Desk-size models of teacher's materials.

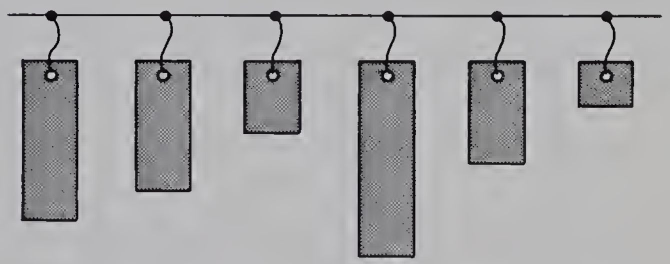
The Lesson:

To introduce this page, you might wish to give your pupils more experience with identifying largest, smallest, and in-between shapes. Draw circles, squares, triangles, or rectangles of different sizes on the board; or make gay-colored shapes to be attached to magnets and used on a magnetized board.

Put a series of shapes in random order, and let pupils arrange them in sequence—largest to smallest or smallest to largest, shortest to longest or longest to shortest:



Or make a group of eight or nine tags of different sizes and hang them in random order. Have pupils rearrange the tags according to a progression in size:

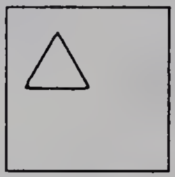


Pupils should not be asked at this time to compare the sizes of different shapes. These comparisons are considerably more difficult and require a clearer definition of what is meant by "same size" than we

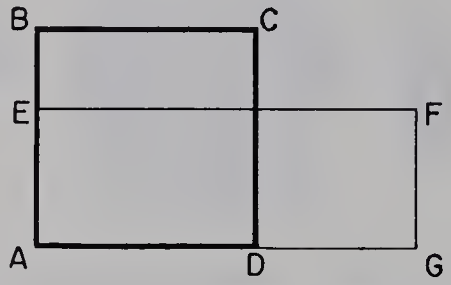
more

Headline for the Day: Harry is 5 inches taller than Bob. If Bob is ___ inches tall, then Harry is . . . inches tall. Make a table.

wish to give at this time. Of course, in talking about larger and smaller we mean larger or smaller in area —in amount of the page covered by the shape (or contained within the boundary of the shape). Comparisons of different shapes can be made easily in some cases. In the following the square is clearly larger than the triangle:

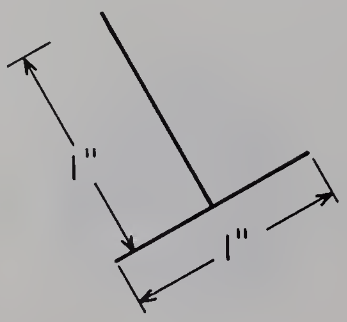


However, some such comparisons are more difficult. Which is larger—the square or the rectangle—is not easy to determine immediately.



These more difficult comparisons and more precise definitions will be considered later. For now we wish to give practice in recognizing the common shapes and in comparing the sizes of the two or more figures of the same shape.

Each example should be contrived so that not all comparisons are obvious. If possible, include some examples that involve an optical illusion. For example, two long thin objects of the same length do not appear to be so when one is seen approximately perpendicular to but not across the other, as:



These lines are both one inch long, but they appear unequal.

Do some work on the chalkboard or a flannel board to make the record keeping for pupil page 9 clear. Use 2 × 2 or 2 × 3 tables:

	Larger	Smaller
□		
○		

	Largest	In-Between	Smallest
△			
⬡			

Transitivity

Take three rectangles of the same shape but of different sizes and give one to each of three pupils without allowing the rest of the class to see the rectangles. Let the three pupils decide among themselves who has the biggest, who has the one in-between, and who has the smallest. Then announce to the class: Mary's rectangle is bigger than Sue's, and John's rectangle is bigger than Mary's. Whose rectangle is the bigger, John's or Sue's? Who has the biggest rectangle? This experience helps pupils develop an awareness of the fact that the greater-than relation is *transitive*. (If object A is greater than object B and object B is greater than object C then object A is greater than object C.)

Two-Dimensional Tables

For additional practice in working with two-dimensional tables, we suggest the following activity. Draw the following grid on the board:

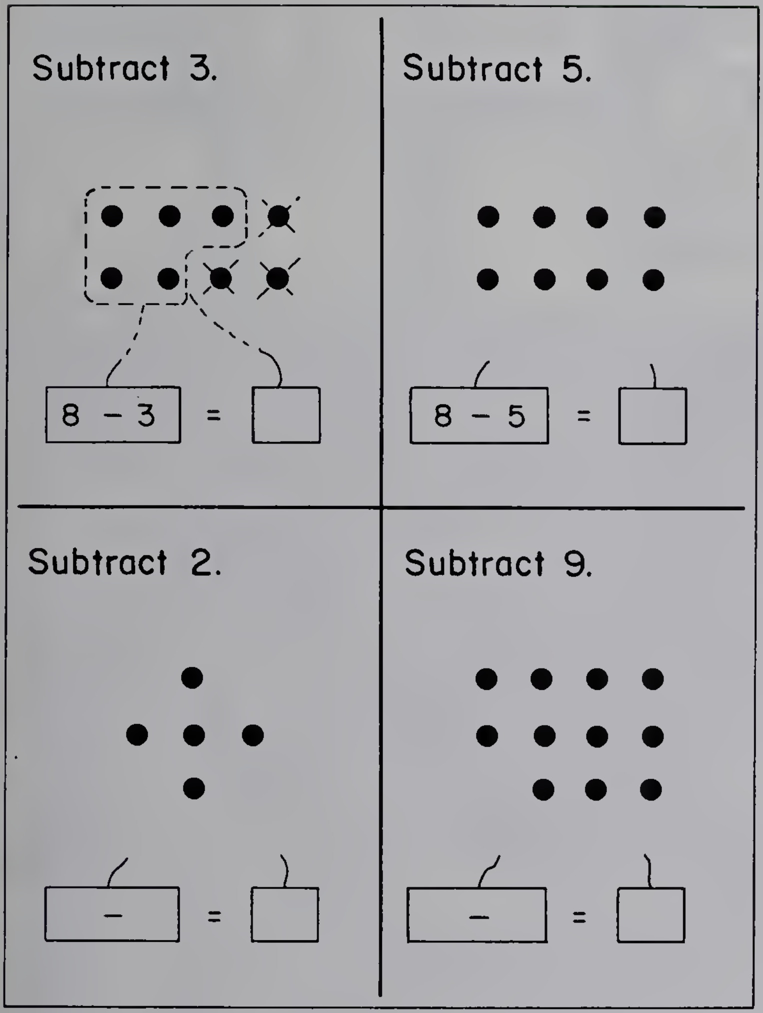
	A	B	C
D	●	● ● ●	● ●
E	●	●	● ● ● ●
F	● ● ● ●	● ●	

Then ask: How many dots are there in row D? In column A? How many dots are there in the cell formed by row D and column B? How many dots are there in column A and column B together?

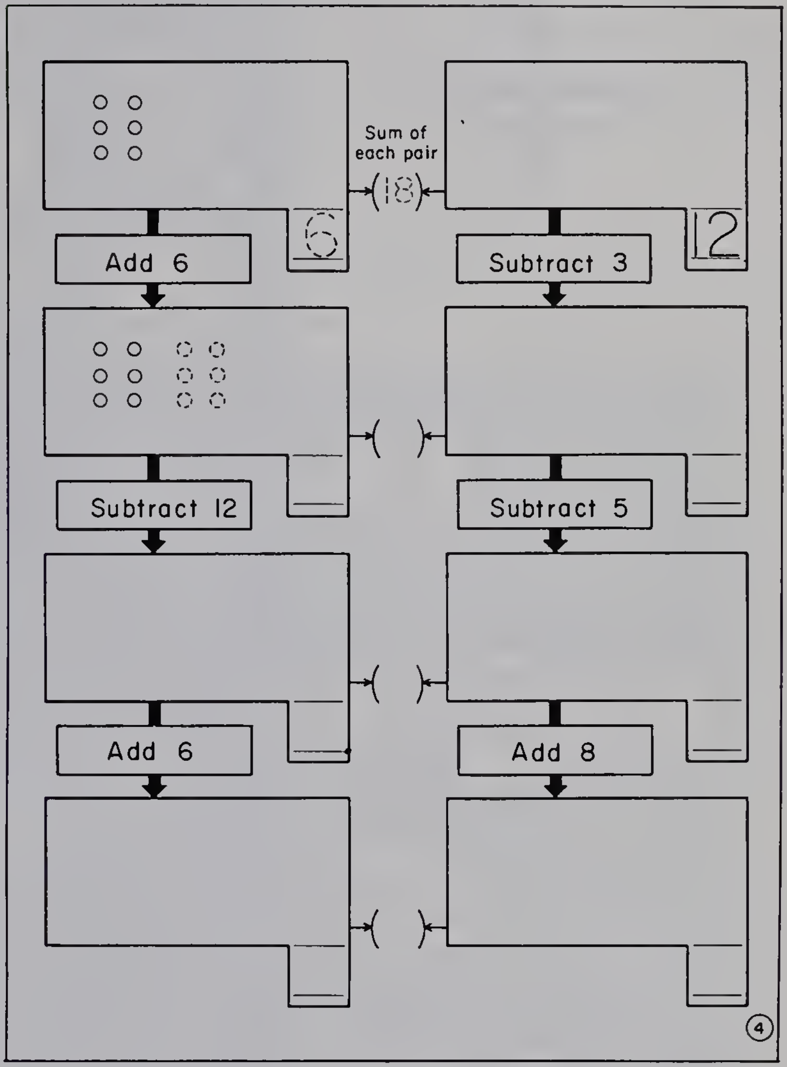
This grid can also be used as a game board. There are two teams—Column and Row. Secretly, the Column team chooses A, B, or C and the Row team chooses D, E, or F. Then they reveal their choices, and the number of dots in the cell formed by the chosen row and column is the score for the Column team. The result of the next play is the score for the Row team. The score of each is cumulative. Whichever team reaches 10 first, wins.

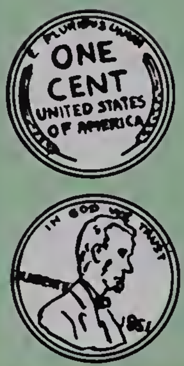
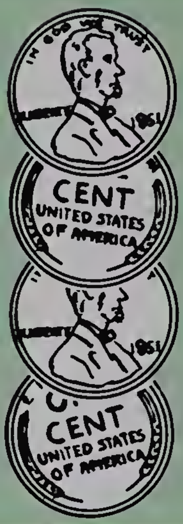
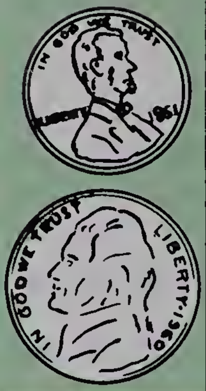
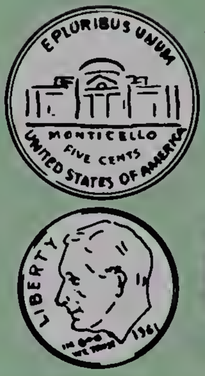
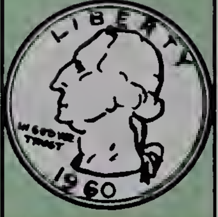
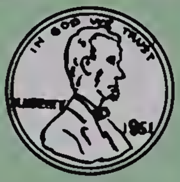
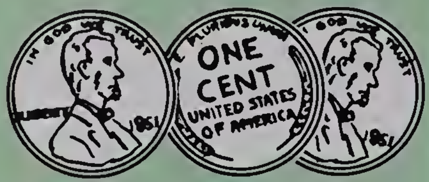
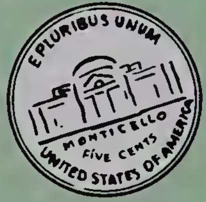
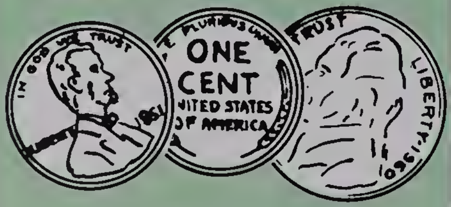
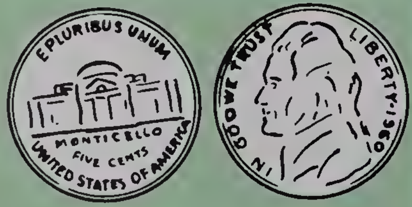
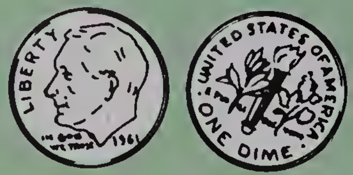
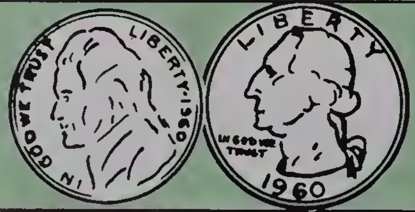
Drill

Here is a suggested drill sheet:


Subtract 3.
8 − 3
=
Subtract 5.
8 − 5
=
Subtract 2.
−
=
Subtract 9.
−
=

Also consider using page 4 of Math Workshop Supplementary Activities, Level B as a drill sheet:


Sum of each pair
18
6
12
Add 6
Subtract 3
Subtract 12
Subtract 5
Add 6
Add 8
④

+					
	3¢	5¢	7¢	16¢	26¢
	5¢	7¢	9¢	18¢	28¢
	7¢	9¢	11¢	20¢	30¢
	9¢	11¢	13¢	22¢	32¢
	12¢	14¢	16¢	25¢	35¢
	22¢	24¢	26¢	35¢	45¢
	32¢	34¢	36¢	45¢	55¢

Notes on Pupil Page 10

Tasks: *Pupils record in each blue cell the total value (in cents) of the coins displayed in the green sections at the beginning of the row and at the head of the column that contain the blue cell.*

Purpose: (a) To review and extend the pupils' knowledge of the values of coins; (b) to practice addition of number with sums less than sixty.

Unifying Ideas: Addition and subtraction; measurement.

Materials for Teacher: Ditto masters.

Materials for Pupils: Play money; Dittoed sheets; Cuisenaire rods.

The Lesson:

This page offers another opportunity to help pupils keep and interpret records in the form of a chart. Prepare pupils by displaying on the chalkboard a chart like the following:

+	1¢ 1¢	1¢ 5¢	5¢ 10¢	25¢
1¢	3¢			
5¢				
10¢				
25¢				

Give each child a Dittoed copy of the chart. Point to the 3¢ in the first cell, and ask why this is written there. Answer the question by saying that 1 penny (point to it) and (point to the plus) 2 pennies (point) are worth 3 cents. How much are 1 penny (point) and (point to the plus) 1 penny and 1 nickel (point) worth? Write 7¢ in the second cell of the first row. Fill the cells in the first row and then proceed with the second row. Allow pupils to finish the chart. Check to see that they know how to use the given information to get the sums for the cells.

If pupils need help with combinations, use play money or Cuisenaire rods: white for pennies; yellow for nickels; orange for dimes; trains of 2 orange rods and a yellow for quarters.

After pupils have completed the chart on page 10, reproduce it on the chalkboard and ask them to give number sentences (without units) suggested by the chart (1 + 2 = 3, 5 + 6 = 11, etc).

Remind pupils that this addition chart suggests subtraction sentences. Point to the 13¢ and to the penny and nickel at the head of that column; then point to the 2 pennies and 1 nickel at the beginning of that row. As you do this, say 13 minus 6 equals . . . 7. Write 13 − 6 = 7.

A Store Chart

As a further activity with charts, use the following story. In a store a clerk noticed that some of the items he sold were 3¢ each, some 4¢ each, some 5¢ each, some 6¢ each, and some 7¢ each. He made a chart to refer to when someone made a purchase:

	3¢ each
1	3¢
2	6¢

One item at 3¢ would cost how much? How much would you pay if you bought two items at 3¢ each? Record the results in the table.

Add columns to record amounts for the items costing 4¢; 5¢; 6¢; 7¢ each.

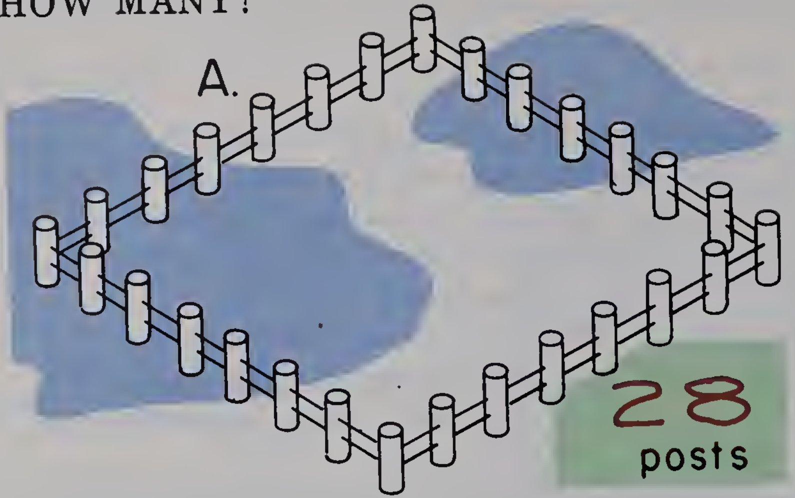
	3¢ each	4¢ each	5¢ each	6¢ each	7¢ each
1	3	4	5	6	7
2	6	8	10	12	14

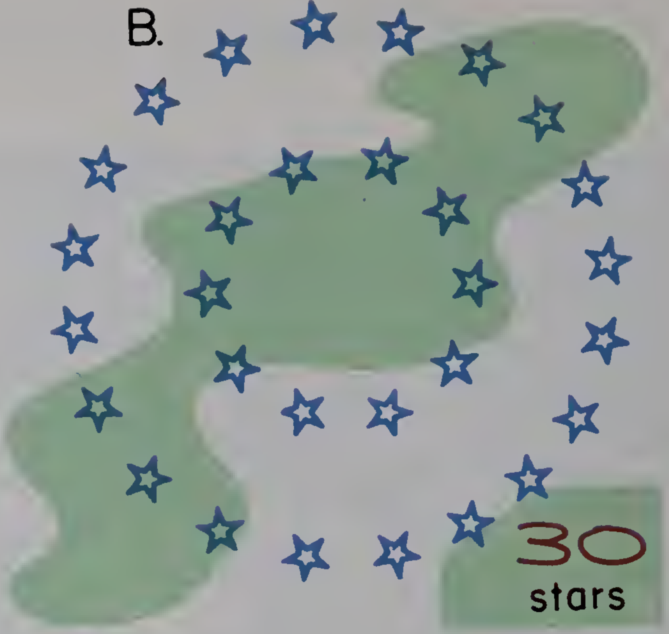
Extend the record until it shows the cost of five of each item (or more if your class does it easily).

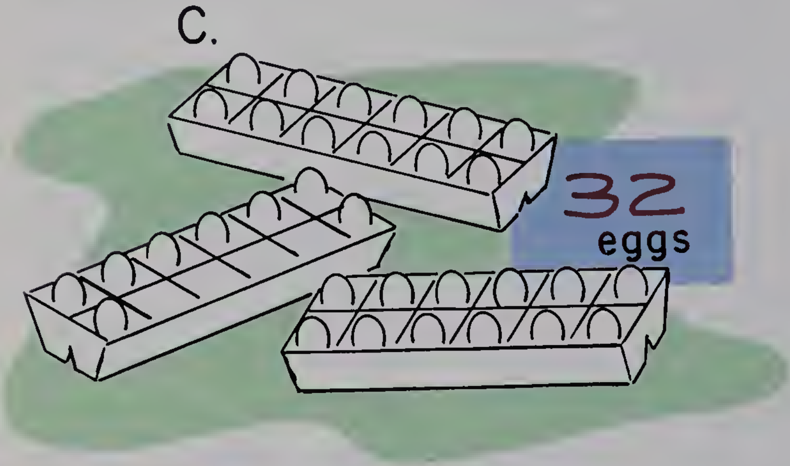
Headline for the Day: Keep a record of how many ways we can make 5¢, 10¢, 15¢, etc. Which way uses the fewest coins?

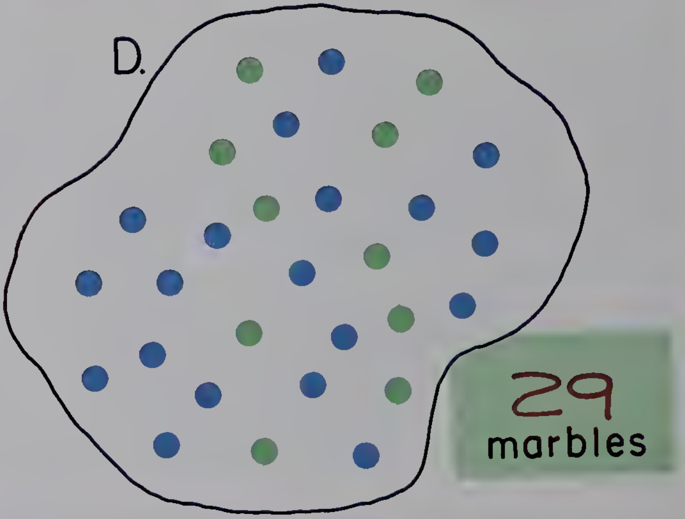
	How Many Ways?	Fewest Coins
5¢	2 (5 pennies or 1 nickel)	1 nickel
10¢	4 (10 pennies; 2 nickels; 5 pennies, 1 nickel; 1 dime)	1 dime
15¢		

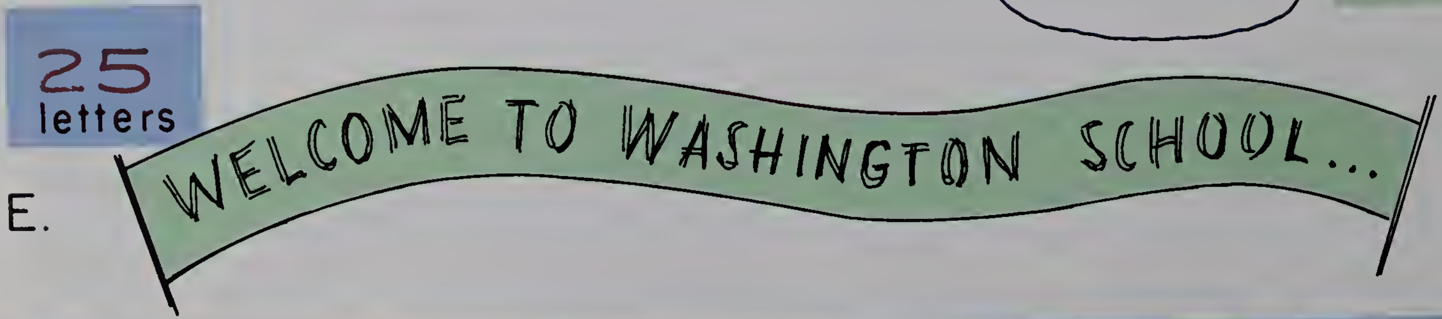
HOW MANY?



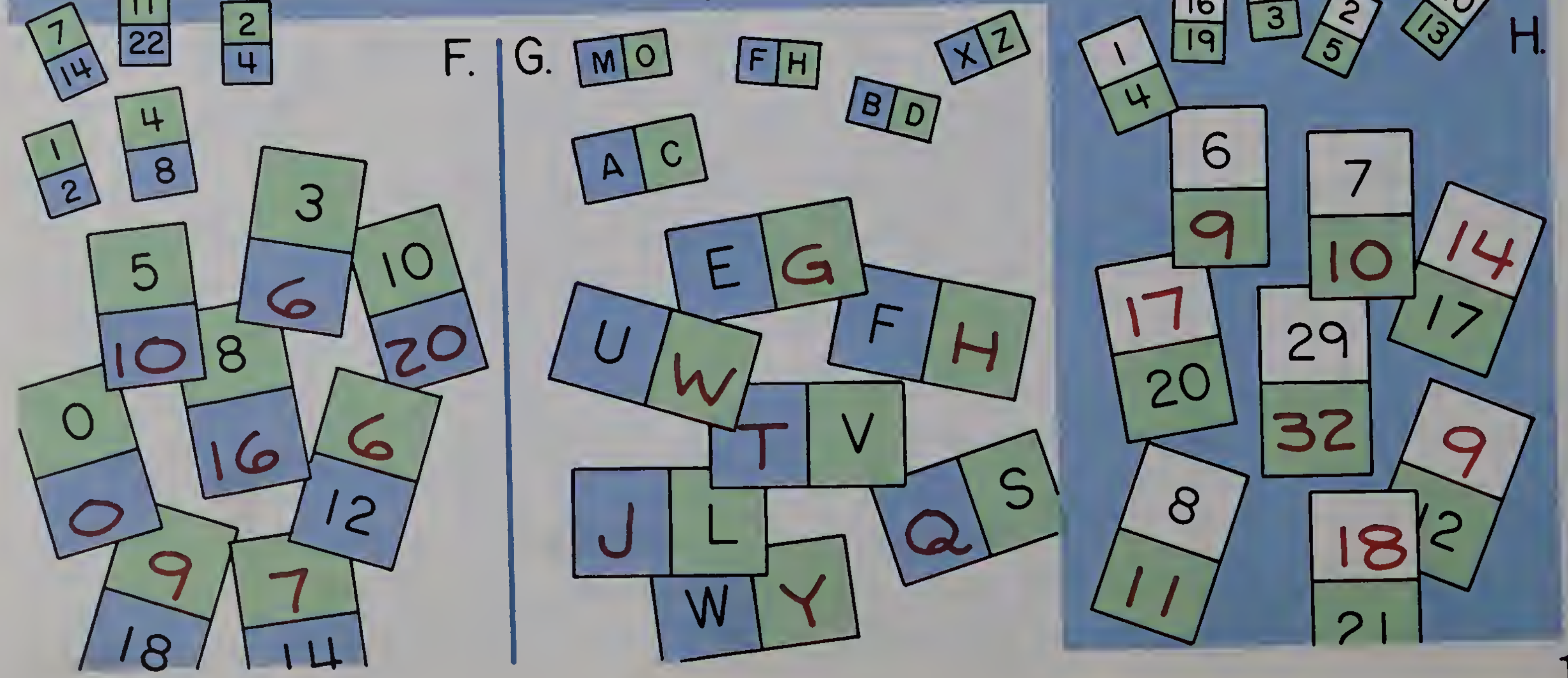





D.
29
marbles


25
letters
E.
WELCOME TO WASHINGTON SCHOOL...

What's My Rule?


F.
7 14
11 22
2 4
1 2
4 8
3 6
5 10
10 20
8 16
0 0
6 12
9 18
7 14
G.
M O
F H
X Z
B D
A C
E G
F H
U W
T V
Q S
J L
W Y
H.
16 19
0 3
2 5
10 13
1 4
6 9
7 10
14 17
17 20
29 32
9 12
8 11
18 21

Notes on Pupil Page 11

Tasks: *In exercises A-E pupils record totals in the shaded boxes. In exercises F-H pupils demonstrate their proposed rules by writing entries where required.*

Purpose: (a) To encourage pupils to search for shortcuts to counting; (b) to give practice in discovering patterns; (c) to give practice in doubling, halving, adding 3, and subtracting 3.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division; functions and relations.

Materials for Teacher: Demonstration-size counters; stars, etc.; flannel board.

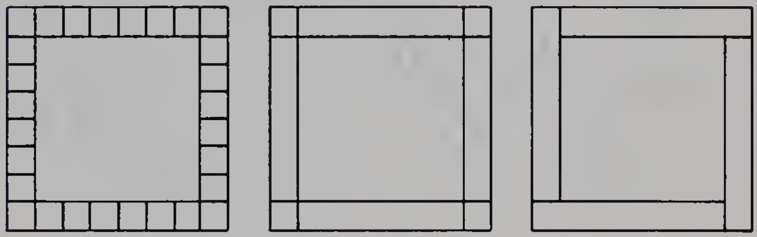
Materials for Pupils: Counters; Cuisenaire rods.

The Lesson:

A major objective of the work in arithmetic is to help pupils develop strategies for shortcutting the job of counting. Exercises A-E are designed to stimulate pupils to find shortcuts.

In exercise A the most primitive way of finding out the total number of posts is to count one by one. There are more sophisticated ways which pupils should be encouraged to use. For example, one determines by counting that there are 6 posts between the end posts on each side. So there are (6 + 6 + 6 + 6) side posts and 4 end posts. Use counting by 6's or number line jumps to find the sum. Another way is to note that the fence could have been assembled by putting together 4 pieces of fence, each with 7 posts.

Cuisenaire rods can be used to advantage here. The fence can be made with 28 white rods, with 4 dark green rods and 4 white rods, with 4 black rods, with 2 brown rods and 2 dark green rods, etc:



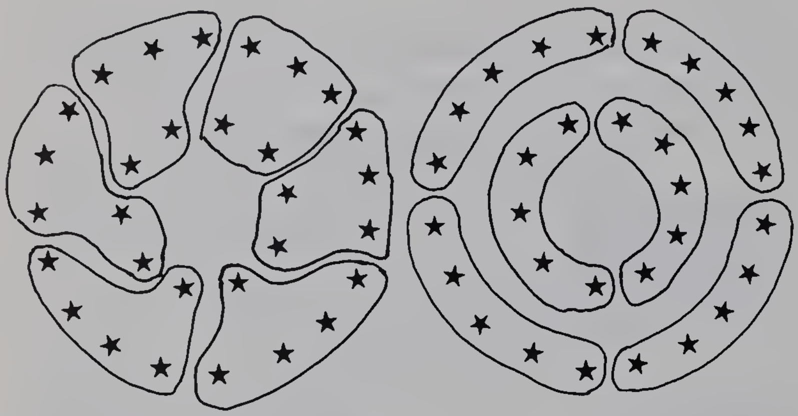
By asking the pupils to build the fence with rods,

more

Headline for the Day: Jim has twice as much money as his sister. If Jim has ___ cents, his sister has . . . cents. Chart possible combinations.

you give them a chance to see shortcuts to counting.

In exercise B the total number can be obtained by counting the stars on the outer circle, counting the stars on the inner circle, and adding. (It is probably too much to expect a pupil to notice that the outer circle has twice the diameter of the inner circle and to reason that, since the stars are equally spaced and there are 10 stars on the inner circle, there should be. . . .) A better way of counting is to first partition the display into groups of 5:



As the pupil carries out the partitioning he should count 5, 10, 15, . . . Or the pupil could group each star on the inner circle with a pair of adjacent stars on the outer circle and count by 3's.

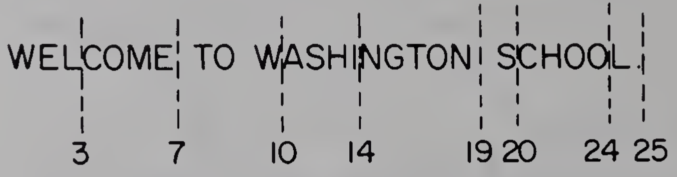
You may wish to use Cuisenaire rods to help prepare pupils for partitioning. Scatter 30 white rods on the table and ask a pupil to exchange them for light green rods (3 whites for a light green). When the pupil has placed 10 light green rods on the table and removed the white rods, ask him how many white rods were on the table. Encourage him to count by 3's as he points to the light green rods. Without such encouragement, he may point to a green rod and say 1, 2, 3, and point to another green rod and say 4, 5, 6, etc. Pupils tend to use counting by 1's when asked to tell how many, since they are most comfortable with this procedure. It is our responsibility as teachers to help them graduate from this primitive procedure. An effective way to encourage them to do this is to tell them to!

In exercise C pupils should spot the dozen grouping. A pupil who tells you that in C there are 2 dozen and 8 more eggs is well on his way toward understanding place value. Your next question should be: How many eggs are there in 1 dozen? When the pupil says 12, you respond with: Yes, 10 and 2 more. So in 2 dozen, there are 10 and 2 more and 10 and 2 more. How many altogether?

Each dozen eggs can be represented by a Cuisenaire rod train consisting of an orange rod and a red rod. The extra 8 eggs can be represented by a brown rod. Using the orange rod as a measure, it is easy to see that there are 32 eggs altogether.

In exercise D one can find the total by counting the blue marbles, then the green marbles, and then adding. But it will be difficult to keep track of the counted marbles. It is much better to partition by drawing loops and to count by 5's.

In exercise E, because of the linear array, it is probably quite easy to count by 1's. But one can readily count by 3's just as if he were jumping on the number line. Or he can utilize his memorized combinations.



We cannot emphasize too strongly that pupils will stay with counting by 1's in a "How Many?" situation unless they are shown and encouraged to use more sophisticated counting procedures. Before letting pupils work on pupil page 11, use a variety of patterned and random arrangements of counters, marks on the board, Cuisenaire rods, etc., to allow pupils to demonstrate fast ways of counting. Let pupils demonstrate their techniques. You should lavishly praise their shortcuts.

What's My Rule?

To anticipate exercise G, put the following on the chalkboard. Can the children finish the examples and make up some of their own that follow the same pattern?

1)	2)	3)
a→b	a→c	a→A
b→c	b→d	b→B
m→n	f→h	c→C
s→___	r→___	m→___
___→s	___→x	___→X
___→r	___→f	___→Y
___→___	___→___	___→___
___→___	___→___	___→___

4) c→b
e→d
t→s
d→___
m→___
___→h
___→i
___→___
___→___

5) c→a
h→f
n→l
r→___
q→___
___→___
___→___
___→___
___→___

6) a→B
f→G
r→S
b→___
d→___
___→T
___→Q
___→___
___→___

You may also wish to include examples using words:

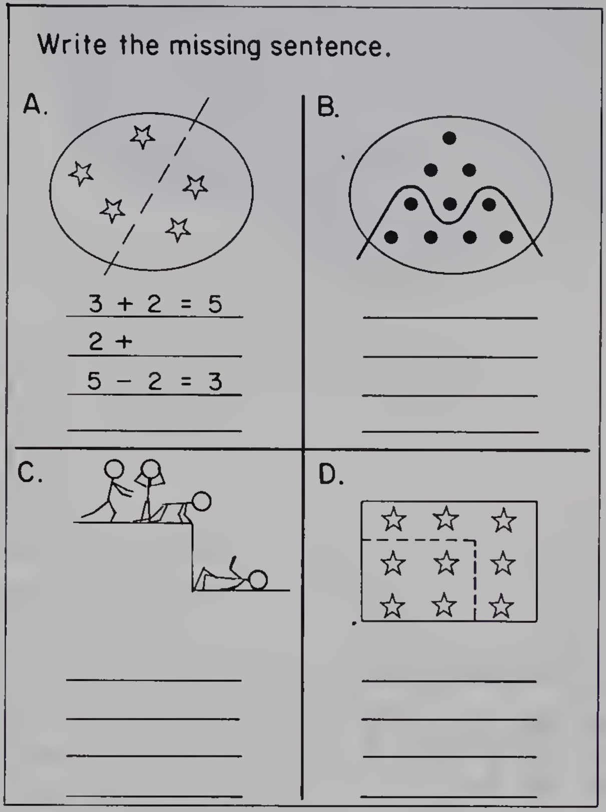
7) man→men
bird→birds
woman→women
mouse→___
___→feet

8) give→gave
have→had
hit→hit
run→___
___→fought
___→___

9) (Reverse first and last letters.)
now→won
saw→was
rat→tar
tip→___
pot→___
___→___

Drill

Here is a suggested drill sheet

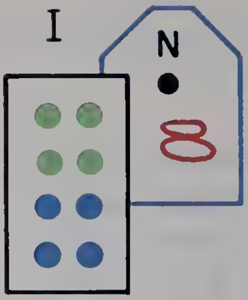

Write the missing sentence.

3 + 2 = 5
2 +
5 − 2 = 3

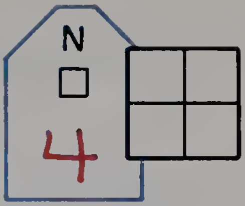
Notice that these exercises require the pupil to write a complete fact team for each picture. You may wish to use the drill sheet to anticipate the work on fact teams on pupil page 13, or you may wish to use it after page 13.

+	2	4	8
2	4	6	10
4	6	8	12
8	10	12	16

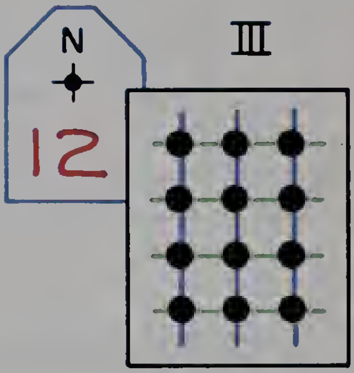
I



II



III



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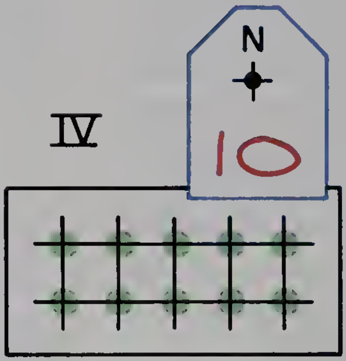

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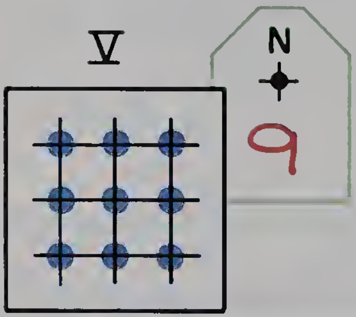
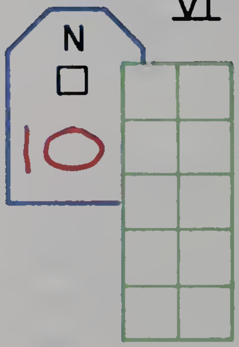
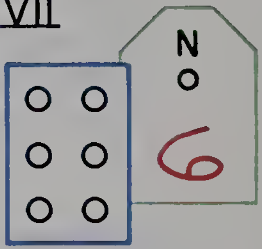


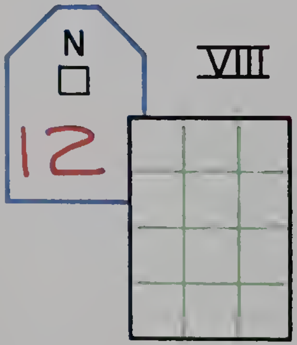
VI



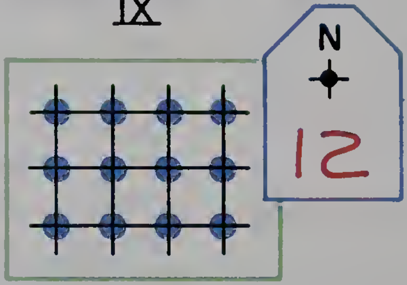
VII



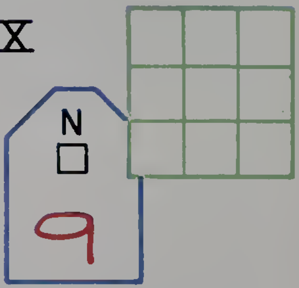
VIII



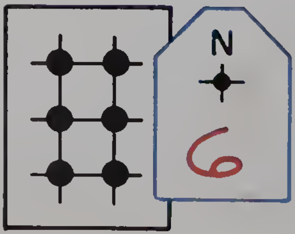
IX



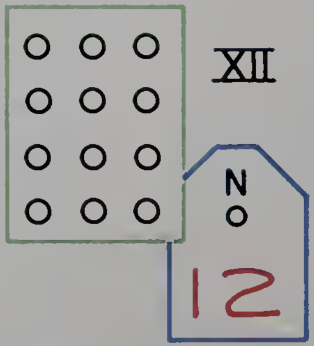
X



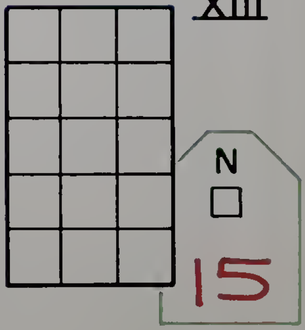
XI



XII



XIII



Notes on Pupil Page 12

Tasks: *Pupils complete the addition table in the section at the upper left. In exercises I-XIII pupils record the number of dots, little squares, or crossing points.*

Purpose: (a) To give pupils practice in addition and subtraction and in the use of two-dimensional tables; (b) to encourage pupils to use shortcuts for counting and to acquaint them with rectangular arrays in developing readiness for multiplication and division.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division.

Materials for Teacher: Demonstration models on tagboard (or oilcloth) of the various arrays shown on pupil page 12; Ditto masters. (See the Notes on Pupil Page 22.)

Materials for Pupils: Small desk-size versions of teacher's materials; Dittoed sheets; Cuisenaire rods.

The Lesson:

The chart at the top left suggests a compact way for providing drill on the facts of addition and subtraction. Similar examples can be created quickly for interesting seatwork. If the addends are patterned, the sums will be patterned also. For example, consider the following addition chart:

+	1	3	5	7
1	2	4	6	8
3	4	6	8	10
5	6	8	10	12
7	8	10	12	14

Patterns

a) Since the addends are odd, the sums are even.
b) Each row and each column shows an "increase by 2" pattern.
c) Each down-to-the-right diagonal shows an "increase by 4" pattern.
d) Each down-to-the-left diagonal shows the same number only.

Try other arrangements of tables for addition as shown on the next page.

more

Headline for the Day: In how many ways can we arrange 12 cans in columns and rows? (1 row with 12 columns, 12 rows with 1 column, etc.)

				7
				5
				3
				1
2	4	6	8	+

10	20	30	40	+
				5
				10
				15
				20

Try reconstructions of tables for addition:

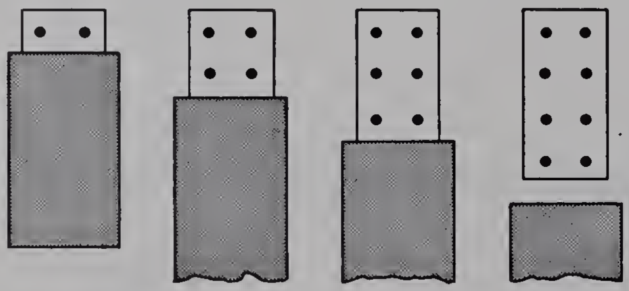
		1		
1				8
		4		
6				
+	3	1	5	

Change to subtraction:

20				
15				
10	8			
5	3	2		
−	2	3	4	5

Rectangular Arrays

Exercises I-XIII provide an opportunity for the pupil to become ready for multiplication and division. It is intended that pupils determine the totals by counting by numbers other than 1. You can help them with this by using models of the arrays shown. For example, consider the array used in exercise I. Expose the dots in stages, asking for the total number exposed at each stage:



If pupils have desk-size models, call out a number and ask them to hold up corresponding models. Gradually increase the tempo of this activity.

Use Cuisenaire rods and ask pupils to make a rectangular "floor" out of 3 light green rods. How many white rods will it take to cover the floor? Make a rectangular floor out of 2 purple rods, etc. (See items 18-20 on page 34 of the booklet *Games and Enrichment Activities.*)

If-Then

As an application of this pre-multiplication work consider the following activity.

How many shoes in a pair? In 2 pairs? In 3 pairs? Keep a record of pupils' statements:

No. of pairs	1	2	3	4	etc.
No. of shoes	2	4	6	8	

Have your pupils act out each of the following situations: How many pupils are there if there are 3 sets of 2 partners? 1 set? 4 sets? This time be sure the list of results is a random one:

No. of sets of partners	3	1	4	2	etc.
No. of pupils	6	2	8	4	

Erase one of the rows from the random record:

No. of sets of partners					etc.
No. of pupils	6	2	8	4	

If there are 6 pupils, how many pairs of partners? If there are 2 pupils, how many pairs of partners? Etc. Now erase the completed list and replace it with:

No. of sets of partners	3	4	1	2	etc.
No. of pupils					

If there are 3 sets of partners, how many pupils in all? Etc.

What Can We Say?

You may wish to introduce the following application situations as follow-up to pupil page 12 and as preparation for pupil page 13.

I have __ cents. I need 6 more cents to buy a set of pencils.

What can we say? Nothing as definite as we would like to, but surely we can say something.

We might point out that if I have a quarter, then a set of pencils must cost 31¢. We might write a headline in this form:

□ + 6 = △

Suppose that we leave in the first number mentioned in a typical story, but omit the second. Our story would read:

I have 10 cents. I need ___ more cents to buy a set of pencils.

We could write a headline for this story such as:

10 + ⬡ = △

We could say that if I need 8¢ more, then the set of pencils must cost 18¢. A chart recording some possibilities might be:

⬡	0	1	2	.....	9	.....	15	etc.
△	10	11	12		19		25	

I have ___ cents. I need . . . more cents to buy a set of pencils that cost 13¢.

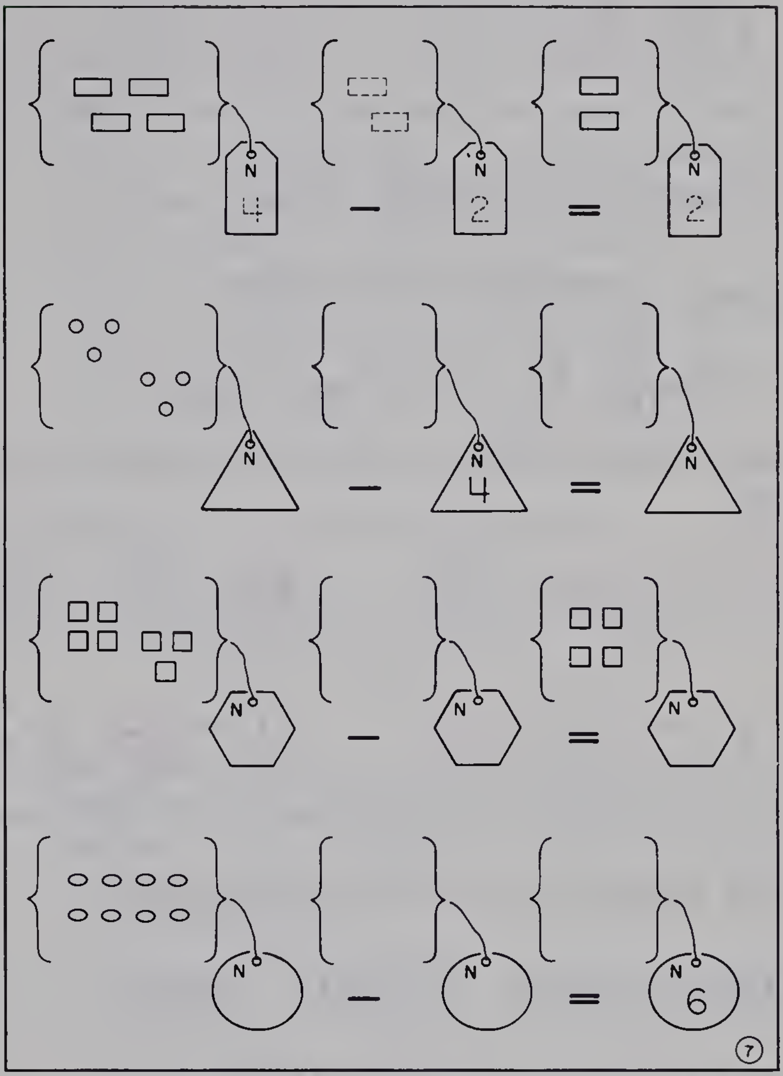
□ + ⬡ = 13

A chart of possibilities might be:

□	0	1	2	3	.....	8	.....	13
⬡	13	12	11	10		5		0

Drill

Consider page 7 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B for drill work:



I.

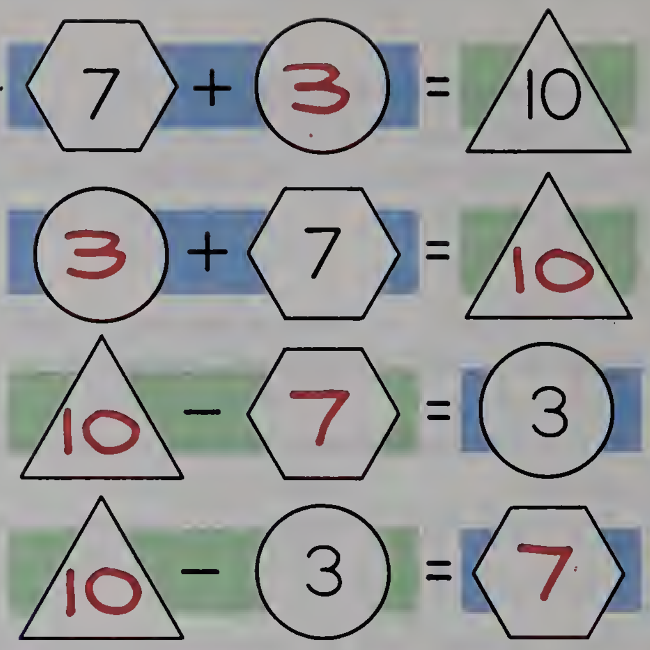
8 + 5 = 13

✓ 7 + 3 = 10

12 − 8 = 4

10 + 5 = 15

I have a nickel and two pennies. It costs a dime for the movies.


7 + 3 = 10
3 + 7 = 10
10 − 7 = 3
10 − 3 = 7

II.

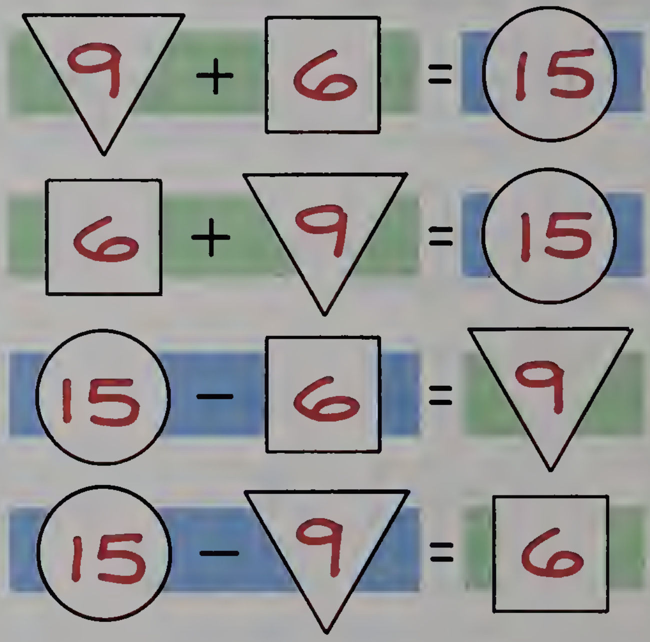
✓ 9 + 6 = 15

8 + 8 = 16

17 − 8 = 9

10 − 8 = 2

At a party of 15 children there were 9 girls. How many boys were there ?


9 + 6 = 15
6 + 9 = 15
15 − 6 = 9
15 − 9 = 6

III.

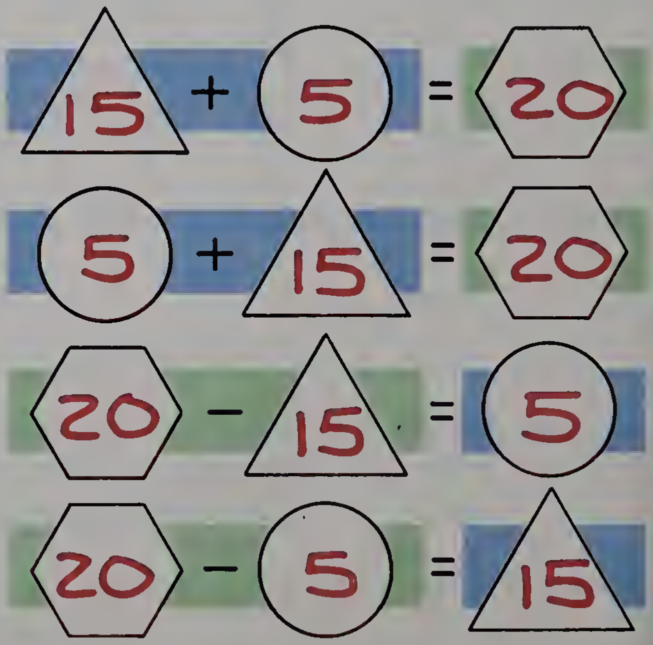
23 − 4 = 19

4 + 13 = 17

11 + 8 = 19

20 − 5 = 15 ✓

We had 20 miles to go to get to the motel. Our first stop was 5 miles away.


15 + 5 = 20
5 + 15 = 20
20 − 15 = 5
20 − 5 = 15

Notes on Pupil Page 13

Tasks: Pupils complete the four number sentences above each story. They then select the one sentence most fitting as a headline for the story, and write that sentence as well as the other members of the same fact team.

Purpose: (a) To give pupils experience with applications of addition and subtraction; (b) to review pupils' understanding of the relation between addition and subtraction.

Unifying Ideas: Structure; addition and subtraction; measurement.

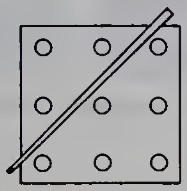
Materials for Teacher: Flannel board or cork board; demonstration-size counters; pointer; colored chalk; Ditto masters.

Materials for Pupils: Counters; Dittoed sheets of number line chunks and various rectangular arrays of dots; Cuisenaire rods.

The Lesson:

Put the partitioned display below on your flannel board or cork board alongside the list of number sentences as shown:

3 + 5 = ___
1 + ___ = 12
7 = ___ - 4
9 = 3 + ___

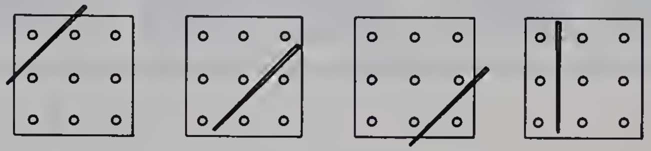


Invite pupils to turn each number sentence into a true one. Ask the pupils to use their counters to duplicate your flannel board display. Which of the completed number sentences is an appropriate title for the partitioned display? Most pupils would agree that $9 = 3 + 6$ would fit the display well. (However, some pupil may present a plausible reason for another selection. If he does, he should not be discouraged, but complimented.)

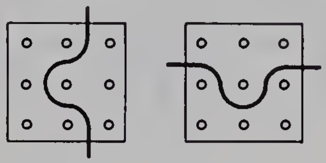
While $9 = 3 + 6$ is an appropriate title, note that all members of this fact team are equally fitting:

$9 = 3 + 6 \quad 6 + 3 = 9 \quad 9 - 3 = 6 \quad 6 = 9 - 3$
$9 = 6 + 3 \quad 3 + 6 = 9 \quad 9 - 6 = 3 \quad 3 = 9 - 6$

Develop several situations similar to the one above by moving your pointer. Pupils should duplicate your partitioning with their pencils and may suggest their own. Some possibilities:



A sketch on the chalkboard would allow:

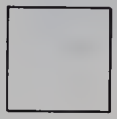


Each situation offered should be accompanied by several number sentences, one of which is an appropriate title.

Now present the following situation for pupils to work on at their desk:

3 + 7 = ___
___ + 3 = 10

10 - 3 = ___
___ - 7 = 3



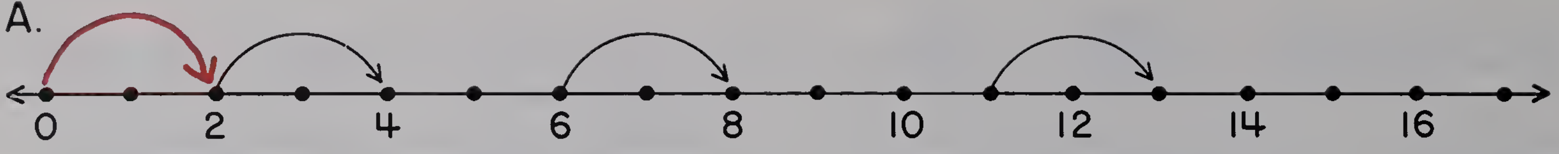
They must (a) complete each number sentence and then (b) construct an appropriate model for the entire fact team. Invite different pupils to draw sketches of their partitioned displays on the chalkboard.

See page 56 of the booklet *Games and Enrichment Activities* for an activity involving Cuisenaire rods and fact teams.

Headline for the Day: Encourage pupils to tell stories from real life that contain the same conditions as the three stories on pupil page 13. Encourage pupils to include unusual twists: $10 - 7 = 3$ . . . I've got a nickel and 2 pennies; when I save 3 more pennies, I can exchange all the coins for a dime.

WHAT'S MY RULE?

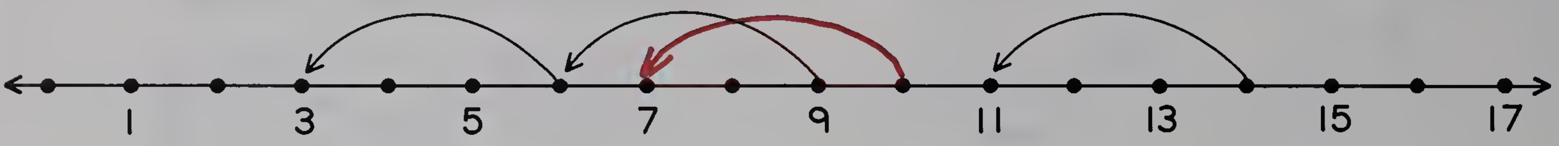
A.


0
2
4
6
8
10
12
14
16

2 → 4
6 → 8
11 → 13
0 → 2
10 → 12

14 → 16
3 → 5
8 → 10
13 → 15
15 → 17

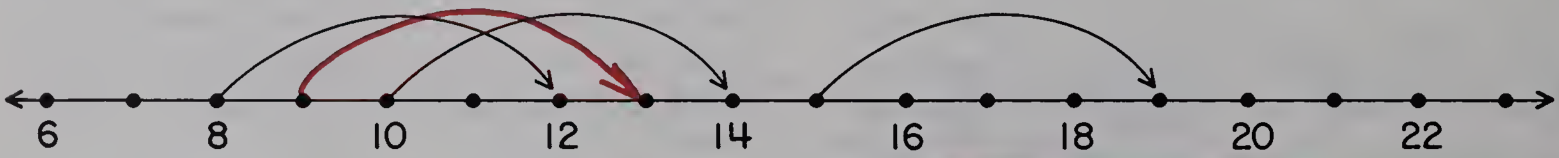
B.


1
3
5
7
9
11
13
15
17

6 → 3
9 → 6
14 → 11
10 → 7
4 → 1

16 → 13
12 → 9
8 → 5
17 → 14
18 → 15

C.

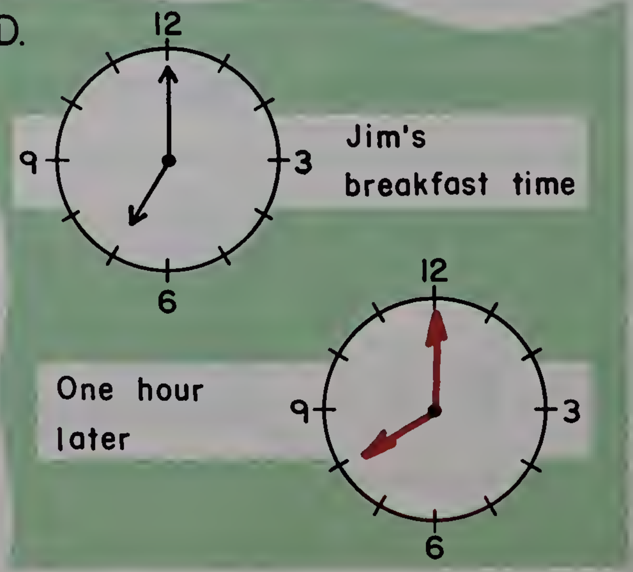

6
8
10
12
14
16
18
20
22

8 → 12
10 → 14
15 → 19

9 → 13
7 → 11
11 → 15

19 → 23
17 → 21
20 → 24

D.


12
9
3
6
Jim's breakfast time
One hour later
12
9
3
6

Notes on Pupil Page 14

Tasks: *Pupils discover rules by studying given pairs of numbers or jumps on number lines and apply the discoveries by filling in the shaded boxes. In exercise D pupils draw clock hands to indicate 8 o'clock.*

Purpose: (a) To provide practice in addition and subtraction of 2, 3, and 4; (b) to use jumps on the number line as representations of addition and subtraction; (c) to provide practice in clock reading.

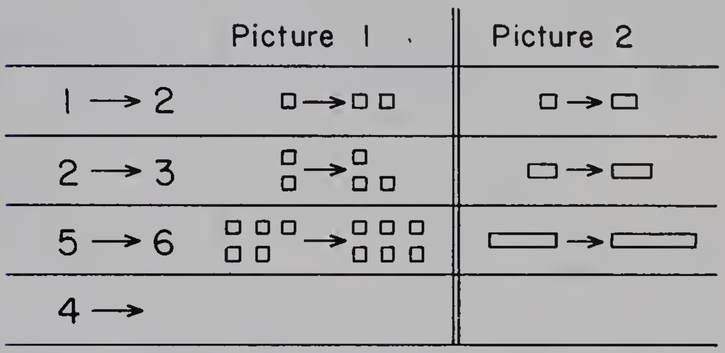
Unifying Ideas: Addition and subtraction; functions and relations; measurement.

Materials for Teacher: Square counters; masking tape models of number line chunks; colored chalk; rectangular strips whose areas are multiples of a unit square; Ditto masters; demonstration clockface.

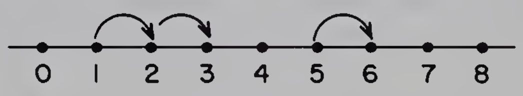
Materials for Pupils: Dittoed sheets of number line chunks.

The Lesson:

What's My Rule? games can be pictured in many ways, as we have noted earlier—by counters, by areas, by jumps on the number line. Consider reviewing some of these pictorial representations:



Picture 3

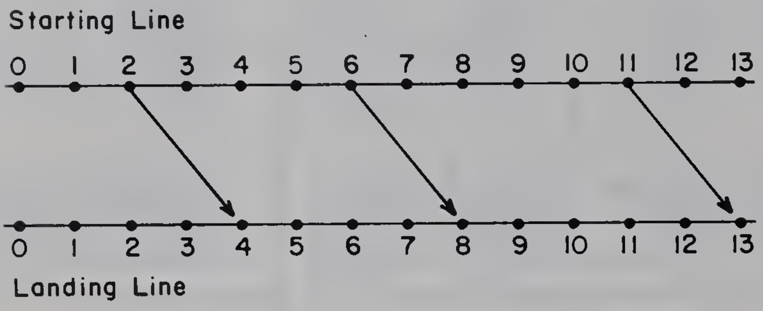


more

Headline for the Day: Make up stories that go with each *What's My Rule?* pattern on pupil page 14. Examples: (1) Bob has 2 cents more than his brother; (2) Bob arrived 2 minutes later than his brother; (3) Bob is 2 inches taller than his brother.

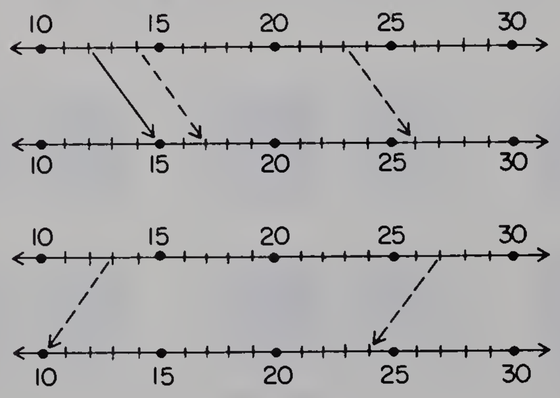
Pairs of Number Lines

After pupils have completed page 14, show them how to represent each rule on two parallel number lines. The first pattern on page 14 gets underway like this:



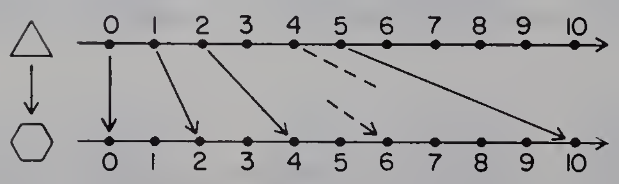
Have pupils diagram the other number pairs. Work backward by drawing an arrow and asking for the number pair.

The following diagrams suggest pictorially that adding 3 can be reversed by subtracting 3:



Now invite the class to use their number lines to explore a "doubling" rule:

△ → ⬡
2 → 4
5 → 10
1 → 2
4 → __
__ → 6



Pupils like to invent their own *What's My Rule?* games, giving a few clues and challenging the others to discover the pattern.

Patterns

There are a number of interesting patterns that can be observed by studying the pairs on page 14. Here are just a few:

1) You can get a new pair that follows the same rule by adding (or subtracting) the same number from both numbers of the given pair. For example, if 2 → 4, then 2 + 1 → 4 + 1, 2 + 7 → 4 + 7, 2 − 1 → 4 − 1. (This relationship holds for any addition or subtraction rule.)

2) Given two pairs that fit the same addition or subtraction rule, subtracting the first numbers gives you the same result as subtracting the second numbers. For example, if 11 → 13 and 8 → 10 then 11 − 8 = 13 − 10.

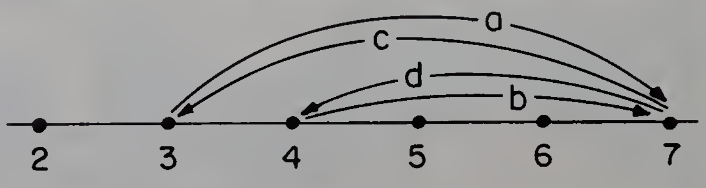
3) Given two pairs that fit the same addition or subtraction rule, adding the first number of the first pair and the second number of the second gives the same result as adding the first number of the second pair and the second number of the first pair.

4) What happens when, given two pairs that fit the same addition or subtraction rule, you add the first numbers and you add the second numbers?

Number Lines and Fact Teams

Jumps on a number line can be used for many activities other than *What's My Rule?* For example, this may be a good time to ask your pupils to tell you what is going on in the following pattern and record.

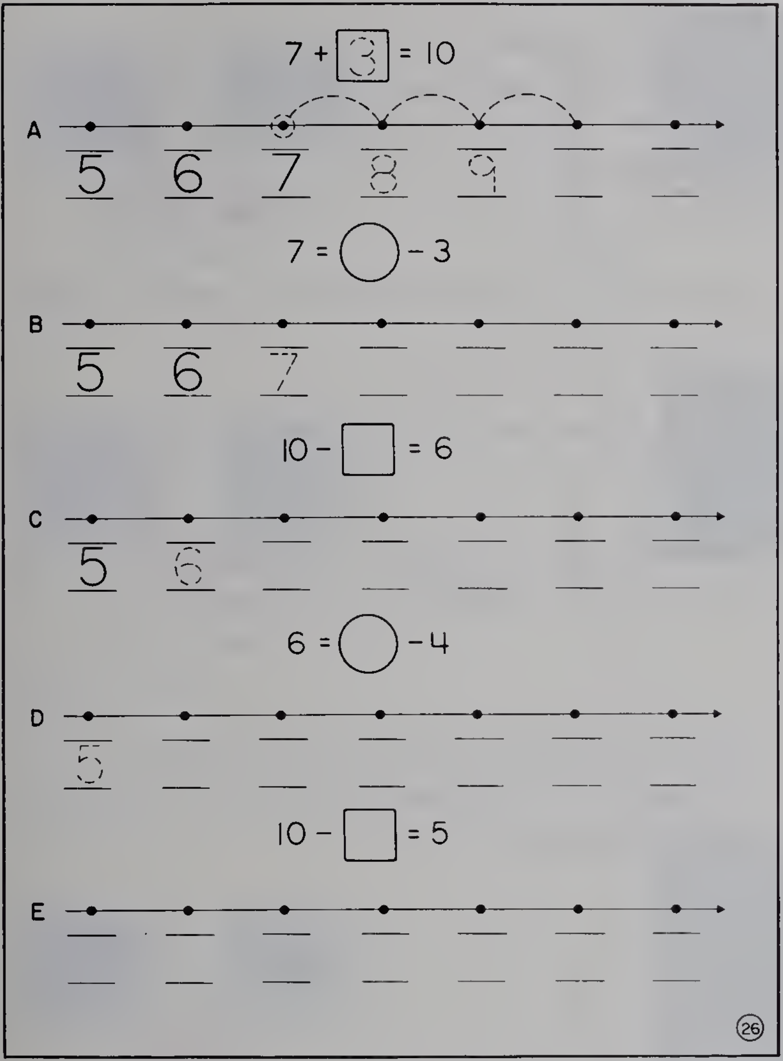
(It illustrates in another way the inverse relationship between addition and subtraction, and the commutative principle of addition. Of course, we are not interested in these words. The important thing is the exploration, discussion, and investigation of similar patterns.)



(a) 3 + 4 = 7
(b) 4 + 3 = 7
(c) 7 − 4 = 3
(d) 7 − 3 = 4

Drill

Consider using as drill sheets pages 26 and 27 of Math Workshop Supplementary Activities, Level B:



$7 + \boxed{3} = 10$

A 5 6 7 8 9 ___ ___

$7 = \bigcirc - 3$

B 5 6 7 ___ ___ ___ ___

$10 - \square = 6$

C 5 6 ___ ___ ___ ___ ___

$6 = \bigcirc - 4$

D 5 ___ ___ ___ ___ ___ ___

$10 - \square = 5$

E ___ ___ ___ ___ ___ ___ ___

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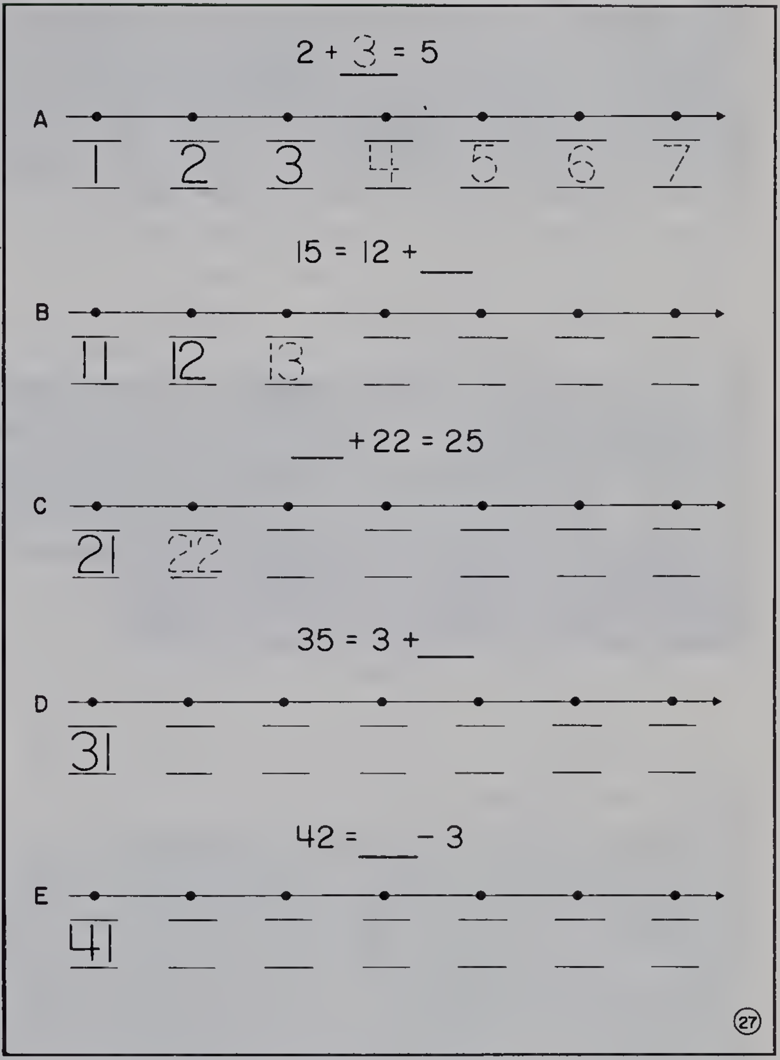


$2 + \underline{3} = 5$

A 1 2 3 4 5 6 7

$15 = 12 + \underline{\quad}$

B 11 12 13 ___ ___ ___ ___

$\underline{\quad} + 22 = 25$

C 21 22 ___ ___ ___ ___ ___

$35 = 3 + \underline{\quad}$

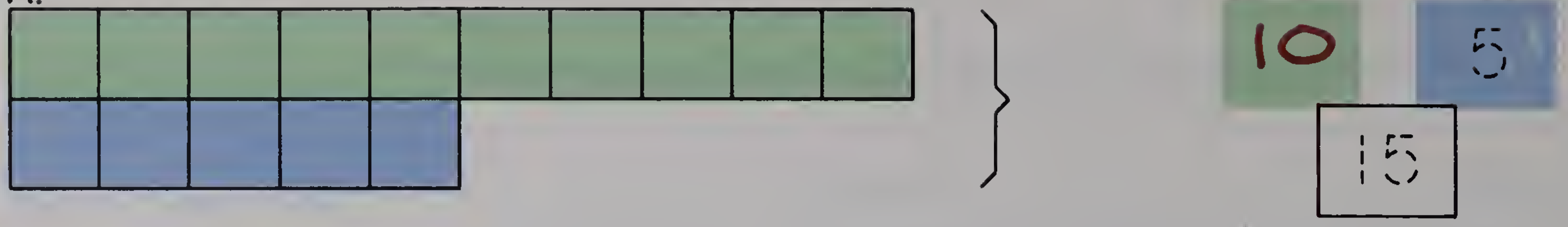
D 31 ___ ___ ___ ___ ___ ___

$42 = \underline{\quad} - 3$

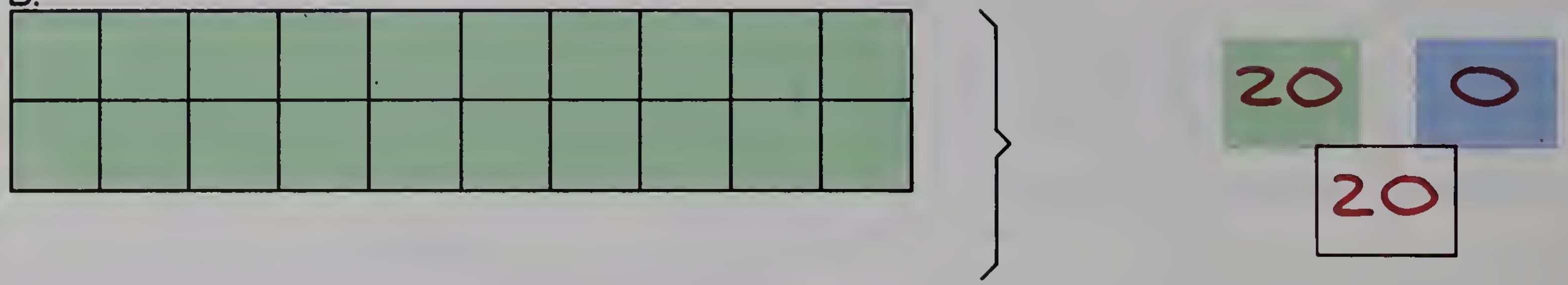
E 41 ___ ___ ___ ___ ___ ___

㉗

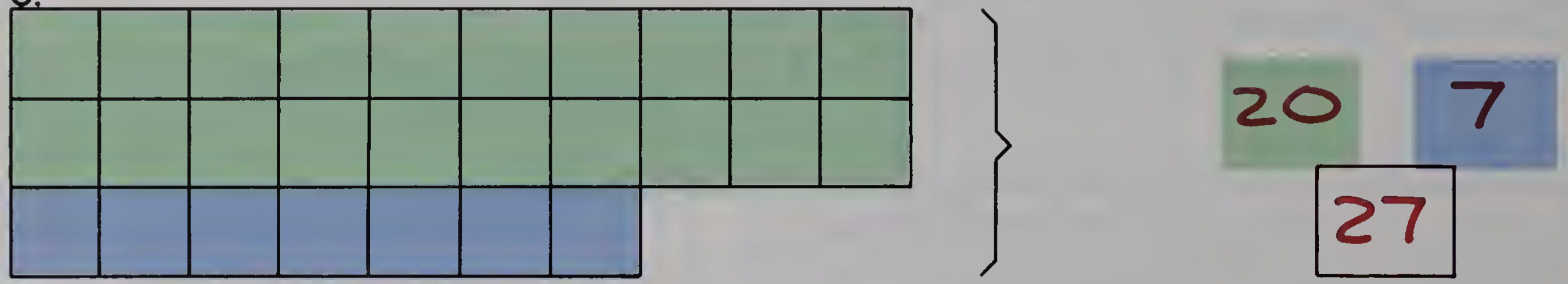
A.



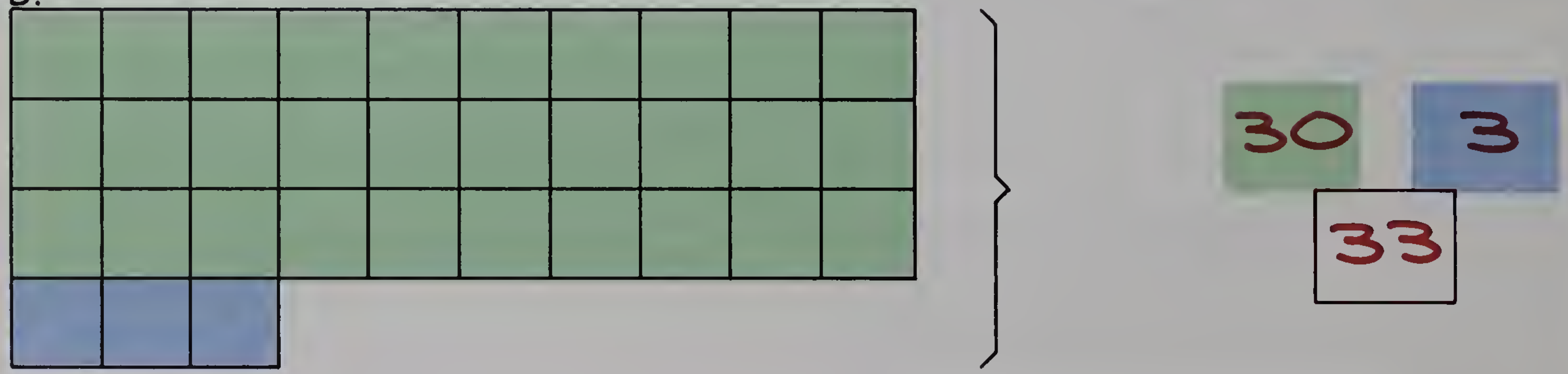
B.



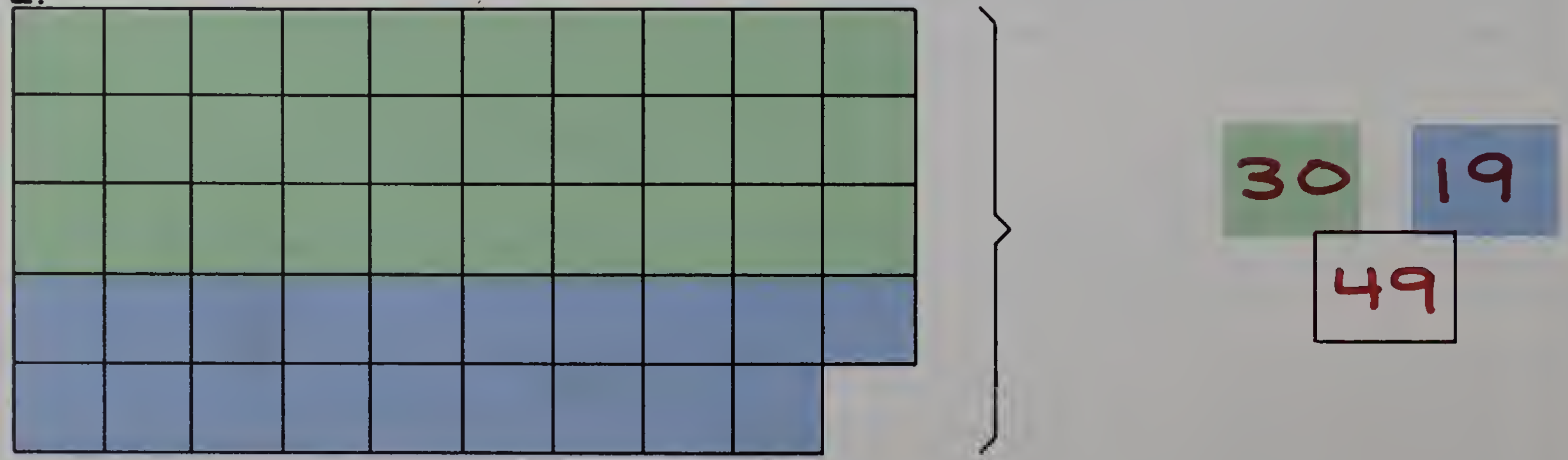
C.



D.



E.



Tasks: *Pupils count (by tens where possible) the number of green tiles and the number of blue tiles and record these numbers and their sums in shaded and white boxes at the right.*

Purpose: (a) To help pupils develop an awareness of the fact that partitioning into 10-groups facilitates counting; (b) to provide practice in the use of the decimal system of numeration in forming names of upper-decade numbers.

Unifying Ideas: Numeration; addition and subtraction.

Materials for Teacher: About 2000 sticks; rubber bands; Ditto masters.

Materials for Pupils: Cardboard boxes; Dittoed sheets; Cuisenaire rods; graph paper.

The Lesson:

The tasks on pupil page 15 are relatively simple once pupils have some feeling for the usefulness of partitioning into 10-groups in counting and recording totals. A pupil who can do exercises A-E only by counting individual tiles is not making satisfactory progress and should be given special help.

Lots of Things to Count

Young children need a great many experiences in seeing and handling large numbers of objects so that they will see the need for some kind of organization in counting. Provide each two or three pupils with a box of 200 sticks (such as Popsicle sticks or tongue depressors) and a supply of rubber bands. Forty of the sticks should be bundled with rubber bands in groups of 10. Ask each team to bring 14 sticks; 37 sticks; 48 sticks; 59 sticks; 78 sticks; 93 sticks. Praise those pupils who use the bundles in carrying out your requests. Encourage pupils to make additional bundles so that they will be able to fulfill your requests quickly and accurately.

When a pupil hands you 4 bundles and 8 more sticks, say: 4 tens and 8 more—40 plus 8—48. Make similar remarks for the other two-digit numbers.

Have the pupils count by tens, saying: One ten, 10; two tens, 20; three tens, 30; . . . ten tens, 100.

Let a team of pupils face the class holding a total

more

Headline for the Day: Mary has _____ pennies. This is the same as . . . dimes and _ _ _ pennies. Chart the result.

of 37 sticks in their hands (3 bundles and 7 extra). Put 1 stick in the hand of one of the team members and ask the class how many sticks the team now has. Give one of the team members a bundle of 10 and ask how many sticks the team has now. Pull 1 stick from one of the team's bundles and ask for the new total.

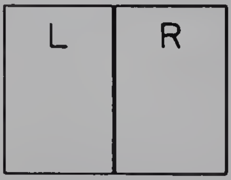
Cuisenaire Rods

Cuisenaire rods are useful in teaching pupils the decimal system of numeration. Here are some activities in which rods are used.

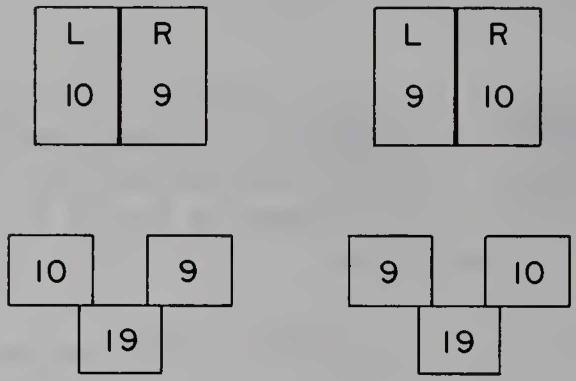
Place all your white rods on your desk. Roger, how many do you have on your desk? Are you sure? If you didn't know how many you had, how could you be sure? Gary, pile your white rods on Roger's desk, but don't tell him how many you gave him. Now, Roger, how many do you have? Can you think of a faster way to tell us without counting them one by one? (There are many possible answers to this question.) Suppose that we use the method of making trains as long as the orange rod. Each train would be how many white rods? (10) Let's use this method to tell how many white rods we have without counting them one by one. Jane, how many tens do you have? How many ones do you have left over? Suppose that Jane says that she has 3 tens and 6 ones. What is a short way to say 3 tens and 6 ones? (36) Let's keep a record on the chalk board:

pupil	tens	ones	number
Jane	3	6	36
Billy	2	9	29
Sarah			
Joel			

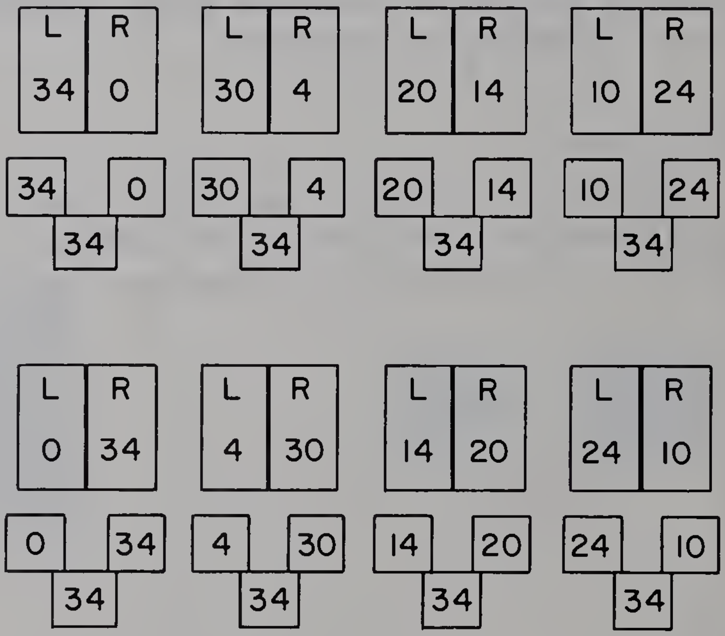
Build a rod staircase from white to orange and place it at the upper right corner of your desk. Now let's play the cross-over-the-line game. Take out your Dittoed sheet:



Take 1 orange rod and any rod from your staircase. Place them on your desk. Suppose that we all take 1 orange rod and 1 blue rod to start. What are the different ways we could place them on the left and right side of the line on our Dittoed sheet? List the suggestions on the chalkboard as shown below:



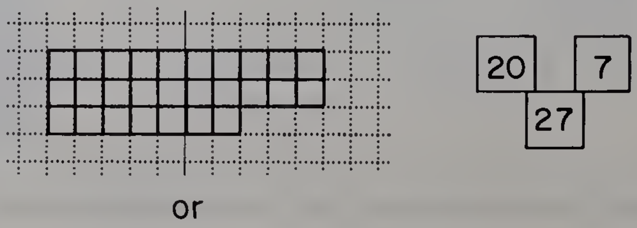
Take 3 orange rods and 1 purple rod. What are all the different ways we could place our rods on the Dittoed sheets on either side of the line? A summary of the investigation would reveal the following:

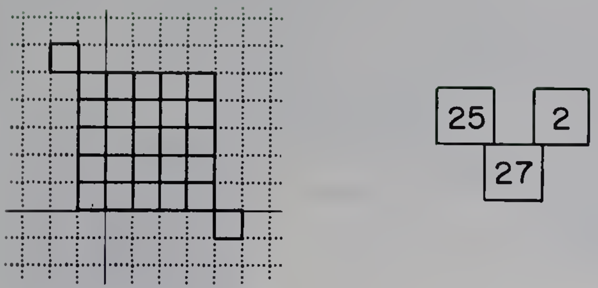


Proceed with several other similar situations.

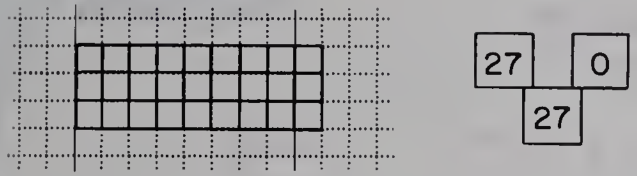
Graph Paper

Distribute a sheet of graph paper (squared paper) to each pupil. Have them mark out 27 squares. There may be many shortcuts your pupils will devise. Discuss some of them with the class.

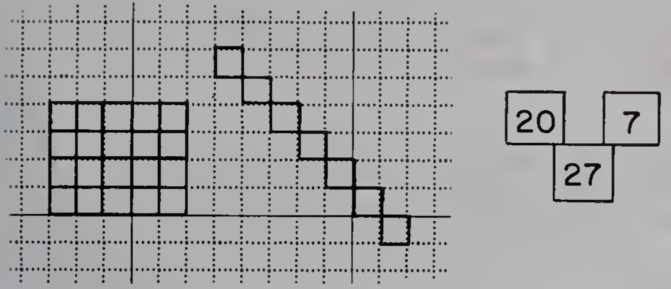




or

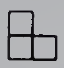
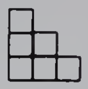
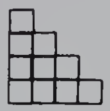


or

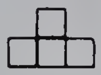
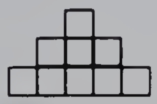
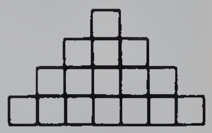


Select other numbers for which the pupils must construct graph paper models. Encourage them to construct (a) the usual one—so many tens and some more as indicated on pupil page 15, and (b) as many other models of each number as seem interesting and pleasing to them.

You might wish to have your "front-runners" investigate to determine which numbers can be represented by square arrays (1, 4, 9, 16, etc.), rectangular arrays with two rows (2, 4, 6, etc.), rectangular arrays with three rows (3, 6, 9, etc.), or triangular arrays such as those below:

They might also consider which numbers can be represented by other triangular arrays:

There are many other special types of arrays which can be studied. Emphasis should be placed on looking for patterns among the numbers which can be represented by each special type of array.

NUMBERS HAVE MANY NAMES

1 + 4

1 + 5

0 + 4

fourteen

1 + 3

I

sixty

64

II

1 + 2 + 3

V

3 + 1 + 0

III

Add

$$\begin{array}{ccccc} 2 & 2 & 2 & 2 & 2 \\ \underline{1} & \underline{2} & \underline{3} & \underline{4} & \underline{5} \end{array}$$

IV

4

0 + 4

1+3

3+1+0

$$\begin{array}{r} 2 \\ \underline{+2} \end{array}$$

IV

0+1+3

4+0

four

VII

3 + 2 + 0

5 + 1

six

0 + 1 + 3

sixteen

6

1 + 5

1+2+3

$$\begin{array}{r} 2 \\ \underline{+4} \end{array}$$

5+1

six

4+2

VI

four

4 + 0

4 + 1

VI

4 + 2

4 + 3

4 + 4

46

twenty

Notes on Pupil Page 16

Tasks: Pupils circle names for 4 or for 6 and write each name in the appropriate circle.

Purpose: (a) To provide review in addition combinations; (b) to familiarize pupils with certain Roman numerals and word names for numbers.

Unifying Ideas: Numeration; addition and subtraction.

Materials for Teacher: Tagboard strips with masking tape backing and with various names for numbers written on them; colored boxes and small objects; demonstration strips.

Materials for Pupils: Crayons; square counters; round counters; Cuisenaire rods.

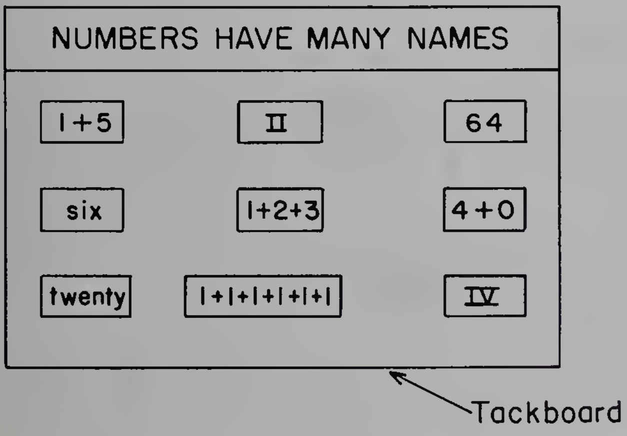
The Lesson:

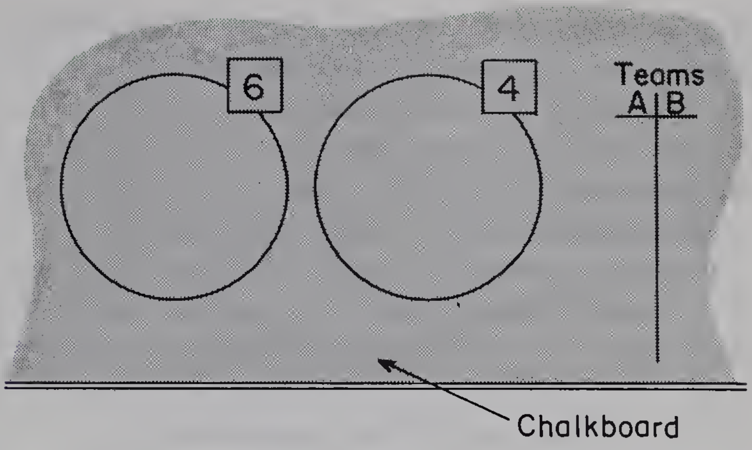
The major stress in this lesson should be placed on the fact that displays for a given number can be partitioned in many ways. Each partitioning leads to an addition fact.

The following activity will prepare the pupils for pupil page 16. Some of the other activities described below should be used as follow-up.

Activity 1

Arrange a display similar to the one shown below on your tackboard and chalkboard:





Cut out tag strips and write on them each of the names given on the green field on pupil page 16. Tack the strips on your tackboard or use scotch tape to fasten them to the tackboard as shown in the sketch above. Cover up the entire display before beginning the lesson.

Today we are going to play detective. We'll need two teams. Divide the class into two groups and appoint team leaders for team A and team B. Uncover the tackboard and ask for volunteers to describe what is on the board. Uncover the chalkboard. What do you think we shall do with the strips from the tackboard? (Place names for 6 in the 6-circle and names for 4 in the 4-circle.) A team leader selects a team member to pick a strip from the tackboard and place it in the appropriate circle on the chalkboard. If the team member's placement is correct, he scores a point for his team; if not, the opposing team scores a point. The teacher keeps the score on the chalkboard. Interrupt the game occasionally and ask someone to tell the class what the score is up to that point in the game.

The game is finished when the pupils have correctly classified all the strips into three sets—those that name 4, those that name 6, and those that name neither 4 nor 6. The team with more points wins the game.

When the game is completed, have the class turn

Headline for the Day. Two names for the price of one:

4 + 7	and	7 + 4
8 + 9	and	9 + 8
	etc.	

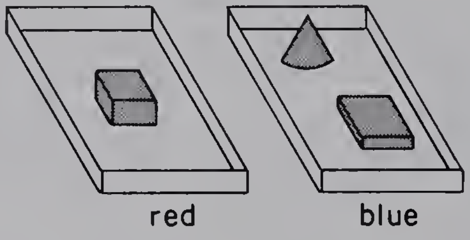
to pupil page 16 and complete it as follows. Have them use a colored crayon to circle names for 4 on the green field. Then have them record in pencil in the 4-circle all the names for 4 they found. Using a different-colored crayon for names for 6, they proceed in a fashion similar to that for names for 4.

How many different names for 4 did you find? (8) How many different names for 6 did you find? (7) How many names did you find that were not for either 4 or 6? (19) Who knows some other names for 4? For 6? (Make a list of the suggestions.)

Activity 2

Here are two boxes—one red and one blue. We have 3 objects. We can put them all in the boxes in several ways.

How many different ways are there (using all of the objects each time)?



Anne volunteers one arrangement—1 object in the red box and 2 in the blue. A record keeper (one of the pupils or the teacher) fills in the top line of a chart.

	red	blue	
Anne	1	2	1 + 2 = 3

(or $\begin{array}{r}1\\+2\\\hline 3\end{array}$)

Other pupils are urged to find another arrangement. Each proposal is checked against the record to see whether or not it is different (different numbers of objects in the red and blue boxes) from arrangements already noted. If it is, its originator is given credit.

Eventually the record may look like this:

	red	blue	
Anne	1	2	1 + 2 = 3
Fred	2	1	2 + 1 = 3
Ruth	0	3	0 + 3 = 3
Bill	3	0	3 + 0 = 3

4 ways

Are they any more different ways?

When we place 3 objects in the boxes, can we have the same number of objects in each box? Can we have two more in one box than in the other? Can we have twice as many in one box as in the other?

Let's try again with another set of objects. (Leave the original record on the board so it can be compared with the next one.)

Activity 3

Another interesting exploration gets under way when we have three boxes—red, blue, and white. In how many ways can we put 1 object in the boxes? 2 objects? 3 objects?

1 object:

	red	blue	white	
Barbara	1	0	0	1 + 0 + 0 = 1
Jim	0	0	1	0 + 0 + 1 = 1
Harry	0	1	0	0 + 1 + 0 = 1

2 objects:

	red	blue	white	
Barbara	2	0	0	2 + 0 + 0 = 2
Jim	1	0	1	1 + 0 + 1 = 2
Harry	0	0	2	0 + 0 + 2 = 2
Etc.				

3 objects:

	red	blue	white	
Barbara	3	0	0	3 + 0 + 0 = 3
Jim	0	3	0	0 + 3 + 0 = 3
Harry	0	0	3	0 + 0 + 3 = 3
Etc.				

Activity 4

Discuss with pupils such questions as these:

1) If Jim and Dick were looking for 4 lost marbles and together found all 4, how many did each find? A record will help.

JIM		DICK	
1	+	3	= 4
2	+	2	= 4

etc.

2) If I have 4 cents in my two pockets, how many do I have in each?

LEFT		RIGHT	
3	+	1	= 4
0	+	4	= 4
	etc.		

Activity 5

Ask the pupils to take out 8 counters, 4 each of two kinds. Tell them they are to discover all the ways they can put 4 counters in a row on their desks, using these counters. Help each pupil keep a record of what he finds out, as shown below (of course, the order of the entries may differ).

Arrangements of counters:

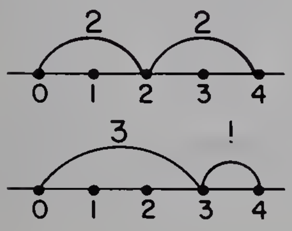
☐ ☐ ☐ ☐
☐ ☐ ☐ ○
☐ ☐ ○ ○
☐ ○ ○ ○
○ ○ ○ ○
etc.

Record:

N ☐	N ○	N all
4	0	4
3	1	4
2	2	4
1	3	4
0	4	4
	etc.	

Activity 6

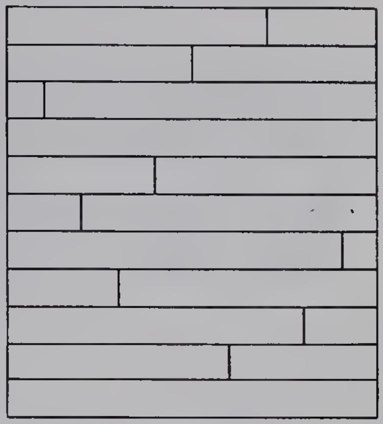
Do the equivalent of Activity 5 in terms of jumps on a number line. You may have the pupils actually do the jumping on a number line on the floor, or you may use pictures on the chalkboard or on a piece of paper.



First jump	Second jump	Land
2	2	4
3	1	4
	etc.	

Activity 7

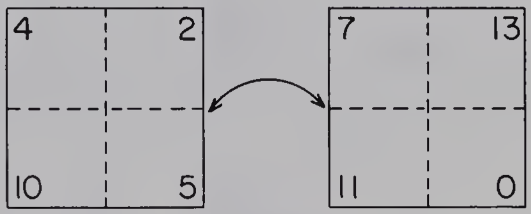
Use your demonstration strips to help the pupils build strips 10 units long out of no more than two strips or rods. Do not suggest any particular order; rather, encourage each pupil to keep a record of each row he has constructed.



10 = 7 + 3
10 = 5 + 5
10 = 1 + 9
10 = 10 + 0
10 = 4 + 6
10 = 2 + 8
10 = 9 + 1
10 = 3 + 7
10 = 8 + 2
10 = 6 + 4
10 = 0 + 10

Activity 8

Have pupils divide both front and back of a piece of paper into four parts. Next, have them write their favorite four numbers on one side, one number in each square. Then, have them turn the paper over and write the four numbers they like least, one in each square.



Divide the class into two or more teams. Then say: Now, for 10 minutes (or any other period) write next to each of your favorite numbers as many other names as you can think of for that number. Go!

The game can be scored by having the members of one team correct the papers of the other. Each paper is scored by giving 1 point for each correct name. Points for any incorrect names go to the team that finds them.

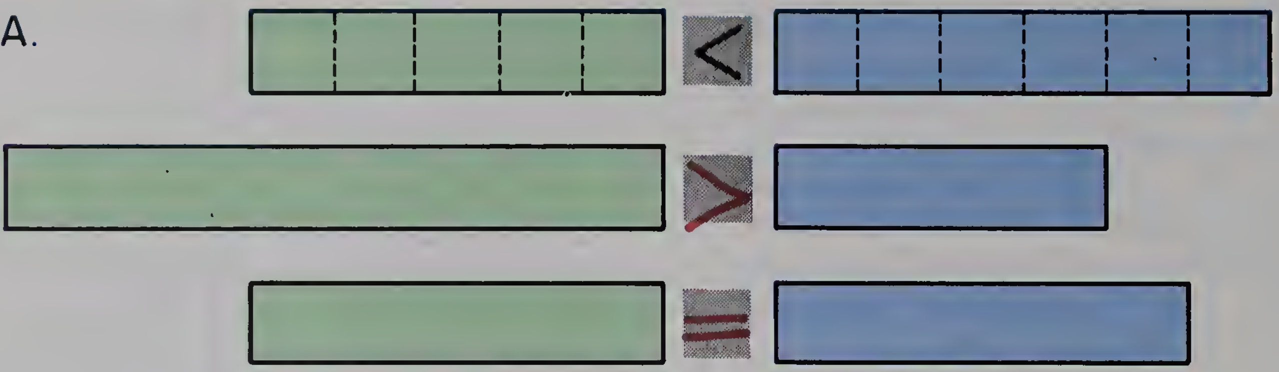
Do not discourage children from employing patterns, such as the following which they might use in listing names for 7.

7 — 0
8 — 1
9 — 2
5 + 1 + 1
4 + 1 + 1 + 1
7 + 0
6 + 1
5 + 2

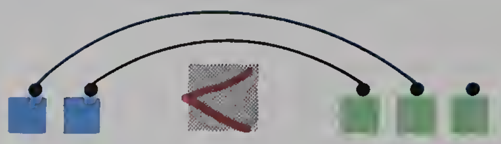
10 — 3
11 — 4
12 — 5
1 + 1 + 1 + 1 + 1 + 2
1 + 1 + 1 + 1 + 2 + 1
12 — 12 + 7
100 — 100 + 7

> Greater than < Less than = Equal to

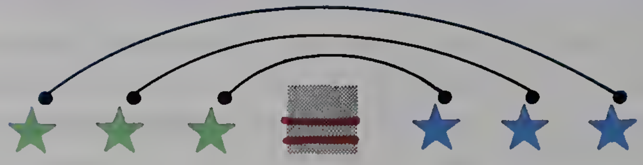
A.



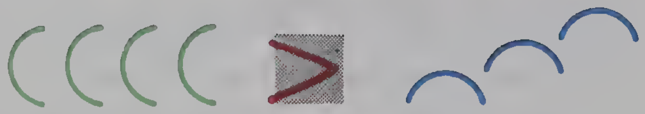
B.



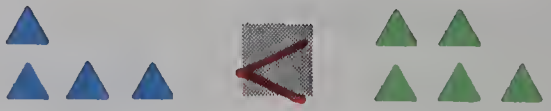
C.



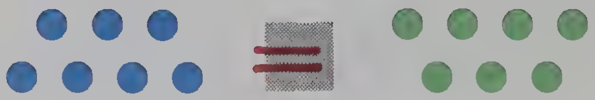
D.



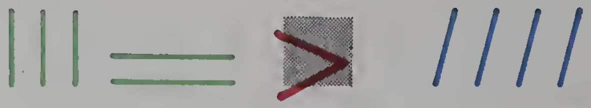
E.



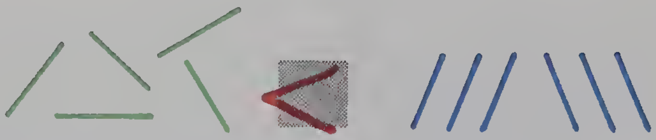
F.



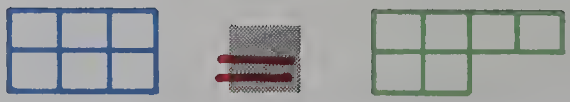
G.



H.



I.



J. 8 > 3

K. 11 < 12

OR LESS

L. 7 = 7

M. 4 < 9

N. 12 > 11

OR LESS

O. 10 < 11

OR LESS

P. What's My Rule?

□	13	50	11	15	10	17	20	25	21	31	46
□ -10	3	40	1	5	0	7	10	15	11	21	36

Notes on Pupil Page 17

Tasks: *In exercise A pupils compare the lengths of strips and record results in the empty squares. In exercises B - I pupils compare the number of objects on the left of each empty gray square with the number on the right. In exercises J - O pupils fill in frames to make true sentences. In exercise P pupils discover a rule and use it in filling in the empty cells in the table.*

Purpose: (a) To review pupils' knowledge of the equality and inequality relations; (b) to provide practice in adding 10 and subtracting 10.

Unifying Ideas: Measurement; functions and relations; addition and subtraction.

Materials for Teacher: Several strips of cardboard of varying lengths, labeled as indicated below; demonstration-size tagboards showing various displays of dots or X's as described below; Ditto masters; two demonstration-size boxes, one with a triangular lid and one with a square lid, as described in the Notes on Pupil Page 5.

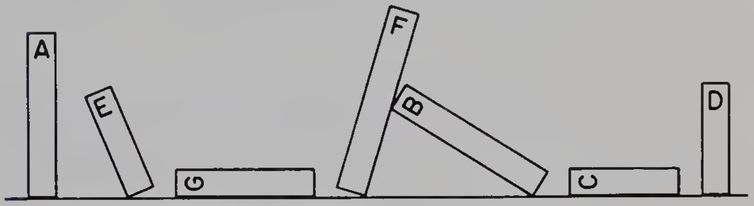
Materials for Pupils: Counters; Dittoed sheets; a desk-top tagboard for each pupil with △ > □ drawn on one side and △ < □ on the other side—both drawings large enough for pupils to place counters in the frames.

The Lesson:

Some of the varied activities described below may be used in preparation for pupil page 17; the rest may be used as follow-up.

Activity 1

Display several labeled strips on the chalkrail, arranging them in random order and at several different angles.



Is the length of A greater than the length of G, or is the length of A less than the length of G? How can we be sure? The pupils should consider at least 3 ways to check this: (1) move A and G together and notice which is longer; (2) select a unit such as a block, then count the number of blocks it takes to represent the length of A and then the number of blocks needed

more

Headline for the Day: 8 □ 4

to represent the length of G; (3) stretch a string along A and note the length, move to G and repeat the process, and then compare the string length of A with that of G.

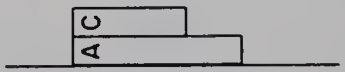
Introduce a shorthand to record the 3 possible results.

$A > G$ (A's length is greater than G's length)
$A < G$ (A's length is less than G's length)
$A = G$ (A's length is the same as G's length)

The point of the hook ($<$, $>$) points to the label for the smaller quantity.

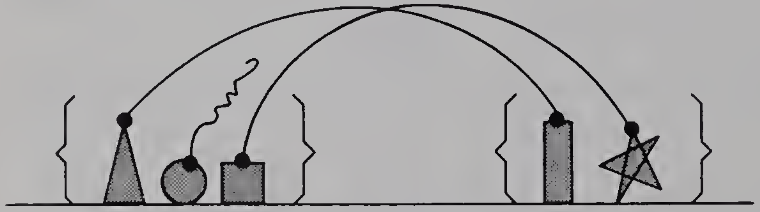
Compare other pairs of strips and record results:

$A > C$ $\qquad$ $C < A$



Activity 2

When we compare the number of objects in two displays, how can we be sure which display has the greater number? Could we be sure without counting the number of objects in each display? (Yes, by establishing a one-to-one correspondence until the matching has accounted for all the objects in one of the displays.)

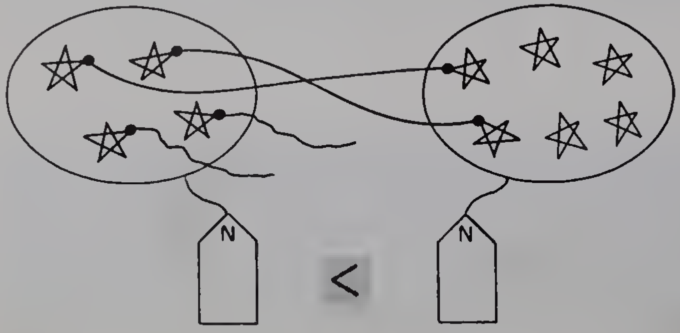


Let's call the display pictured on the left L and the display on the right R. If we are only interested in comparing the numbers of objects in the displays, we can tell from the diagram that

$N(L) > N(R)$ and $N(R) < N(L)$.

Read N(. . .) as the number of objects in . . . .

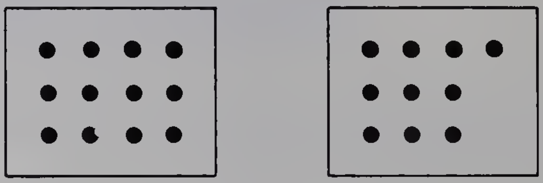
In which display below are there more stars?



Without filling in the tags indicating the number of objects in each display, we know that the "less than" symbol will be appropriate. Why?

Activity 3

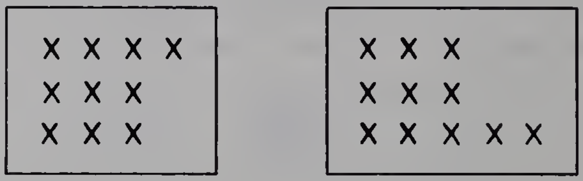
Show pairs of your demonstration displays on the board, such as:



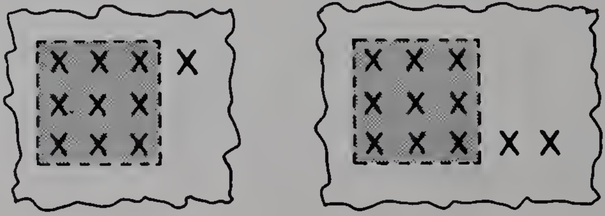
Are there the same number of dots in each display? If not, which contains more? Do you need to count the dots in both displays before you can answer these questions? No! The two designs would be alike, except that there are two dots missing from the display on the right. So, we can say that the number of dots on the left is greater than the number on the right. Then, after counting, we can write:

$$12 > 10$$

Consider another pair of displays:



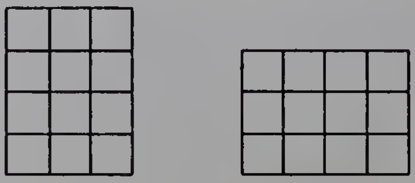
Are there the same number of X's in each display? If we notice the 3-by-3 component of each display, the question can be answered by noting that 1 is less than 2:



After counting, we can write two sentences:

$9 + 1 < 9 + 2$ $\qquad$ $10 < 11$

Still another example:



Are there the same number of small squares in each display? Yes! You need only notice that a simple turn of one display would reveal that it is just like the other. After counting each, we can write:

$$12 = 12$$

Ask the pupils to make up a story for each inequality or equality statement displayed above so that it might serve as a headline. Examples:

> Mother had a full carton of a dozen eggs. She used 2. Now she has fewer eggs because 10 eggs are less than 12 eggs.
>
> Mary is 10 years old and John is 11. Mary is younger than John.

Activity 4

Put on the board or prepare seatwork activities like these:

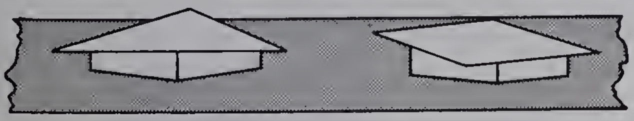
$1 < 2$	$13 = 13$	$7 > 4$
2 3	7 0	$\square > \triangle$
5 1	2 4	$\diamond < \bigcirc$
6 6	0 1	$\square = \square$
3 8	9 9	$\triangle <$ ⬡

Have the pupils fill in the missing relation signs in the first two columns. Have them write appropriate numbers in the frames in the last column. They should notice that in this last column there is an endless list of correct answers.

Activity 5

Place some objects in each of the two demonstration boxes so that there are more objects in the one with the triangular lid. Invite several pairs of pupils to determine which box has more objects in it. Now write this sentence on the board above the boxes:

$$\triangle > \square$$



We use this sentence in getting ready to say that the number of things in the box with the triangular lid is more than the number of things in the box with the square lid.

Have your helpers pass out the desk-top tagboards with $\triangle > \square$ facing up. Ask each pupil to place some of his counters in each frame so that he can state a true number sentence to report his situation. Record some of their sentences in this form:

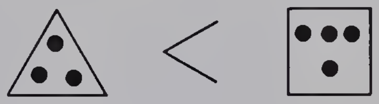
$\triangle > \square$
Bill $5 > 3$ true

Now ask the class to construct a situation which would suggest a false number sentence:

$\triangle > \square$
Mike $2 > 3$ false

Make further constructions and records on the other side of the tagboard.

Now have all pupils construct the following situation:



Ask the class to figure out a way to add some counters to those in the frames so that a number sentence describing the new situation is still true. (Example: $5<8, 4<5$)

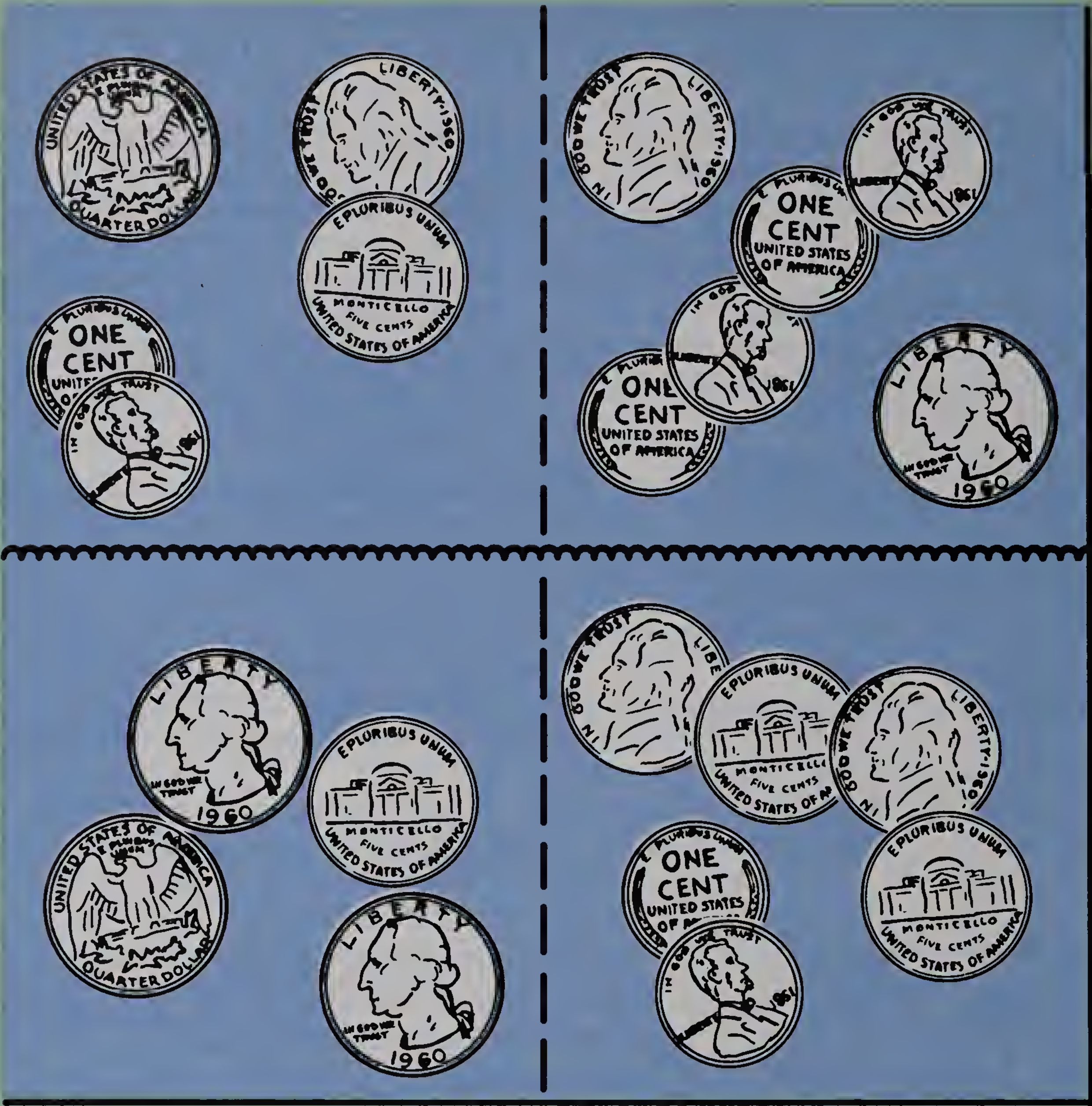
Write the following two sentences on the board:

$$\triangle < \square \qquad \triangle + 3 < \square + 3$$

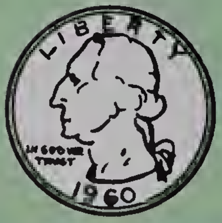
Have each pupil put counters in the frames in his copy of the first sentence to make a representation of a true sentence. Now have him include 3 more counters in each of the displays in the frames. Is the result a representation of a true statement?

$\triangle < \square$ $\triangle + 3 < \square + 3$
Doug $6 < 8$ $6 + 3 < 8 + 3$
true   true

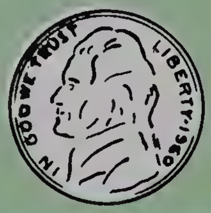
Our purpose here is to have pupils discover the fact that if a first display has fewer objects than a second and if both displays are increased in size by the same amount then the augmented first display has fewer objects than the augmented second display.



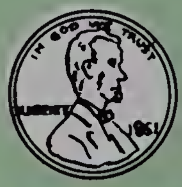
		11
		10
9	12	21



1	1	2
3	0	3
4	1	5



2	1	3
1	4	5
3	5	8



2	4	6
0	2	2
2	6	8

Notes on Pupil Page 18

Tasks: *For the large display pupils record at the right the number of coins above the wavy line and the number of coins below the wavy line. At the bottom of the large display they record the number of coins to the left of the dashed line and the number of coins to the right of the dashed line. In the lower right-hand corner they record the total number of coins in the large display. The three displays at the bottom of the page represent a partition of the large display.*

Purpose: (a) To provide practice in addition; (b) to study *Cross Number Puzzles* and thus review the notion that addition can be carried out in any order and with any grouping without affecting the sum.

Unifying Ideas: Structure; number and counting; addition and subtraction.

Materials for Teacher: Demonstration-size counters for mounting on a chalkboard; Ditto masters.

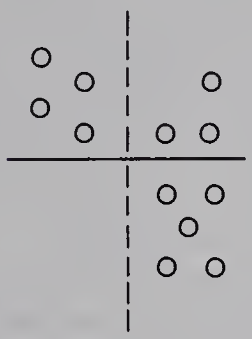
Material for Pupils: Counters or checkers; Dittoed sheets.

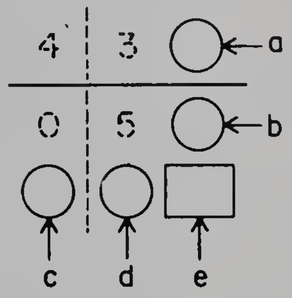
The Lesson:

This is the first appearance of the *Cross Number Puzzle* on the pupil pages. As we have noted in the Introduction, the *Cross Number Puzzle* is used throughout the text as a device for helping pupils discover the common algorithms for addition and subtraction. We present below a description of several "above-below" and "right-left" games which may be used to prepare your pupils for *Cross Number Puzzle* activity. You may present the games at the chalkboard or prepare the special record forms on Dittoed sheets.

Game 1

Draw a horizontal line on a piece of paper, and then a vertical line (broken, perhaps, to facilitate talking about the placement of objects). Place checkers in some or all of the four regions created by the crossed lines. At the side of the paper, record answers to questions in the proper place in a diagram:





How many checkers are there in the upper left-hand corner—above the solid line and to the left of the broken line? (4) Make a record of this in the diagram on the right.

How many in the upper right-hand corner—above the horizontal line and to the right of the vertical line? (3) How many in the lower right-hand corner? (5) How many in the lower left-hand corner? (0) All these answers should be recorded as shown in the diagram above.

How many checkers are there above the horizontal line? (7) Record this in the circle above the line—labeled *a* in the diagram above.

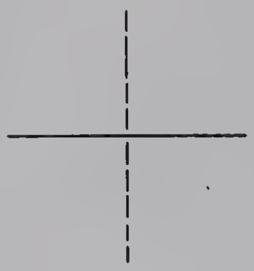
How many are there below the line? (5) How many are there to the left of the vertical line? (4) How many are there to the right of the vertical line? (8) These results should be recorded in the circles indicated by *b*, *c*, and *d*, respectively.

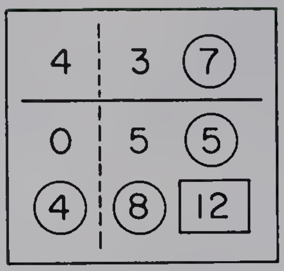
How many checkers are there altogether? (12) Record this in the box (*e*) in the lower right-hand corner of the diagram.

Now the game begins. Remove the checkers.

Can you put the checkers back the way they were, with the same number in each section as at the beginning? (See illustration on the next page.)

Headline for the Day: $(1 + 2) + 3 = 1 + (2 + 3)$



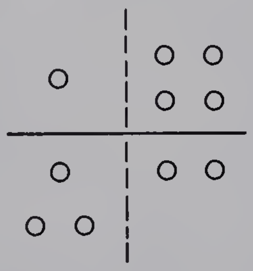


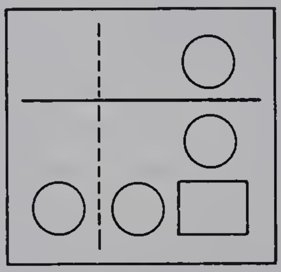
This ought to be easy. As the checkers are placed, ask: How do you know that there were 4 in the upper left-hand section? If the answer is "I remember," then ask: If you had forgotten, is there any way to be sure how many there were in that corner?

We wish to make pupils aware of the fact that the diagram at the right is a reliable record and that we don't need to remember the information we have recorded.

Game 2

We start another game with a new arrangement of checkers:



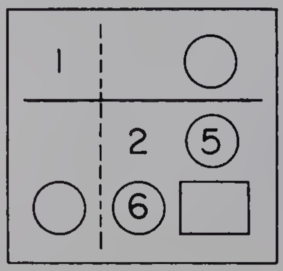


Ask only four questions this time, and record the results before removing the checkers:

How many checkers are there in the upper left-hand corner? (1) How many below the line? (5) How many in the lower right-hand corner? (2) How many on the right side of the vertical line? (6)

When these four results are noted, remove the checkers.

Can you put the checkers back as they were, with the same number in each section as at the beginning of the game? The situation in Game 2 now looks like this:



This is much more challenging than any of the earlier questions.

As the checkers are replaced, ask whether or not the pupil can be sure of his conclusions. Suppose that you came back to this record after you had forgotten all about the checkers. Could you be sure where the checkers were?

Another line of questioning might be used. As soon as the four entries have been made in the record on the right and the checkers have been removed, ask: How many checkers were there altogether in that game? What number should we write in the box in the lower right-hand corner of the record of Game 2? (10) But don't expect an answer right away; it may not come out until the checkers have been replaced.

How many were there on the left side of the vertical line? How many were there above the line? How can you be sure of the answers without depending on your memory?

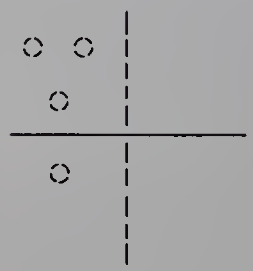
We are looking for an argument that runs something like this: the record shows that there were 6 checkers on the right side of the vertical line; it also shows that there were 2 in the lower right-hand corner; so there must have been 4 in the upper right-hand corner, because $4 + 2 = 6$ (or $6 - 2 = 4$ or $2 + 4 = 6$ or $6 - 4 = 2$).

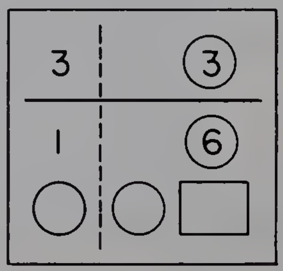
Further, if there were 4 in the upper right-hand section then there must have been 5 above the line, since the record shows that there was only 1 above the line and to the left of the vertical line, and $4 + 1 = 5$ (or $1 + 4 = 5$, etc.).

Such reasoning can support each conclusion about how many checkers there were in each section at the beginning of Game 2. When the checkers are replaced, the record can be completed, including the recording of the fact that there were 10 altogether.

Game 3

Succeeding games can begin with a partial record of a game. Now memory won't help a bit. The record tells us where some of the checkers go:





How many must there have been in the upper right-hand corner? None, because the record shows that there were only 3 above the line and that they were all in the upper left-hand corner, so there must have been none in the upper right-hand corner.

How many were there in the lower right-hand corner? 5, because the record shows that there were 6 below the line and that only 1 was on the left of the vertical line, so there must have been 5 in the lower right-hand corner.

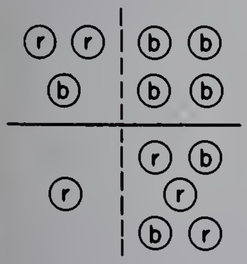
So there were 4 on the left side of the vertical line and there must have been 5 on the right side. Finally, there must have been 9 altogether. (See "Reconstruction Exercises" in Notes on Pupil Page 26.)

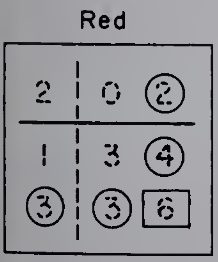
More Challenging Games

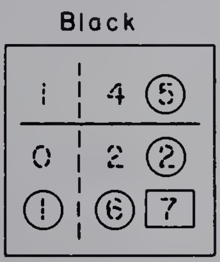
The opportunities of asking less-than-obvious questions in these games are many and they range across almost every level of difficulty.

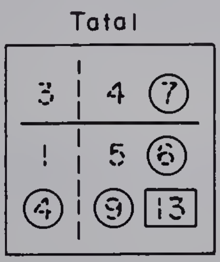
The game changes as we consider not only the number of checkers but also their color—red or black. In the examples to follow, *r* and *b* will indicate the colors of the checkers. (This is direct preparation for pupil page 18.)

Three separate records are kept—one of the red checkers, one of the black checkers, and still another to show the total, regardless of color.







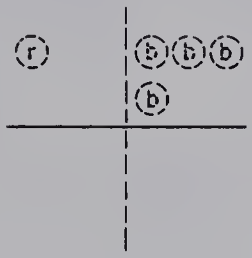


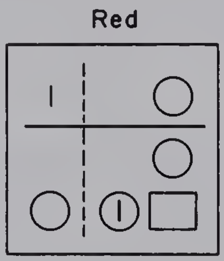
How many red checkers in the upper left-hand corner? (2) How many red ones in the upper right-hand corner? (0)

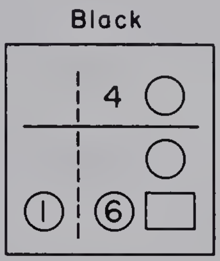
As each question is asked, the result is noted in the proper place. There are 27 pieces of information to record. When this is completed, remove the checkers.

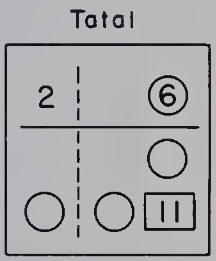
Can we put the checkers back as they were, so that each section has as many red and as many black checkers as it had at the beginning? Yes, by referring to the complete records we kept.

Later, consider situations in which we start with only a partly completed record. Here is an advanced example:









How many red checkers are in the upper right-hand corner? None, because in the total record we can see that there was a total of 6 checkers above the line and 2 were in the upper left-hand corner, so there must have been a total of 4 checkers in the upper right-hand corner. The record shows that all 4 of these were black checkers, so there must have been no red checkers in the upper right-hand corner.

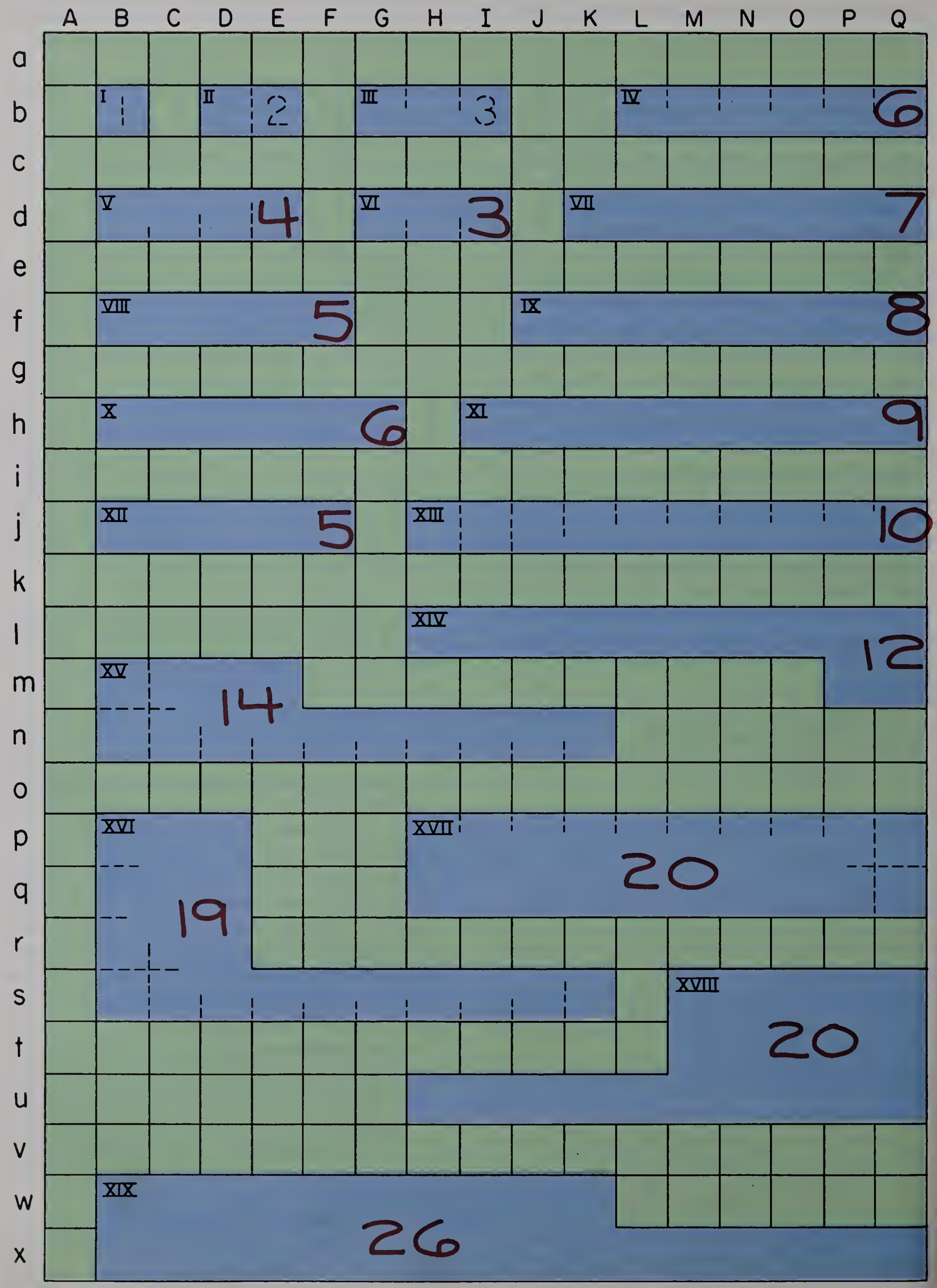
Similar reasoning can be used to determine the number of checkers of each color there are in each of the regions.

Pupils can advance at different rates to problems requiring this much reasoning. They should not be pushed beyond the point where they can succeed with a minimum of help. A feeling of achievement, of having solved a challenging problem, will spark more intensive and extended efforts. Reasoning power is used at every step of the way.

As each example unfolds, pupils are using the addition facts that only understanding and practice will fix in the memory for immediate recall. The connection between the real world and the abstractions of arithmetic is maintained—pupils can always count the checkers when their memory does not provide the answers.

So long as the gamelike spirit is maintained, motivation continues at a high level, and the mind retains new facts more easily.

How many green tiles does each blue card cover?


A B C D E F G H I J K L M N O P Q
a b c d e f g h i j k l m n o p q r s t u v w x
I 1
II 2
III 3
IV 6
V 4
VI 3
VII 7
VIII 5
IX 8
X 6
XI 9
XII 5
XIII 10
XIV 12
XV 14
XVI 19
XVII 20
XVIII 20
XIX 26

Notes on Pupil Page 19

Tasks: *Pupils find and record the number of green tiles that are covered by each of the blue cards.*

Purpose: To give pupils practice in (a) developing strategies for eliminating counting; (b) writing number sentences (equations and inequations); (c) using place value in writing two-digit numbers; (d) finding areas.

Unifying Ideas: Number and counting; numeration; geometry; measurement; functions and relations.

Materials for Teacher: Rectangular strips; graph board; colored chalk.

Materials for Pupils: Graph paper; crayons.

The Lesson:

The principal activity for this page (as for pupil page 2) is to develop strategies for eliminating counting. Once pupils have counted the tiles covered by the blue cards in row *b*, there should be no need to count in order to determine the numbers of tiles covered by the blue cards in row *d*. For example, since card VII is 1 tile longer than card IV and $6 + 1 = 7$, it follows that card VII covers 7 tiles. Similar strategies should be used for rows *f*, *h*, and *j*. The remaining blue cards should be "measured" by noting that each of them covers one or more strips of 10 tiles. The notion of place value makes it easy to give the tile counts for these cards.

There are many interesting questions that could be asked when pupils complete this page. Here are a few such questions (write designations like *row a* and *column D* on the board so that pupils do not confuse capital letters with small letters):

How many green tiles are in row *a*? (17)

How many green tiles are covered in row *b*? $(1 + 2 + 3 + 6 = 12)$

How many green tiles are shown in row *b*? $(17 - 12 = 5)$

How many green tiles are in row *c*? (17; notice how many pupils count despite the fact that they already counted when they answered your question about row *a*.)

Who can *guess* how many green tiles are shown in row *e*? In row *g*? In row *i*? In row *k*?

How many green tiles are shown in row *p*? (4)

Guess how many green tiles are shown in row *q*. In row *r*.

Find the square that is in row *d* and in column E. Is it blue?

As a follow-up to the place value concept distribute sheets of graph paper and ask pupils to mark out and color shapes (like those on pupil page 19) that cover given numbers of squares. Use a sequence of numbers like 3, 7, 10, 12, 15, 20, 24, 35, 47. Require that no shape be longer than 10 squares.

Require that no shape be longer than 10 squares.

Special Notation

After pupils have completed the page, have them prepare as many true statements as they can based on combining and comparing the blue cards. In doing this, pupils should use the notation N(. . .) to designate the number of tiles covered by a blue card. For example, N(I) = 1, N(II) = 2, N(IV) = 6, and N (XIX) = 26. Read N (. . .) as the number of tiles covered by card . . . .

$$\begin{array}{cc} N(I)+N(II)=N(III) & N(III)-N(II)=N(I) \\ 1 + 2 = 3 & 3 - 2 = 1 \end{array}$$

$$\begin{array}{cc} N(II)+N(I)=N(III) & N(III)-N(I)=N(II) \\ 2 + 1 = 3 & 3 - 1 = 2 \end{array}$$

$$\begin{array}{cc} N(I)<N(II) & N(II)>N(I) \\ 1 < 2 & 2 > 1 \end{array}$$

$$\begin{array}{c} N(V)+N(VI)=N(II)+N(VIII) \\ 4 + 3 = 2 + 5 \end{array}$$

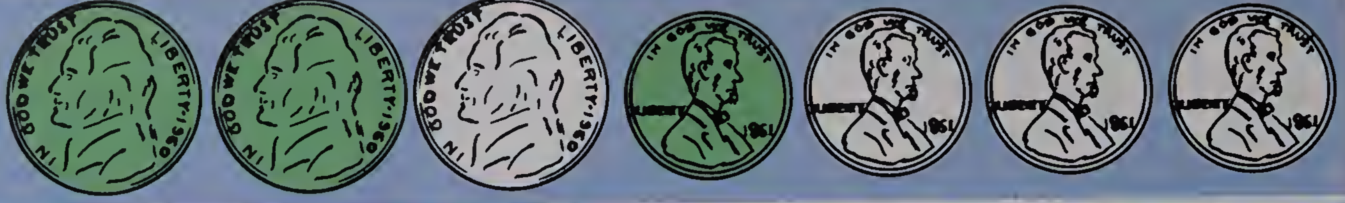
$$\begin{array}{c} N(II)+N(III)<N(V)+N(III) \\ 2 + 3 < 4 + 3 \end{array}$$

Headline for the Day: If each green tile costs ___¢, how much would it cost to replace the tiles covered by each blue card? Set up a table:

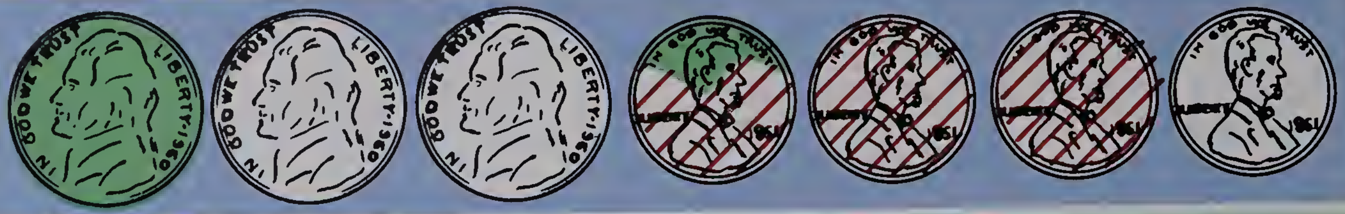
	I	II	III	IV	etc.
1¢	1	2	3	6	
2¢	2	4	6	12	
3¢					
etc.					

A.

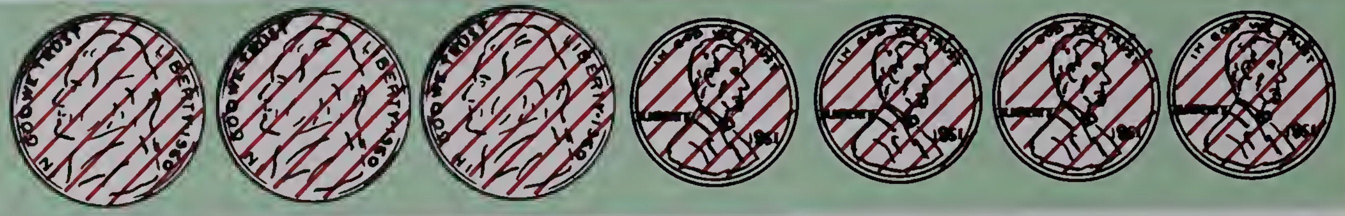
11 cents



8 cents

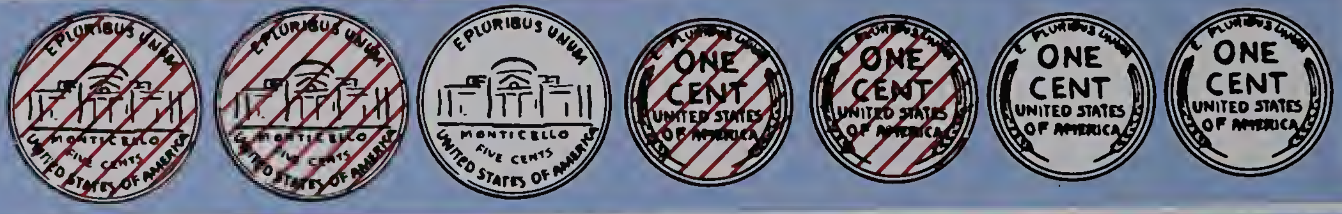


11 + 8 cents

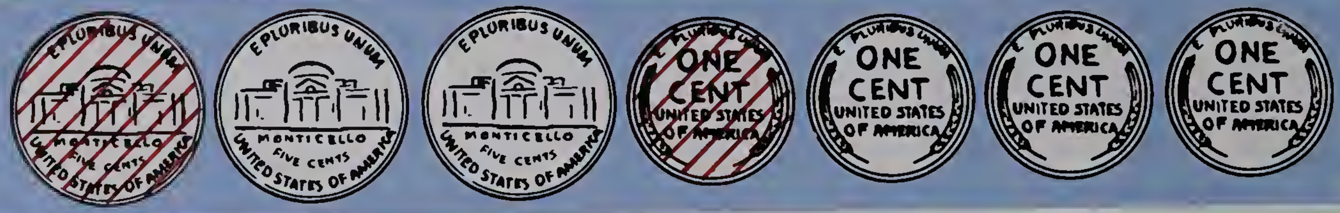


B.

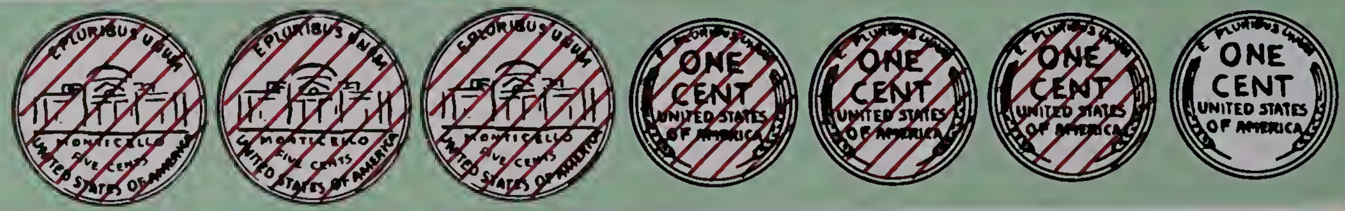
12 cents



6 cents

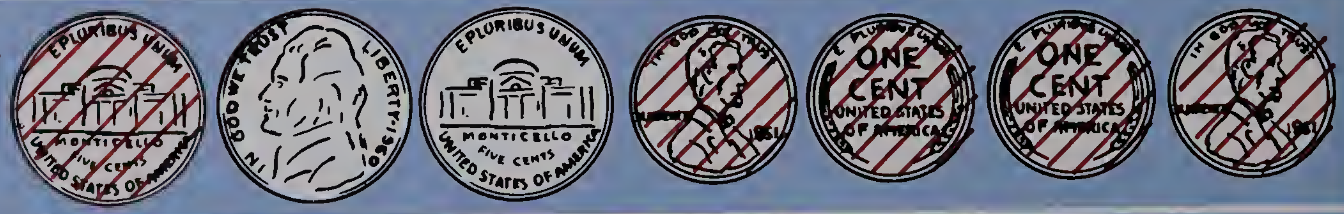


12 + 6 cents

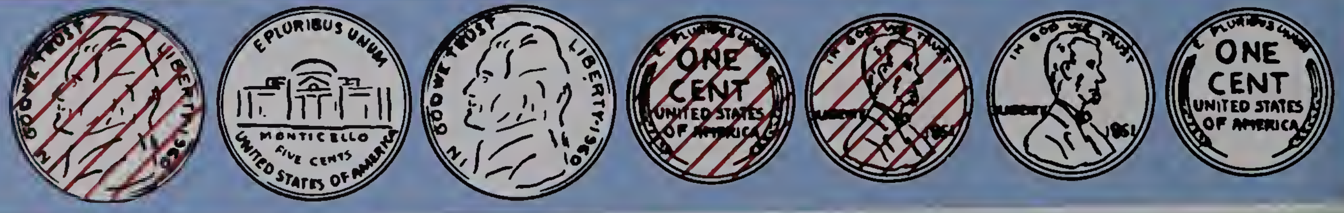


C.

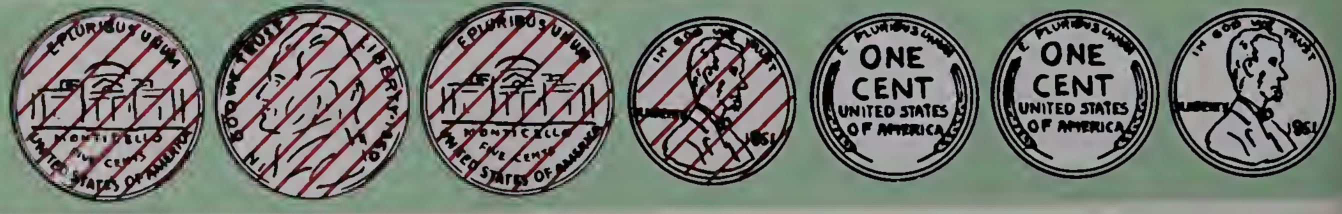
9 cents



7 cents



9 + 7 cents



Notes on Pupil Page 20

Tasks: *In each row of coins pupils shade those coins whose values add up to the value listed at the beginning of the row.*

Purpose: (a) To provide pupils with practice in working with coins; (b) to anticipate the exchanging step in the addition algorithm.

Unifying Ideas: Numeration; addition and subtraction; measurement.

Materials for Teacher: Seven demonstration cards, three to represent nickels and four to represent pennies; four pennies, three nickels, and a jar; Ditto masters.

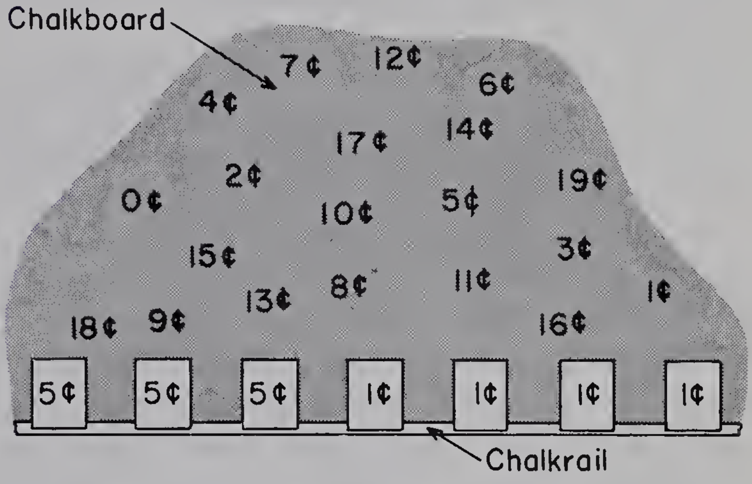
Materials for Pupils: Dittoed sheets.

The Lesson:

(In our discussion here we shall refer to pupil pages 20 and 21.)

Note that one each of pages 20 and 21 exercises A and B are straightforward combining or take-away situations. In exercise C the pupil is required to exchange 5 pennies for 1 nickel. It is a good idea to prepare pupils so that exercise C will be a summarizing activity.

Place the seven demonstration cards on the chalkrail and write in random fashion on the board amounts of money from 0¢ through 19¢:



Call upon a pupil to come to the board, circle an amount of money written on the board, and then re-

more

Headline for the Day: Tell a number story about money for each example on pupil page 20:

11 + 8 = ___
12 + 6 = ___
9 + 7 = ___

move cards from the chalkrail so that the remaining cards "add up" to the amount circled. Continue this activity until all of the amounts have been circled or until you are sure the pupils can handle these problems.

Call upon John to select cards to represent 11 cents. Ask Susan to give John some cards so that John will be holding cards to represent 8 more cents. Repeat this activity for sums such as (12 + 6)¢ and (7 + 7)¢.

Next, call upon Jim to select cards to represent 9 cents. Ask Ellie to give Jim some cards so that Jim will be holding cards to represent 7 more cents. Ellie will be forced to give Jim the two 5¢ cards and demand three 1¢ cards in "change." Repeat this activity for sums such as (8 + 7)¢ and (3 + 7)¢ which require exchanging.

The foregoing constitutes preparation for pupil page 20. Pupil page 21 can be prepared for in a similar manner. First, select a pupil to pick up cards to represent 18 cents. Then select a second pupil to take away cards for 6 cents. Repeat for "nonexchanging" differences such as (19 — 8)¢, (18 — 8)¢, and (12 — 7)¢. Then work with differences such as (16 — 7)¢ and (18 — 9)¢.

Another type of preparation for pupil pages 20 and 21 is the "fewest coins" game like the one in the "Headline for the Day" for pupil page 10.

A related activity is to ask the pupils to keep track of an increasing amount of money. Take 4 pennies and drop them, one at a time, into a jar. As you drop them, have the pupils draw tally marks. They say in unison: 1 penny (/), 2 pennies (//), 3 pennies (///), 4 pennies (////). Now, reach into the jar and replace the pennies with a nickel as the pupils say: 1 nickel (~~////~~). Continue with pennies: One more penny (~~////~~ /), two more pennies (~~////~~ //), three more pennies (~~////~~ ///), four more pennies (~~////~~ ////) —get ready to exchange—two nickels (~~////~~ ~~////~~).

You may wish to follow pupil pages 20 and 21 with the following activity.

In how many ways could we give the exact amount for a purchase that costs 4¢? (Only one—with 4 pennies). In how many ways could we give the exact amount for a purchase that costs 5¢ (Two ways—5 pennies or 1 nickel.) How many ways for a purchase that costs 12¢? (Four ways—12 pennies; 1 nickel and 7 pennies; 2 nickels and 2 pennies; 1 dime and 2 pennies.)

Apply the same basic question to this chart:

Purchase	Quarters	Dimes	Nickels	Pennies
6¢	0	0	0	6
6¢	0	0	1	1
12¢	0	0	0	12
12¢	0	0	1	7
12¢	0	0	2	2
12¢	0	1	0	2
19¢	0			
19¢	0			

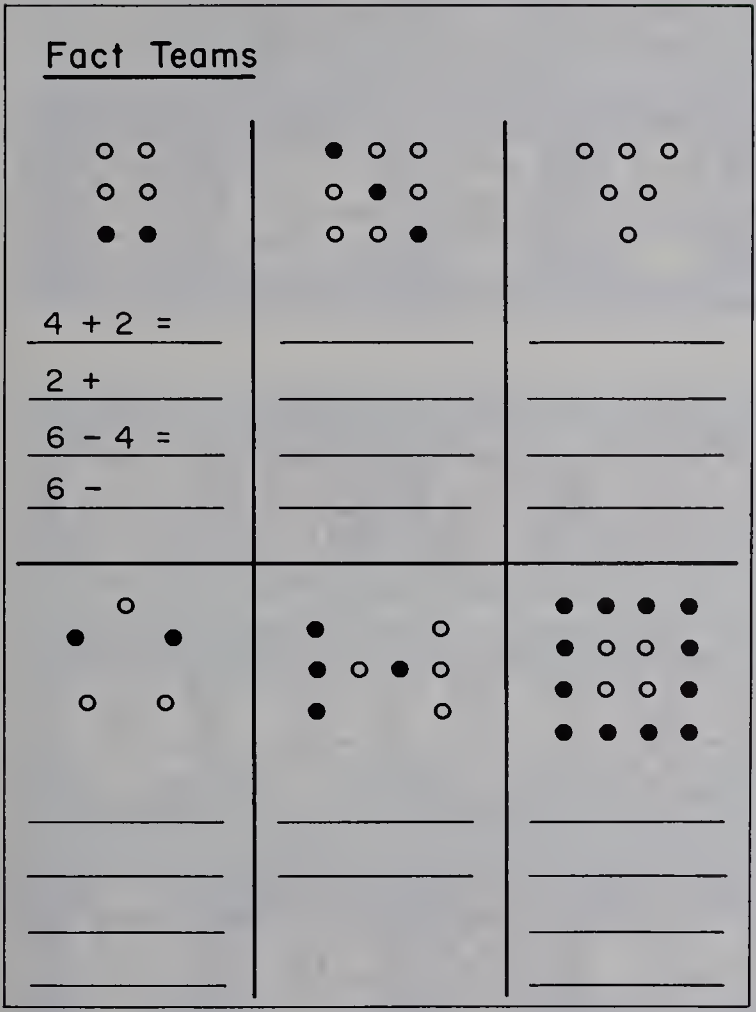
Another follow-up activity is to present the pupils with the following Dittoed sheet:

	Fewest coins				Amounts of money			
	25¢	5¢	1¢	Total	Quarters	Nickels	Pennies	Totol
a.	1	1	1	3	25¢	5¢	1¢	31¢
b.	1	2	3	6	25¢	10¢	3¢	38¢
c.	0	3	1	4	0¢	15¢	1¢	16¢
d.	0	2	4	6	0¢	10¢	4¢	14¢
e.	2	2	4	8	50¢	10¢	4¢	64¢
f.	2	2	2					
g.			0		50¢	20¢		
h.								62¢
i.								49¢
j.				5	25¢		3¢	
k.		3	2		25¢			
l.	2	4		8				
m.								93¢
n.				9		20¢	2¢	

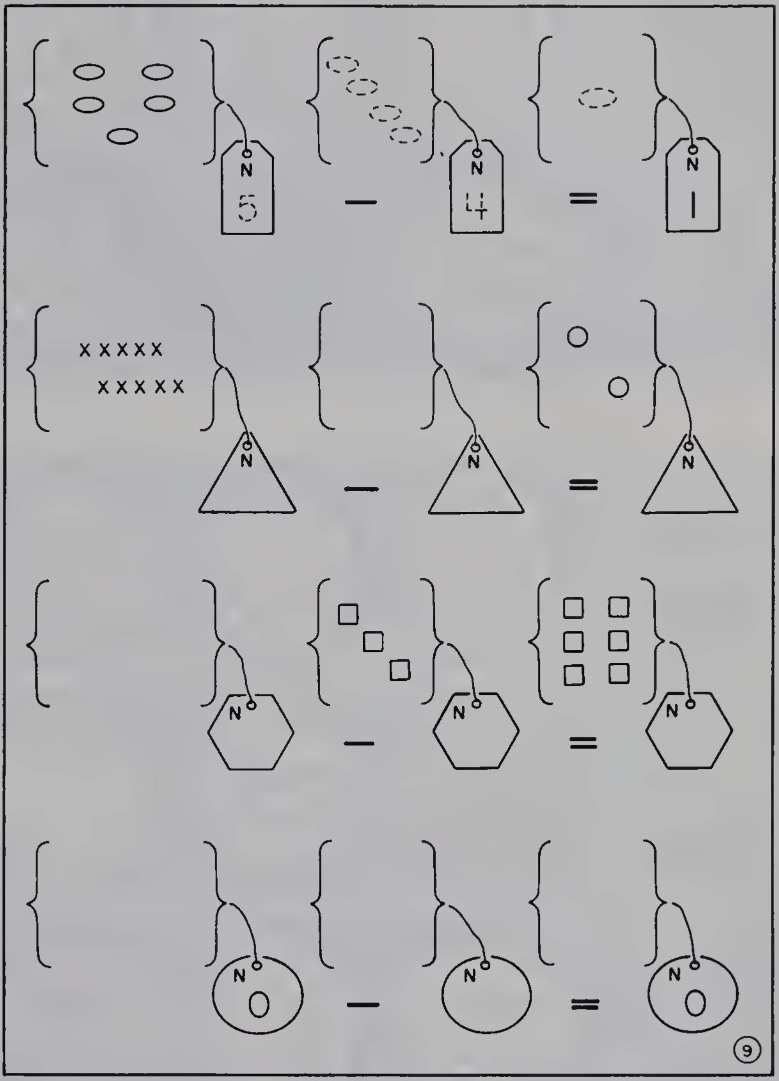
In example *a*, you are told that, given quarters, nickels, and pennies to choose from, the least number of coins needed to make 31¢ is 3—1 quarter, 1 nickel, and 1 penny.

Drill

Here is a suggestion for a drill sheet:

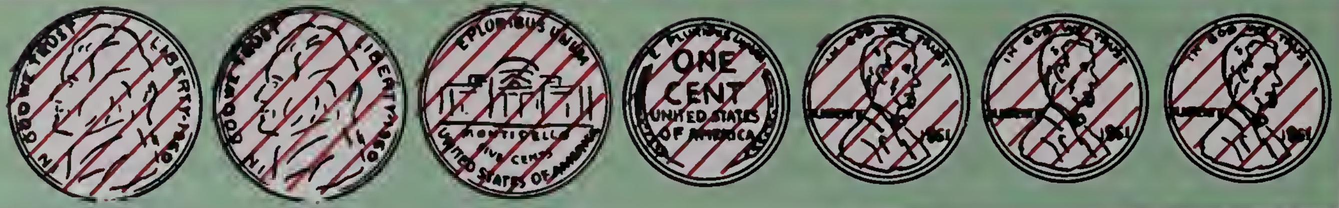

Fact Teams
4 + 2 =
2 +
6 − 4 =
6 −

You may also wish to consider as a drill sheet page 9 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:

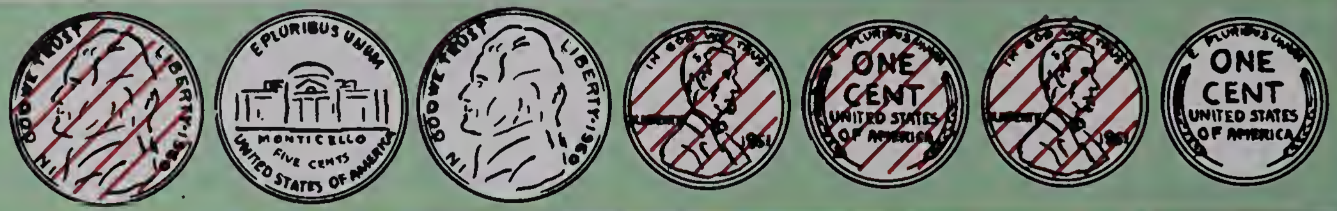

N
5
−
N
4
=
N
1
xxxxx
xxxxx
N
−
N
=
N
N
−
N
=
N
N
0
−
N
=
N
0
⑨

A.

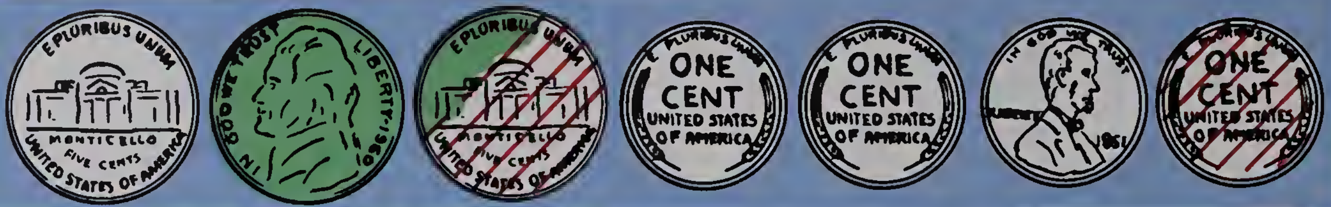
19 cents



8 cents

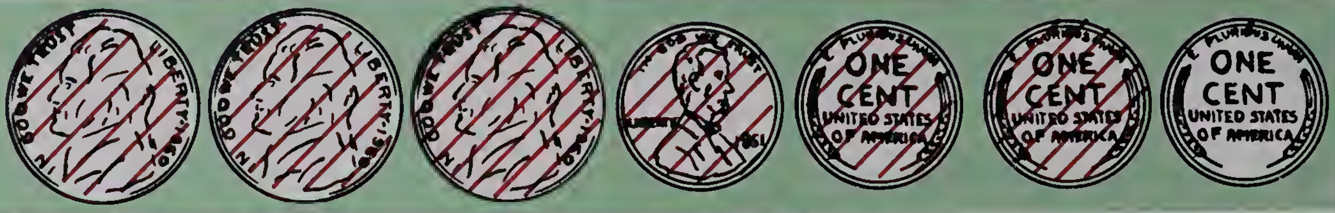


19-8 cents

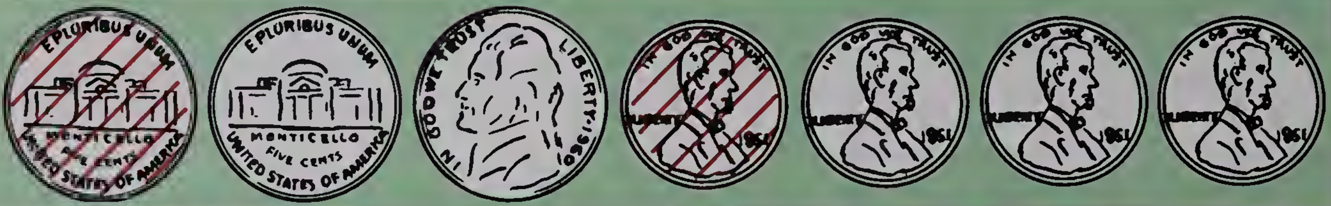


B.

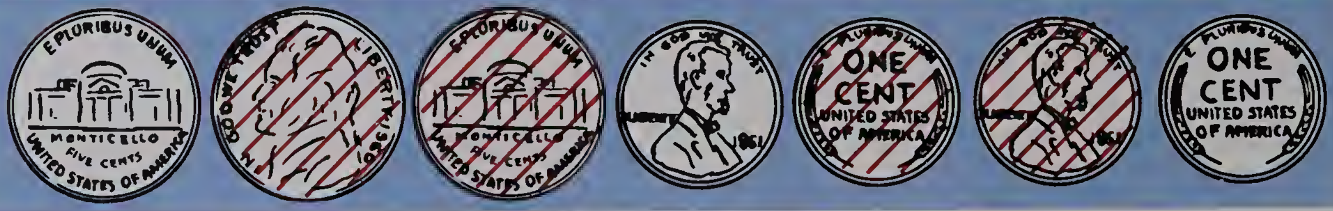
18 cents



6 cents

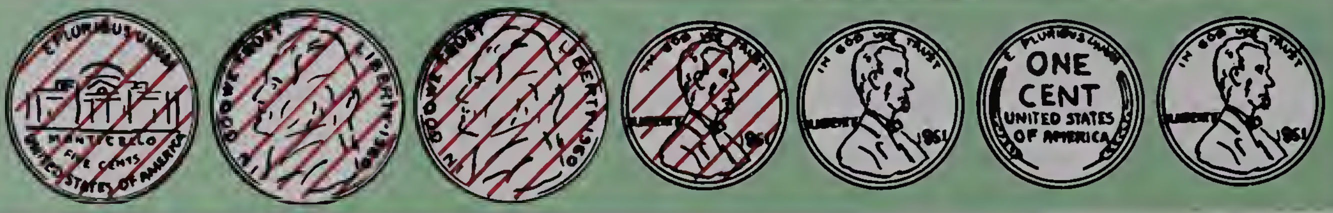


18-6 cents

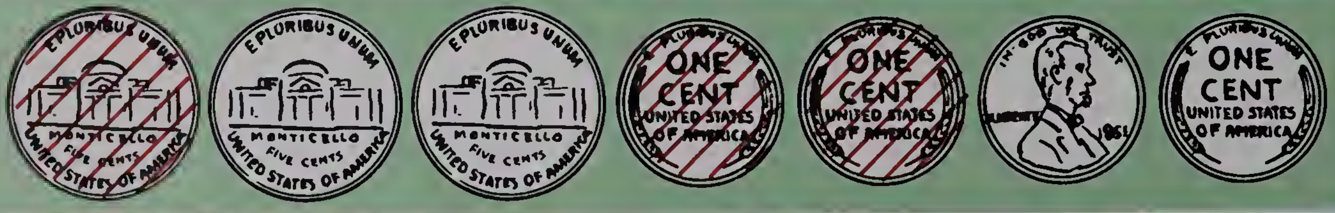


C.

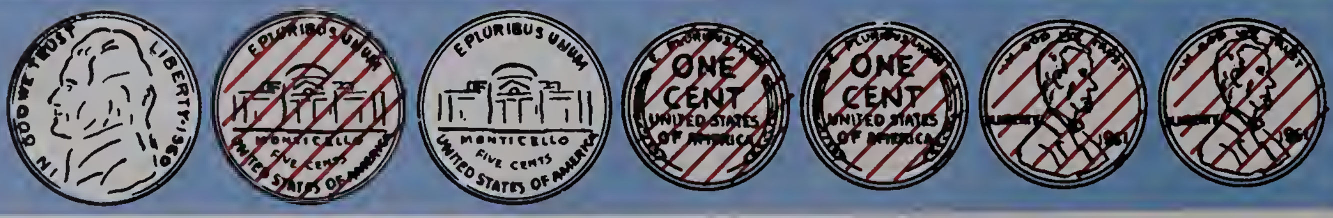
16 cents



7 cents



16-7 cents



Notes on Pupil Page 21

Tasks: *In each row of coins pupils shade those coins whose values add up to the value listed at the beginning of the row.*

Purpose: (a) To provide pupils with practice in working with coins; (b) to anticipate the exchanging step in the subtraction algorithm.

Unifying Ideas: Numeration; addition and subtraction; measurement.

Materials for Teacher: Seven demonstration cards, three to represent nickels and four to represent pennies; four pennies, three nickels, and a jar; Ditto masters.

Materials for Pupils: Dittoed sheets.

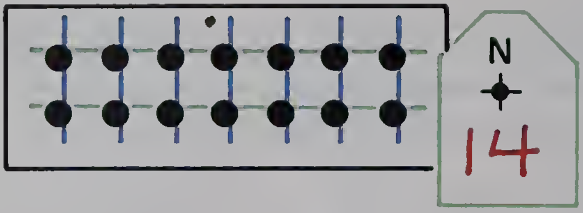
The Lesson:

See the Notes on Pupil Page 20.

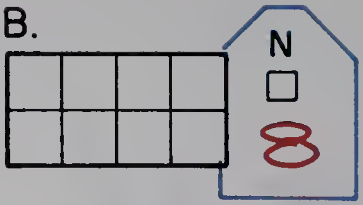
Headline for the Day: Tell a number story about money for each example on pupil page 21:

19 — 8 =____
18 — 6 =____
16 — 7 =____

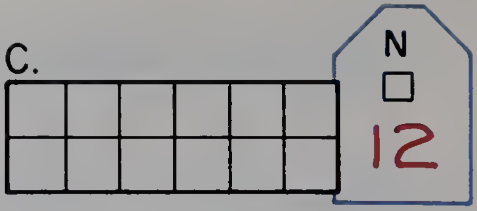
A.



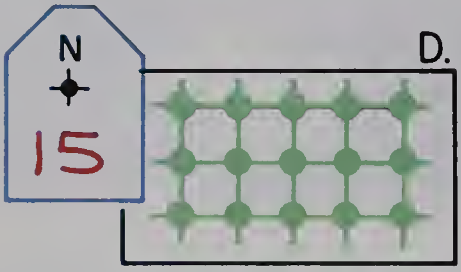
B.



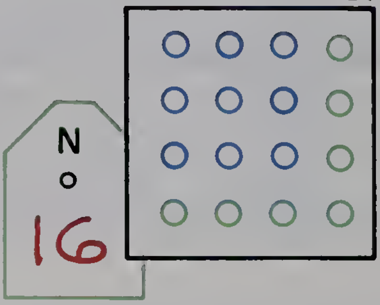
C.



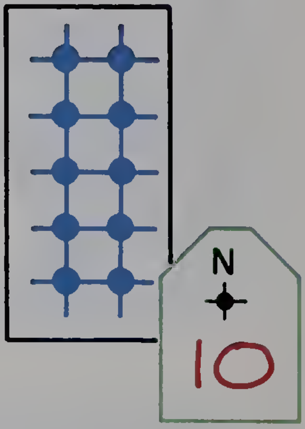
D.



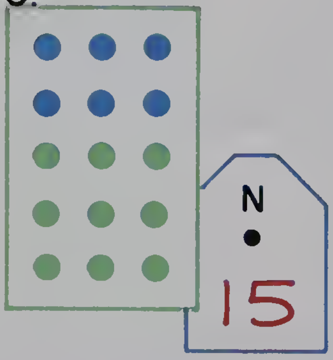
E.



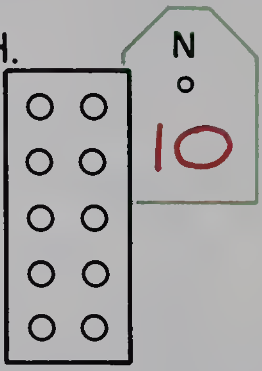
F.



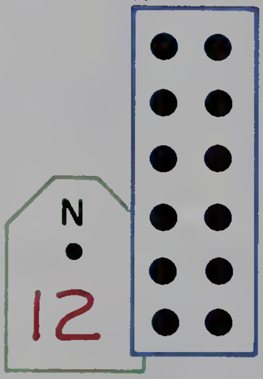
G.



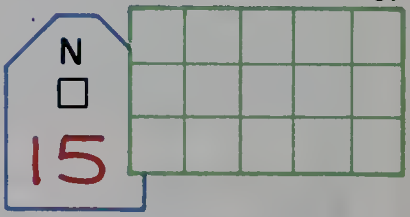
H.



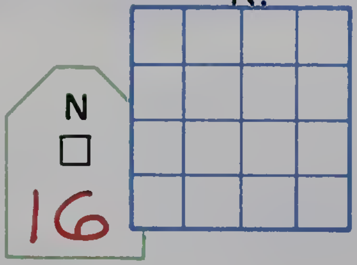
I.



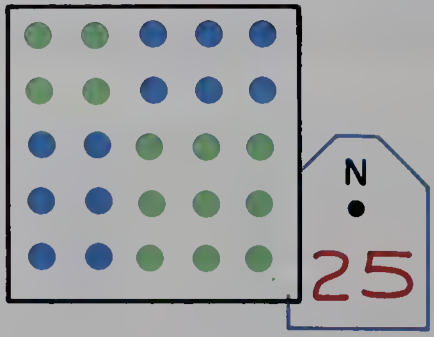
J.



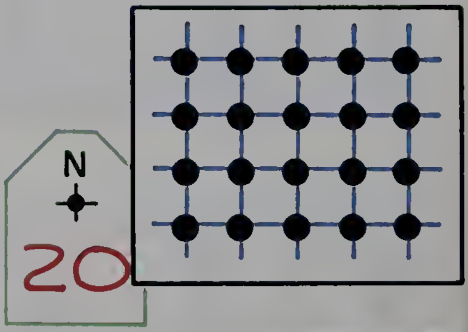
K.



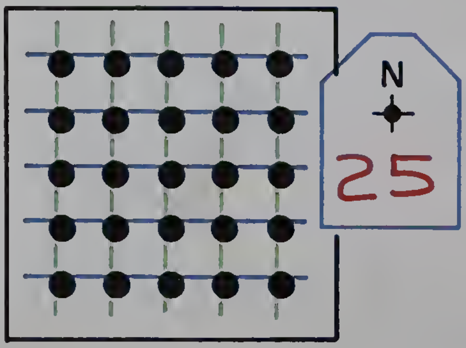
L.



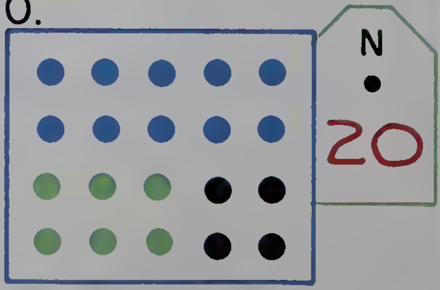
M.



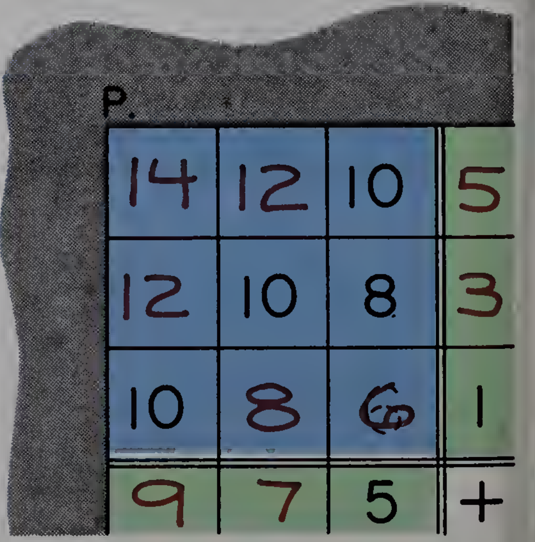
N.



O.



P.



14	12	10	5
12	10	8	3
10	8	6	1
9	7	5	+

Notes on Pupil Page 22

Tasks: *In exercises A-O pupils record the number of dots, little squares, or crossing points. In exercise P they reconstruct the addition table.*

Purpose: (a) To encourage pupils to use shortcuts for counting and to become acquainted with rectangular arrays in developing readiness for multiplication and division; (b) to give pupils practice in addition and subtraction and in the use of two-dimensional tables.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division.

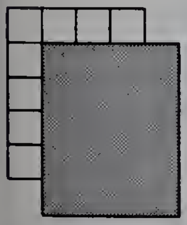
Materials for Teacher: Flash cards showing rectangular arrays.

Materials for Pupils: Graph paper.

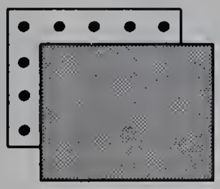
The Lesson:

It will be worth your while to make a set of flash cards showing the arrays on pupil pages 12 and 22. These cards can be used throughout the year. Pupils who can look at a rectangular array and quickly respond with the total are well on the way toward mastering the basic facts of multiplication and division. Set as a goal for Level B the mastery of rectangular arrays up to 5-by-5 arrays.

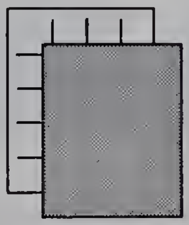
To strengthen the tie to multiplication, include practice with the flash cards. Mask the card so that only a pair of adjacent edges are exposed:



How many little squares altogether ?



How many dots altogether ?



How many crossing points altogether ?

Vary the masking procedure—sometimes expose the right and bottom edges, sometimes the left and bottom edges, etc.

Let pupils use graph paper to draw rectangular arrays. Use expressions like "a 4-by-5 rectangle" and "a 1-by-3 rectangle." For each drawing, the pupil should record the total number of small squares included in the array. Reverse the exercise by asking pupils to draw rectangular arrays to represent given numbers. Let pupils become aware of the class of numbers which can be represented only by "one-by" arrays. (Except for the number 1, these are the prime numbers. See page 21 of the booklet *Games and Enrichment Activities.*) Let them become aware of the numbers which are represented by "△-by-△" arrays—these are the square numbers.

Two-Dimensional Tables

Review the suggestions in the Notes on Pupil Page 12 with respect to the use of two-dimensional tables. Then consider using the following reconstruction exercises:

+	2	3	4	
1				6
2				
		6		
4				

+	10			
8	18			
6		18		
4			18	
2				18

+				
		8		7
3	3		7	
		7		
	7		11	8

Here are some harder ones to challenge your best pupils:

+			9	
				35
			35	
	58	35		
12	35			30

−		8		
	19	22		
		19		
36			19	
		15		19

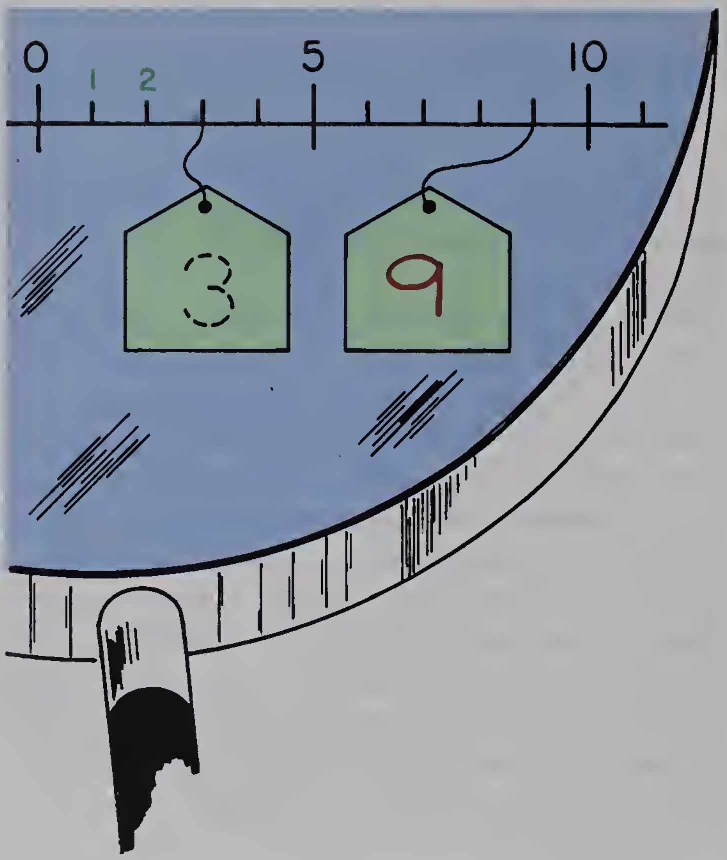
−		17		
21	13	4		
	22	13		11
34			13	
				13

Headline for the Day: Many things are arranged in rows and columns. Ask pupils to draw or find pictures of examples of such things. They might include such items as are found—

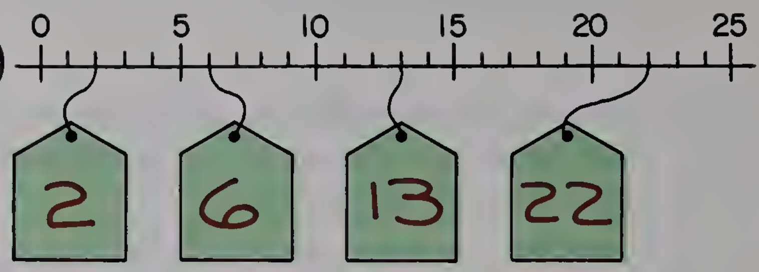
In a store:
an egg carton
a case of cola bottles
cans stacked on a shelf
a chocolate bar divided into squares

Around the home:
composition tile floors
patterns in a tablecloth
the squares on a checkerboard
small panes in a window

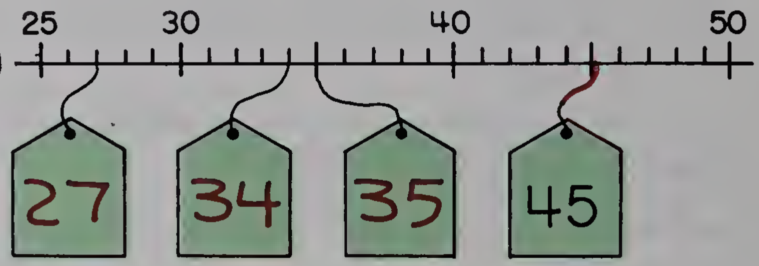
At school:
seats in some classrooms
seats in the auditorium
beads in counting frames
lining up "by twos" or partners



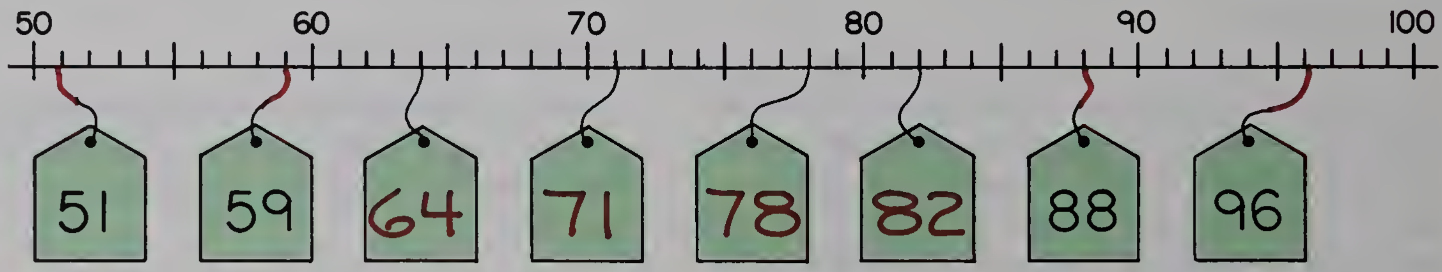
(a)


0
5
10
15
20
25
2
6
13
22

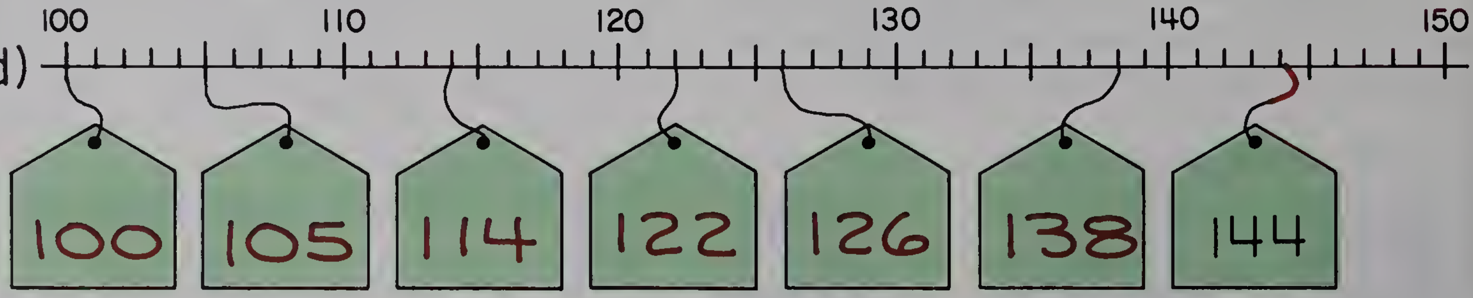
(b)


25
30
40
50
27
34
35
45

(c)


50
60
70
80
90
100
51
59
64
71
78
82
88
96

(d)


100
110
120
130
140
150
100
105
114
122
126
138
144

What's My Rule?

(e) 13, 15, 17, 19, 21, 23, 25, 27, 29, 31 . . .

(f) 26, 24, 22, 20, 18, 16, 14, 12, 10, 8 . . .

(g)

d	1	4	2	3	5	15	25	26	36	37	47
d + 5	6	9	7	8	10	20	30	31	41	42	52

Notes on Pupil Page 23

Tasks: *Pupils use the clue supplied by the left-hand tag under the magnifying glass to write a number on a tag which is attached to a point on the number line, or to "attach" a numbered tag to a point on the line. In the* What's My Rule? *exercises at the bottom of the page, pupils apply their proposed rules by filling in blanks.*

Purpose: (a) To review and extend pupils' understanding of order; (b) to help pupils gain insight into place value; (c) to provide pupils with practice in observing patterns; (d) to provide pupils with practice in adding and subtracting 2, and in adding and subtracting 5.

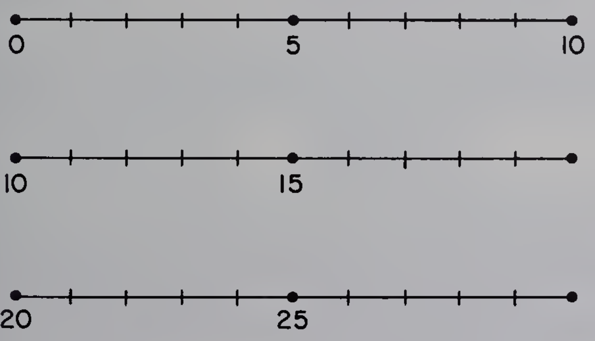
Unifying Ideas: Number and counting; numeration; addition and subtraction; functions and relations.

Materials for Teacher: Ditto master.

Materials for Pupils: Dittoed sheets.

The Lesson:

Preparation for this page might begin by exploring number line chunks in this form on the chalkboard or on a Dittoed sheet:



Ask pupils to come up to the board and find the points that correspond with 0, 5, 20, 1, 9, 14, 22, 28.

Have them show the jumps representing
2 + 3 = __, 12 + 3 = __, 22 + 3 = __.

Draw jumps and have pupils write the appropriate number statements.

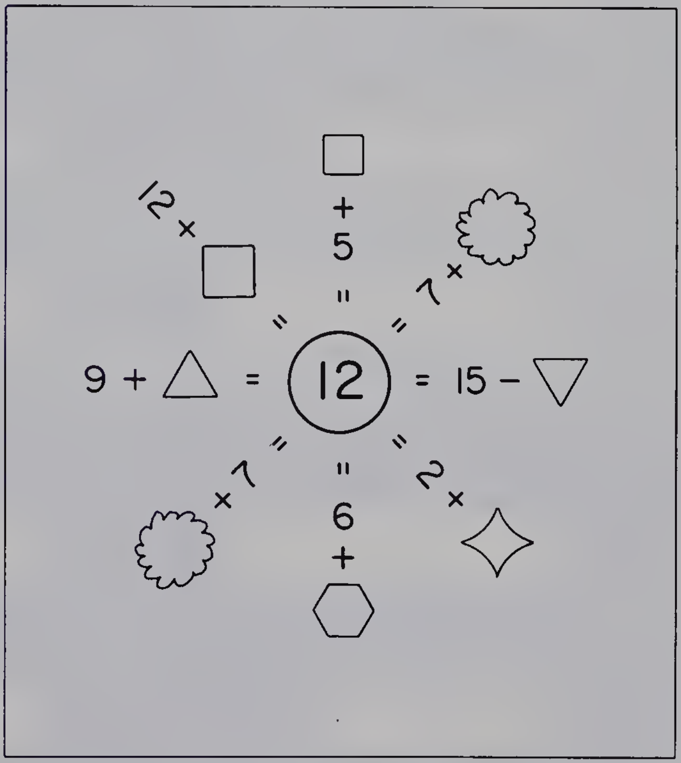
Next, draw on the board the number line chunk marked (c) on pupil page 23. Place the chalk on the point for 51, and ask for the number. Then say: Let's jump 10 to the right. Where do we land? Jump 10 to the right again, and ask for the landing point, etc. Start at 53 and make repeated jumps of 10 to the right. Repeat this activity with the number line chunk marked (d) on pupil page 23.

When pupils have completed pupil page 23, ask how big a jump must they make to go from 50 to 100. (They should be looking at pupil page 23.) How big a jump to go from 60 to 110? From 70 to 120? If you jump 50 from 80, where do you land? (Either at 130 or at 30.) From 90? From 100?

See the section "Number Line Chunks" on pages 18-19 in the booklet *Games and Enrichment Activities.*

Drill

Here is a suggestion for a drill sheet:



Headline for the Day: Theresa has 37 cents in her bank. She earns 10 cents a day by helping her mother. Finish the story and write a headline.

I

1 ☐
2 ☐
3 ☐
4 ☐

II

2 ☐
4 ☐
6 ☐
8 ☐

III

12 ☐
9 ☐
6 ☐
3 ☐

IV

10 ☐
5 ☐
15 ☐
20 ☐

V

10 ☐
20 ☐
30 ☐
40 ☐

VI

400 ☐
300 ☐
200 ☐
100 ☐

VII

16 ☐
12 ☐
8 ☐
4 ☐

VIII

18 ☐
6 ☐
12 ☐
24 ☐

IX

50 ☐
25 ☐
75 ☐
100 ☐

Notes on Pupil Page 24

Tasks: *Pupils write a number to the left of each white rectangular strip. These numbers are multiples or factors of the ones given.*

Purpose: (a) To give pupils practice in counting by 1, by 2, by 3, by 5, by 10, by 100, by 4, by 6, and by 25; (b) to anticipate multiplication.

Unifying Ideas: Number and counting; multiplication and division.

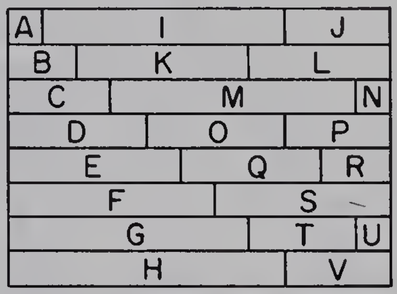
Materials for Teacher: Demonstration-size rectangular strips as described below; Ditto masters.

Materials for Pupils: Dittoed sheets; Cuisenaire rods; counters.

The Lesson:

One way in which pupils can handle these exercises is to pretend that the numbers to be written indicate the prices to be paid for tiling the strips with tiles the size of the smallest strip. For example, in exercise III notice that the smallest strip has a 3 printed next to it. This tells us to assume that a tile costs 3¢. So, it costs 3¢ to tile the smallest strip. Hence, since it takes 2 of the smallest strips to cover the next larger one, it must cost 6¢ to tile the latter.

To prepare pupils for this approach, we suggest the following activity. Use your demonstration set of rectangular strips to construct the pattern sketched below. (You may wish to provide pupils with a copy of it on a Dittoed sheet.)



If tiles the size of A cost 4¢ each, what would it cost to cover B? C? F? G? H?

Next, how much would it cost to cover J, L, N, P, R, U, and V?

Finally, how much would it cost to cover I, K, M, O, Q, S, and T?

Whenever there is any question about the number of tiles, have pupils come to the board and use blocks of cardboard cut to unit size to determine the

more

Headline for the Day: If it costs ____¢ for a tile to cover the smallest rectangle in exercise I, then it costs . . . . . ¢ to cover all four rectangles. Make a chart.

number used. Ask the class to decide on a way of organizing their information about the costs of tiles of different sizes. A chart similar to the following is most likely to result from their discussions:

No. of tiles	1	2	3	4
Cost	4¢	8¢	12¢	16¢

The class might also decide to make a chart recording the various sizes of the labeled areas:

No. of tiles	1	2	3
Sections covered	A, U, N	B, R	C, J, P, T, V

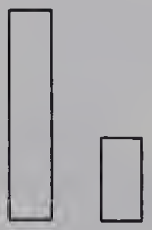
Ask again for the cost of covering sections picked at random. Pupils can use the diagram as before or can find the result from their charts.

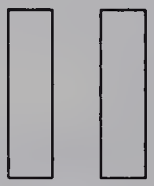
Remove the pattern of rectangular strips from the board but leave the charts on the board. Ask again for the cost of covering sections picked at random. Pupils must now rely solely on their charts for information. Vary the activity by assuming that tiles the size of A cost 5¢ each, or 25¢ each, or $1.00 each. Also vary the activity by having the pupils work in groups of two or three with each member designing his own pattern using Cuisenaire rods.

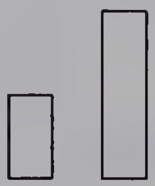
For another related activity with Cuisenaire rods, see "Establishing a Unit of Measure" on pages 40-41 of the booklet *Games and Enrichment Activities*.

Inequality Relations

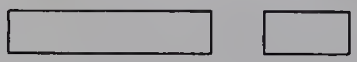
Use the flannel board, the chalkrail, or a table top (depending upon the kind of materials you are using for demonstration) to display pairs of rectangular strips, one pair at a time:



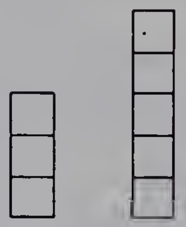




Each time, ask the pupils what sign they would use to express the relation between the lengths of the two objects. (You may find it necessary to review pupil page 17.) Then conduct a similar activity, displaying the strips horizontally:



Have each pupil use his counters or rods as bricks to build two small "houses" side by side on a piece of paper. Ask him to count the number of bricks he used to build each house and to write the number beneath it. Then have him write the appropriate relation sign between the numbers.

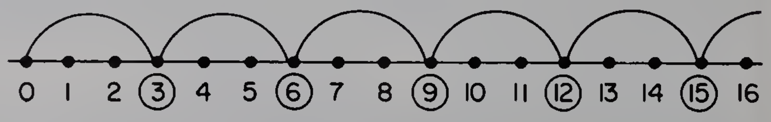


$3 < 5$

Vary this last activity by specifying one of the three relation signs and having the pupils build houses so that their brick counts will fit the sign.

Number Line Jumps

Here is a follow-up activity for pupil page 24. Draw a number line on the board. Pupils should have a copy of it on a worksheet. Starting at 0 on a number line and taking jumps of 3, what would our landing points be? Have pupils help you complete the picture on your demonstration number line:



Have the class decide on the method of keeping a record of landing points:

Landing points	3	6	9	12	etc.
No. of jumps	1	2	3	4	

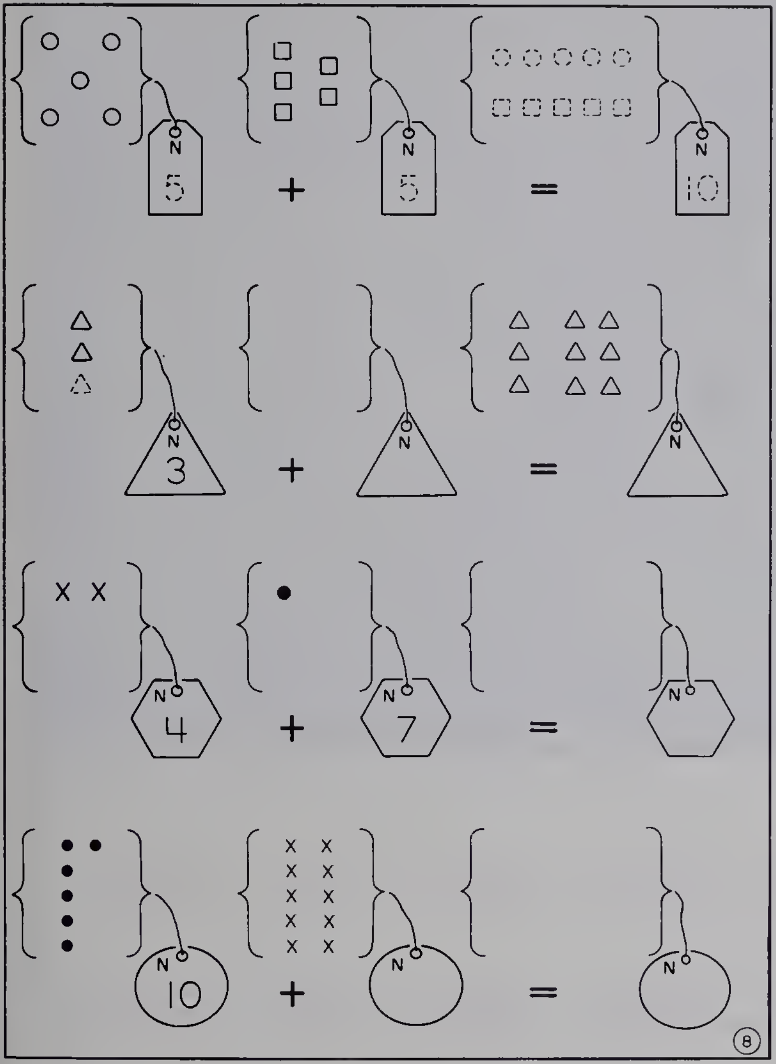
Erase the completed record, leaving just the number line record on display. Rearrange in random order the record just erased so that one number of each pair is missing:

Landing points	6			9	etc.
No. of jumps		1	4		

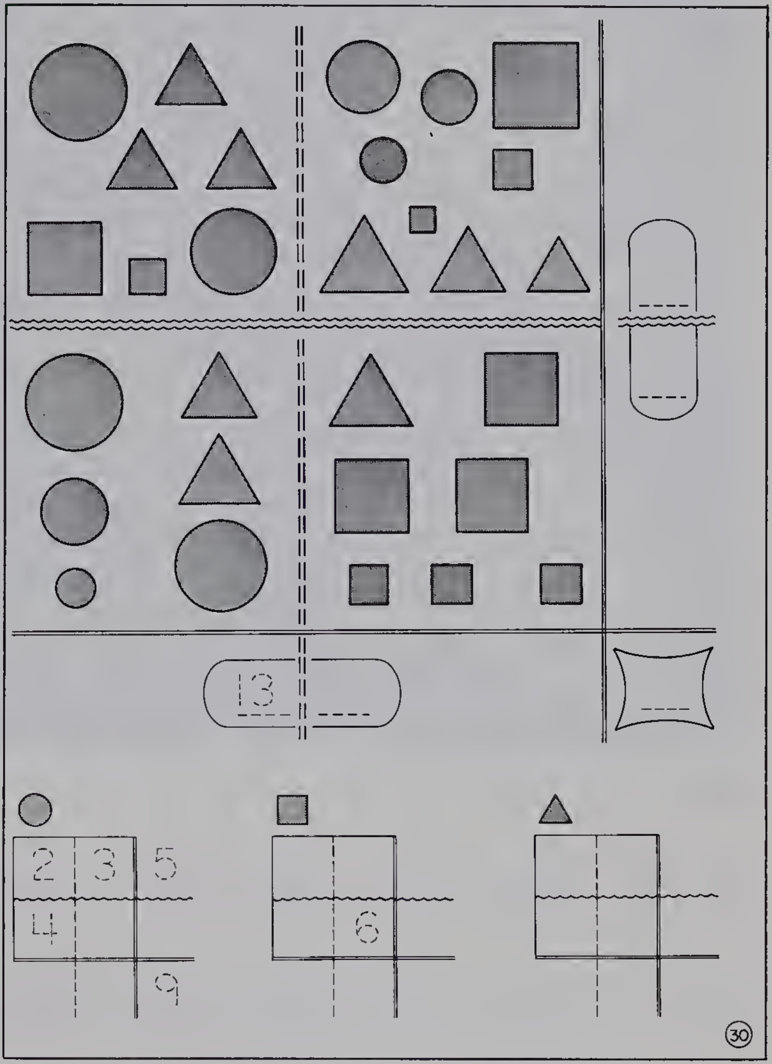
Invite pupils to complete each pair and to verify their results on the number line. Vary the activity by using jumps of 4 or of 6.

Drill

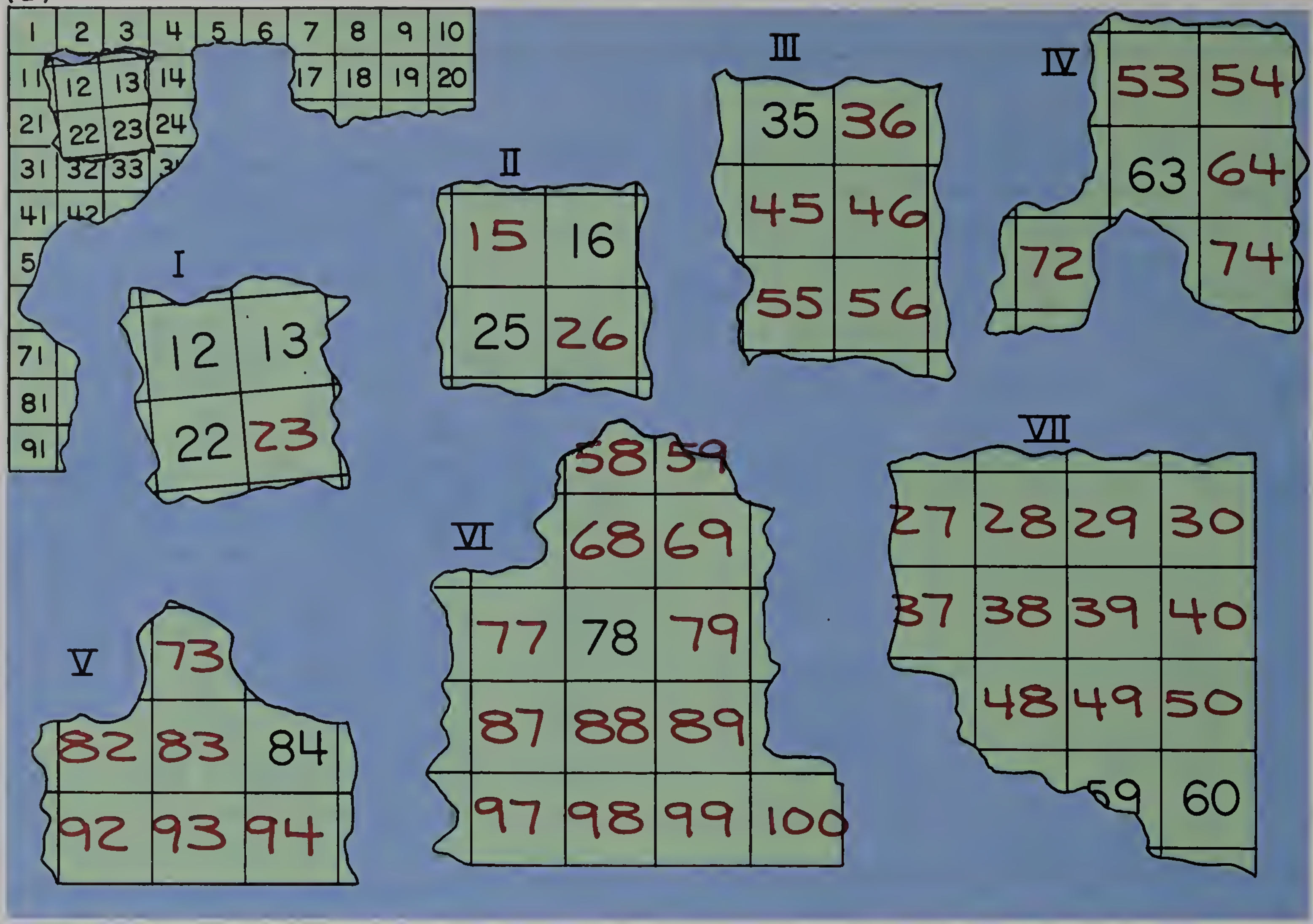
Consider using as a drill sheet page 8 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:


N
5
+
N
5
=
N
10
N
3
+
N
=
N
N
4
+
N
7
=
N
N
10
+
N
=
N
8

Also consider using as a drill sheet page 30 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:


13
2
3
5
4
9
6
30

(a)


1 2 3 4 5 6 7 8 9 10
11 12 13 14 17 18 19 20
21 22 23 24
31 32 33
41 42
71
81
91
I
12 13
22 23
II
15 16
25 26
III
35 36
45 46
55 56
IV
53 54
63 64
72 74
V
73
82 83 84
92 93 94
VI
58 59
68 69
77 78 79
87 88 89
97 98 99 100
VII
27 28 29 30
37 38 39 40
48 49 50
59 60

What's My Rule?

(b) 30, 40, 50, 60, 70, 80, 90, 100, 110, 120 ...

(c) 5, 10, 15, 20, 25, 30, 35, 40, 45, 50 ...

(d)

□	6	4	10	6	7	13	10	8	18	30	60
△	2	0	2	4	3	7	10	2	12	10	20
□ + △	8	4	12	10	10	20	20	10	30	40	80

(e)

m	12	15	10	13	23	30	10	40	20	11	31
n	9	4	0	8	8	10	8	18	10	9	19
m − n	3	11	10	5	15	20	2	22	10	2	12

Notes on Pupil Page 25

Tasks: *Pupils write the missing numbers in the torn-off sections of the original 10-by-10 rectangular grid of numbers from 1 through 100. For the* What's My Rule? *games, pupils demonstrate their discovery of the rules by supplying the missing numbers.*

Purpose: (a) To review and extend pupils' understanding of the decimal system of numeration; (b) to provide pupils with practice in addition and subtraction.

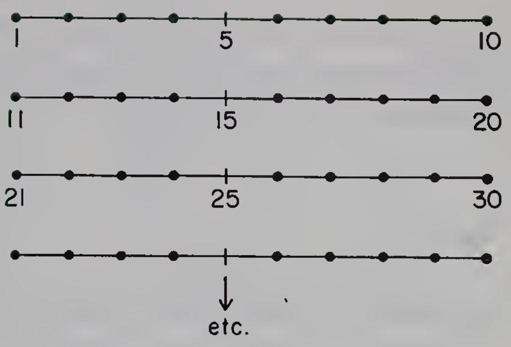
Unifying Ideas: Number and counting; numeration; addition and subtraction; functions and relations.

Materials for Teacher: Ditto master.

Materials for Pupils: Dittoed sheets.

The Lesson:

As preparation for exercise (a) on pupil page 25, we suggest the following activity. Sketch number line chunks on the board and label as indicated:



Begin the patterns for multiples of 2, 3, 4, 5, and 10 and have pupils point on their Dittoed sheets to each number in the sequence on the number line:

2, 4, 6, 8, 10, 12, etc.
3, 6, 9, 12, 15, etc.
4, 8, 12, 16, 20, etc.
5, 10, 15, 20, etc.
10, 20, 30, 40, etc.

Have pupils find points for sequences such as:

15, 16
25, 26
34, 35

2, 12, 22, 32, 42
8, 18, 28, 38, 48
9, 10, 19, 20, 29

more

Headline for the Day: Helen has 57 cents on Monday morning. If she spends 10 cents on Monday, 10 cents on Tuesday, 10 cents on Wednesday, 10 cents on Thursday, and 10 cents on Friday, how much does she have left?

Complete labeling the number line chunks and ask what patterns pupils notice. (There is a pattern in every column, row, and diagonal.)

What's My Rule?

As preparation for exercises (d) and (e), we suggest the following activity. Ask pupils to pick several pairs of numbers, each member of which is less than 10. Put these pairs in a column on the board. Draw several columns of boxes next to the column of numbers (see diagram below).

Ask pupils: Who can think of a secret rule that would make each pair of numbers suggest a third number? Put the first volunteer's name above a column—John, perhaps. Ask John: What number goes with the first pair? What number goes with the second pair? Record his answers in the appropriate boxes. Then ask pupils: Who thinks he knows John's secret rule? What number do you think John would give to go with the third pair? (Let John decide if a pupil's answer is right.) Who else knows John's secret rule?

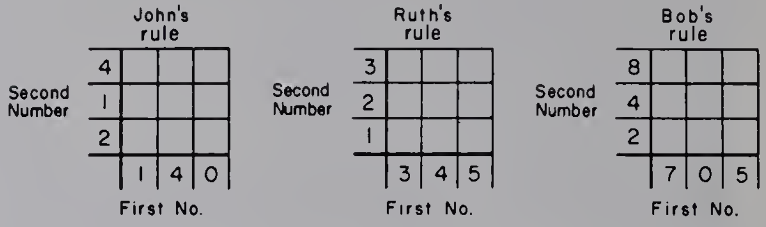
Then it's Ruth's turn. After Ruth has given the third numbers suggested by the first two pairs of numbers, according to her secret rule, ask: Who knows Ruth's rule? Is it the same as John's rule?

After several pupils have volunteered rules, the chart you have been making on the board will resemble the following:

	John	Ruth	Bob	Al	Joon	Debbie
1, 3 →	4	2	1	3	3	5
5, 0 →	5	5	5	5	0	6
3, 7 →	10	4	3	7	7	11
4, 2 →	6					
3, 3 →						
2, 1 →						
	(sum)	(diff.)	(first)	(larger)	(second)	(sum+1)

Verbalization should be kept to a minimum. It might be wise to limit all responses to Yes, No, and numerical answers.

Draw the following charts near the work already on the board. Let the inventor of each rule use any numbers he wishes for the bottom edge and for the left edge.



Have each rule-inventor stand by his chart. Pupils who think they know someone's rule should be invited to suggest how one of the empty boxes in the chart should be filled in. The first number of the pair will be the number written at the bottom of the column for the given box, and the second number of the pair will be the number written at the beginning of the row in which that box is located.

Fewest Questions

An interesting game which has proved to be very exciting with youngsters is *Fewest Questions.* Draw a number line chunk on the board listing the numbers from 1 through 32. Someone selects one of these 32 numbers and the job is to guess the selected number by asking questions which must be answered with either a Yes or a No.

Suppose that a pupil picks the number 9 and informs the class of his choice without letting you in on the secret. You ask:

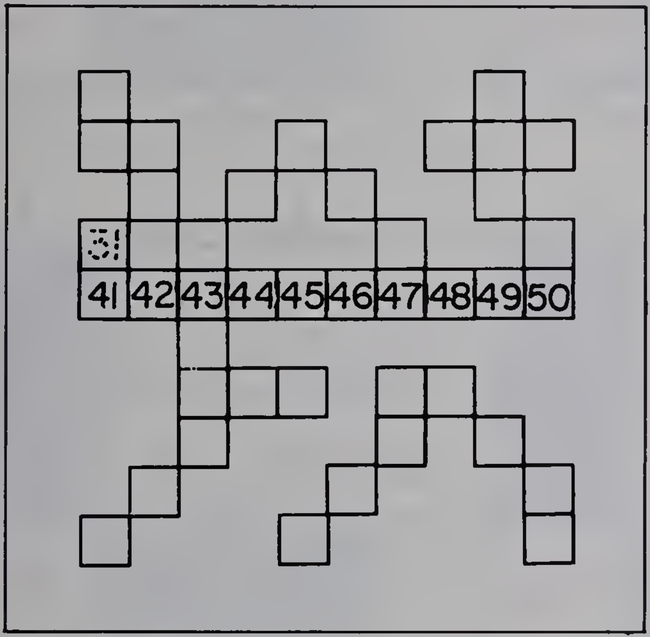
1) Is your number greater than 16?
   (The pupil says No and you erase the 17-32 section of the chunk.)
2) Is your number greater than 8?
   (Erase the 1-8 section when the pupil says Yes.)
3) Is your number greater than 12?
   (No. Erase the 13-16 section.)
4) Is your number greater than 10?
   (No. Erase the 11-12 section.)
5) Is your number 10? (The pupil says No and you say that it must be 9.)

This questioning strategy will guarantee singling out the pupil's choice with exactly five questions. How many questions will guarantee a correct guess for the numbers 1 through 16? 1 through 64? 1 through 8?

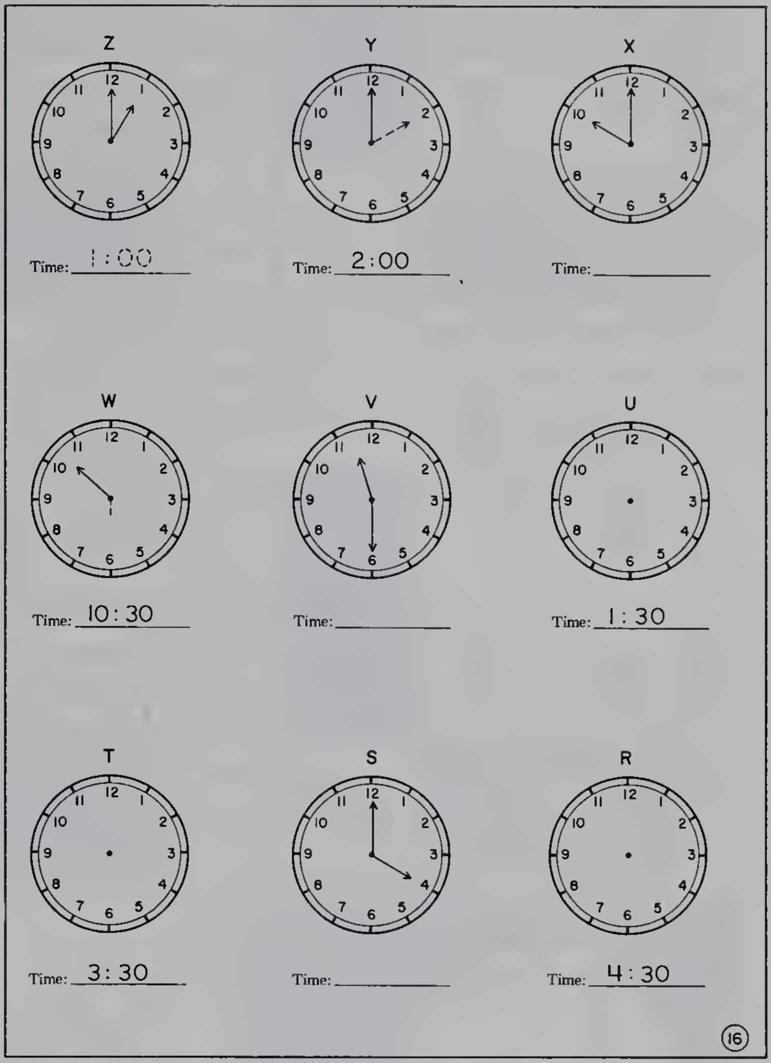
See "The Game of Fewest Questions" on pages 30 and 31 of the booklet *Games and Enrichment Activities.*

Drill

Here is a suggestion for a drill sheet that builds on some of the ideas developed on pupil page 25:


31
41
42
43
44
45
46
47
48
49
50

It may now be time for a drill sheet on clock reading. Consider using page 16 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:


Z
Time: 1:00
Y
Time: 2:00
X
Time:
W
Time: 10:30
V
Time:
U
Time: 1:30
T
Time: 3:30
S
Time:
R
Time: 4:30
⑯

A

1	2	3
4	3	7
5	5	10

B

1	3	4
4	2	6
5	5	10

C

1	2	3
3	4	7
4	6	10

D

0	2	2
6	4	10
6	6	12

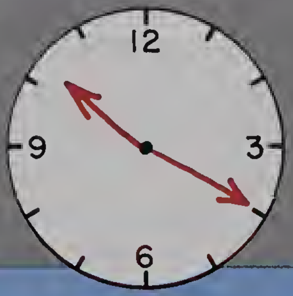
E

1	3	4
7	5	12
8	8	16

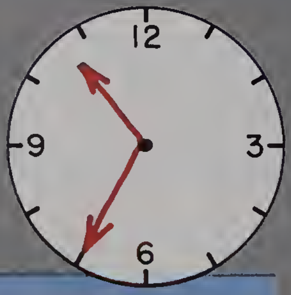
F

2	4	6
8	6	14
10	10	20

K



10:20 .... Time for a 15-minute recess



10:35 ... End of recess

G

10	7	17
0	3	3
10	10	20

H

4	2	6
6	9	15
10	11	21

I

0	0	0
0	0	0
0	0	0

J

3	5	8
0	0	0
3	5	8

Notes on Pupil Page 26

Tasks: *Pupils complete each* Cross Number Puzzle *either by supplying the missing sums or by reconstructing puzzles based on the sums given. For exercise K pupils draw the hour and minute hands to show the time indicated and write the required time statement.*

Purpose: (a) To provide pupils with practice in addition and subtraction; (b) to extend pupils' knowledge of the measurement of time.

Unifying Ideas: Structure; addition and subtraction; measurement.

Materials for Teacher: Demonstration clockface with hour and minute hands; Ditto masters.

Materials for Pupils: Dittoed sheets containing clockfaces on which to draw the hour and minute hands; Dittoed sheets with *Cross Number Puzzles* as suggested below.

The Lesson:

We suggest that you review the discussion of *Cross Number Puzzles* in the Notes on Pupil Page 18. Note that exercises A, B, and C have the same four addends. Of course the total for each puzzle will be the same. Will this be a surprise to some pupils? Let them test other similar cases to see if it will continue to happen.

Note that in each of exercises D, E, and F, the four addends are either all even or all odd. Hence in each exercise the four sums and the totals are even.

Note, also, that the sums of the first and second columns are the same for each puzzle in exercises D-G. Can a pupil make up a new puzzle with this property?

The puzzle in exercise I presents an interesting challenge.

Reconstruction Exercises

In reconstruction exercises such as G and H, the pupil must decide when to subtract and when to add. These opportunities for independent decisions are important in developing problem-solving skills. The fact that both addition and subtraction are involved in this activity reinforces the pupil's awareness that adding a number and subtracting that number are inverses of each other. In addition we are given two addends and we determine the sum. In subtraction

more

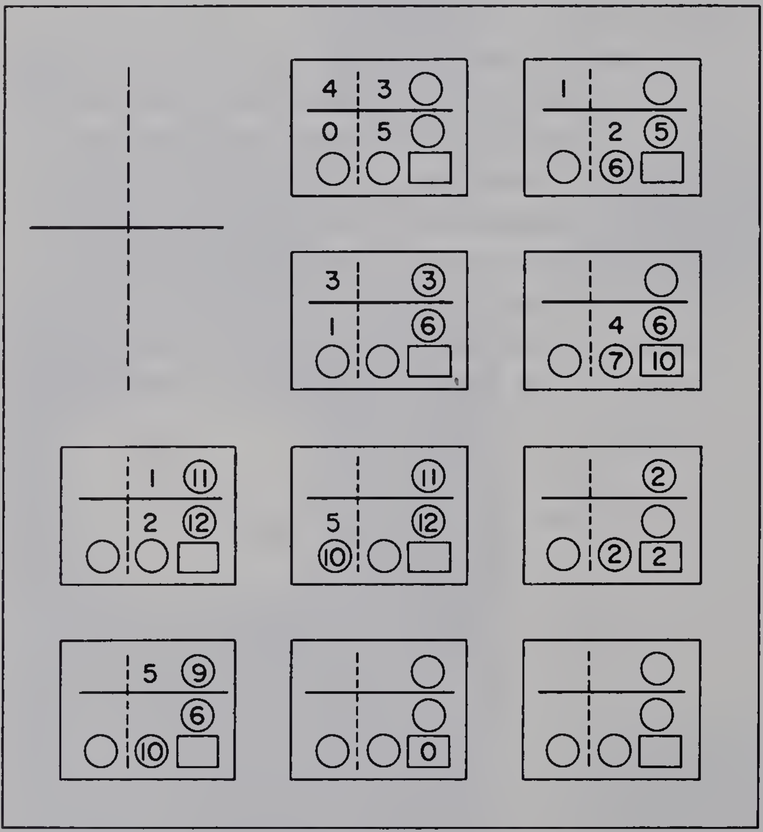
Headline for the Day: One pupil states an addition fact and another pupil must use it as a headline for a time story.

we are given the sum and one addend and we must determine the missing addend:

Addend		Addend		Sum	
3	+	7	=	—	Addition
3	+	—	=	10	Subtraction
—	+	7	=	10	Subtraction

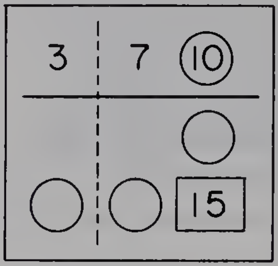
Other Games

Here is a summary of the games that were discussed in the Notes on Pupil Page 18, as well as others:



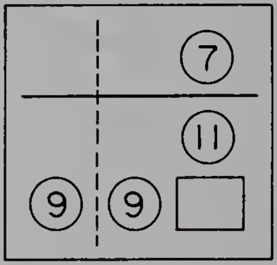
How much information is needed for reconstruction? Do you need at least four entries in the record before you can be sure what numbers were in each of the regions? Are three enough? Generally, four are required, though in special cases fewer are sufficient. An extreme example is that it is sufficient to know merely that 0 is recorded in the bottom right-hand box.

Are any four entries sufficient? No. Each of the four entries must give new information. If the record tells that 3 is in the upper left-hand corner and 7 is in the upper right-hand corner then for the record to show that 10 is the sum above the line yields no new information. So the following record doesn't tell enough, even though there are four entries:

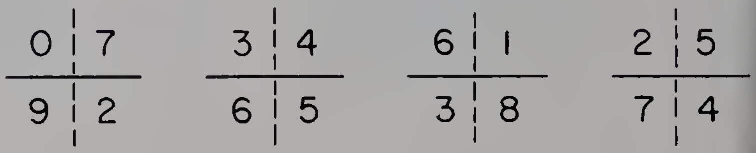


You can find out that the sum of the numbers below the line is 5, but there are six different ways in which 2 numbers that have 5 as their sum might be distributed between the two regions, since $0 + 5 = 5$, $1 + 4 = 5$, $2 + 3 = 5$, $3 + 2 = 5$, $4 + 1 = 5$, $5 + 0 = 5$.

If you have four entries in the circles, showing the numbers above and below the line and the numbers to the right and left of the line, is this enough information? No, except in special cases. Consider this example:



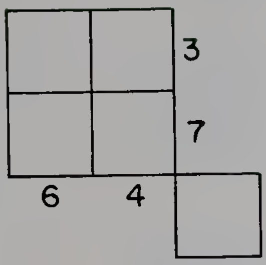
Some of the possible arrangements are:



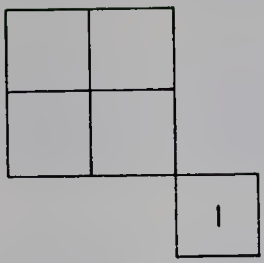
In fact, there are eight possible arrangements. Notice that each arrangement in which the sum above the line is 7 determines the arrangement of the numbers below the line. Or, more specifically, the number in the upper left-hand region (or upper right-hand region) determines the rest, and there can be any of the numbers 0 through 7 there—eight possible arrangements.

Other interesting explorations of *Cross Number Puzzles* include the following:

(1)

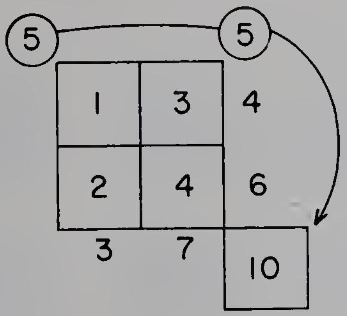


(2)



In how many ways can each of the above puzzles be completed correctly?

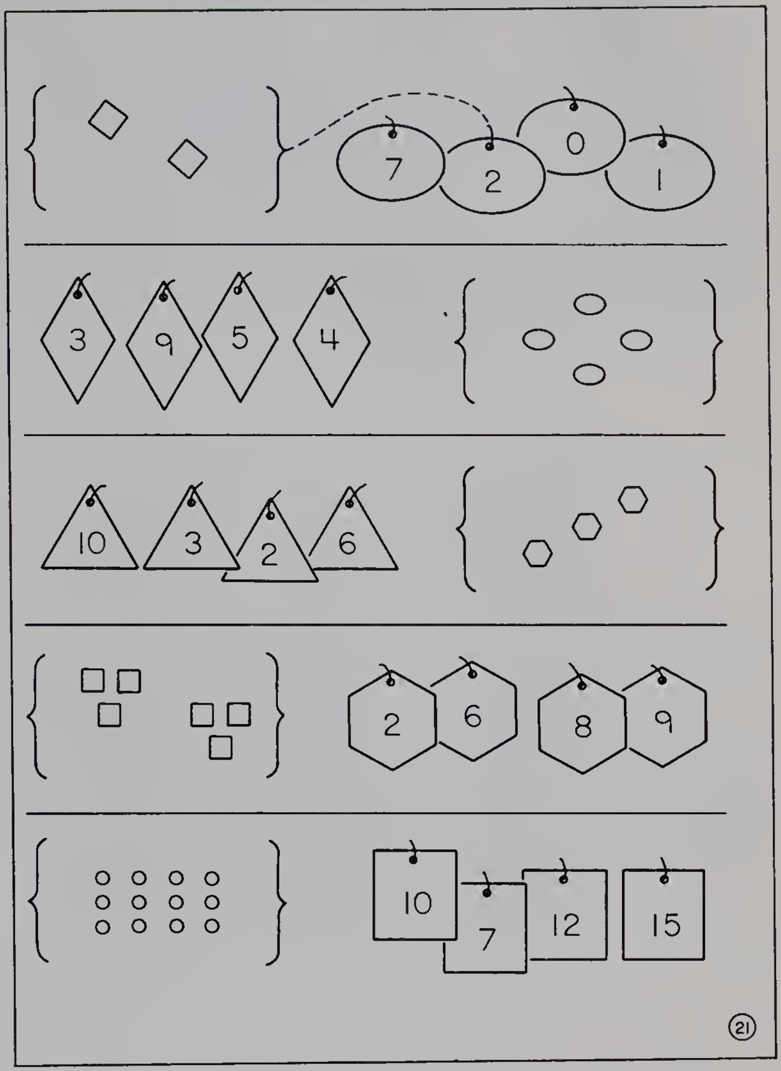
(3)


5
5
1
3
4
2
4
6
3
7
10

The "ears" provide places to record the sums for the diagonals.

Drill

Consider using as a drill sheet page 21 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:


7
2
0
1
3
9
5
4
10
3
2
6
2
6
8
9
10
7
12
15
㉑

WATCH THE SIGNS!

A.

10	+	5	=	15
−		−		−
2	+	3	=	5
=		=		=
8	+	2	=	10

B.

20	+	10	=	30
+		−		+
5	−	5	=	0
=		=		=
25	+	5	=	30

C.

21	−	8	=	13
+		−		+
9	+	1	=	10
=		=		=
30	−	7	=	23

D.

20	+	12	=	32
−		−		−
10	+	9	=	19
=		=		=
10	+	3	=	13

E.

23	−	13	=	10
−		−		−
12	−	9	=	3
=		=		=
11	−	4	=	7

F.

45	−	10	=	35
−		+		−
15	+	2	=	17
=		=		=
30	−	12	=	18

What's My Rule?

G.

□ + △ + 10	15	17	20	24	28	20	35	49	30	10
□	2	6	5	7	10	3	12	39	10	0
△	3	1	5	7	8	7	13	0	10	0

Notes on Pupil Page 27

Tasks: Pupils use the given clues to write in the missing addends, sums, or differences as required. Pupils demonstrate their discovery of the rule in exercise G by writing in the missing addends or sums as required.

Purpose: (a) To provide pupils with practice in addition and subtraction; (b) to provide pupils with practice in adding 10.

Unifying Ideas: Addition and subtraction; functions and relations.

Materials for Teacher: The chart described below; Ditto masters of several colors.

Materials for Pupils: Dittoed sheets.

The Lesson:

The *Watch The Signs!* game is a variation of the *Cross Number Puzzle*. For a detailed discussion of the mathematical principles involved in the *Cross Number Puzzle* and its variations, see the Notes on Pupil Page 63.

Exercise D illustrates precisely what is behind the exchanging (or borrowing) maneuver in the subtraction algorithm. To subtract 19 from 32, one notes that 32 is the sum of 20 and 12 (as well as of 30 and 2), and that 19 is the sum of 10 and 9. Then, one subtracts corresponding addends. Thus:

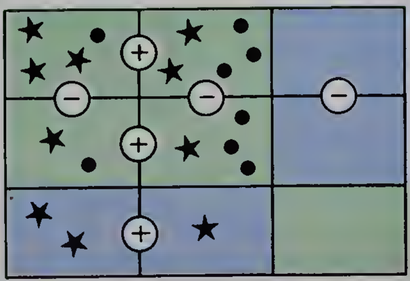
$$\begin{aligned} & 32 - 19 \\ = & (20 + 12) - (10 + 9) \\ = & (20 - 10) + (12 - 9) \\ = & 10 + 3 \\ = & 13 \end{aligned}$$

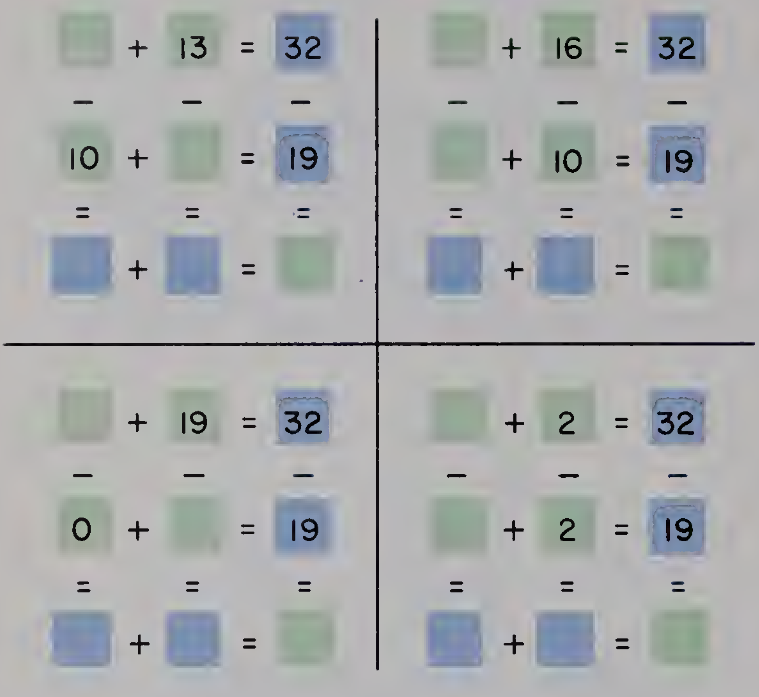
This same principle of rewriting each of 2 numbers as an indicated sum and then subtracting corresponding addends is illustrated in exercise A.

An interesting activity consists of setting up the form for exercise D, filling in the 32 and the 19, and varying the partitioning of 32 and 19, noting that in each case the sum for the bottom blue shaded boxes is 13. (See illustration on the next page.)

more

Headline for the Day:





Since this type of analysis will be useful throughout Level B in dealing with the subtraction algorithm, it will be worth your time to make a large chart of the form for exercise D. Perhaps it can be drawn on tagboard with colored oilcloth rectangles glued in place. The equality signs can be written permanently on the tagboard. If you wish to use the form for all the variations of *Watch The Signs!* games, instead of writing the six operation signs on the tagboard, paste white oilcloth disks in the appropriate places. Then use a felt pen with water-soluble ink to write operation signs and numbers.

If you prepare Dittoed sheets with the same form, be sure to use color shading for the rectangles. If you were to use "hollow" frames all of the same shape and size and expect pupils to fill them with different numbers, you would be expecting pupils to violate the "linking rule" for frames—*frames of the same shape and size must be filled with the same number.* One of our ground rules is that *shaded* frames need not follow the linking rule.

Prepare pupils for this page by placing the following arrangement on the board (or use your *Watch the Signs!* chart):

$$\boxed{\begin{array}{cc}4 & 3\\ \\ 2 & 1\end{array}}$$

We are going to play *Watch the Signs!* Put in the following signs:

$$\boxed{\begin{array}{ccc}4 & + & 3\\ - & & -\\ 2 & + & 1\end{array}}$$

Then complete the number sentences:

$$\begin{array}{ccccc}4 & + & 3 & = & 7\\ - & & - & & \\ 2 & + & 1 & = & 3\\ = & & = & & \\ 2 & & 2 & & \end{array}$$

Now, is it possible to put a sign between the 2 and the 2 and a sign between the 7 and the 3 so that this (point) and that (point) will give the same answer? Yes, it is. Here's how:

I.

$$\begin{array}{ccccc}4 & + & 3 & = & 7\\ - & & - & & -\\ 2 & + & 1 & = & 3\\ = & & = & & =\\ 2 & + & 2 & = & 4\end{array}
\qquad
\begin{array}{ccc}\multicolumn{3}{c}{\text{Sign Pattern}}\\ & + & \\ - & - & -\\ & + & \\ & + & \end{array}$$

Will any other pattern of signs work? Try these but let pupils determine which signs will work:

II.
$$\boxed{\begin{array}{ccc}4 & - & 3\\ + & & +\\ 2 & - & 1\end{array}}$$

III.
$$\boxed{\begin{array}{ccc}4 & - & 3\\ - & & +\\ 2 & + & 1\end{array}}$$

IV.
$$\boxed{\begin{array}{ccc}4 & + & 3\\ + & & -\\ 2 & - & 1\end{array}}$$

V.
$$\boxed{\begin{array}{ccc}4 & - & 3\\ + & & -\\ 2 & + & 1\end{array}}$$

VI.
$$\boxed{\begin{array}{ccc}4 & + & 3\\ - & & +\\ 2 & - & 1\end{array}}$$

VII.
$$\boxed{\begin{array}{ccc}4 & + & 3\\ + & & +\\ 2 & + & 1\end{array}}$$

VIII.
$$\boxed{\begin{array}{ccc}4 & - & 3\\ - & & -\\ 2 & - & 1\end{array}}$$

Here are some sign patterns which work for the given numbers 4, 3, 2, and 1:

I.
$$\begin{array}{cccc} & + & & \\ - & & - & -\\ & + & & \\ & + & & \end{array}$$

II.
$$\begin{array}{cccc} & - & & \\ + & & + & +\\ & - & & \\ & - & & \end{array}$$

III.
$$\begin{array}{cccc} & - & & \\ - & & + & -\\ & + & & \\ & - & & \end{array}$$

IV.
$$\begin{array}{cccc} & + & & \\ + & & - & +\\ & - & & \\ & + & & \end{array}$$

V.
$$\begin{array}{cccc} & - & & \\ + & & - & +\\ & + & & \\ & - & & \end{array}$$

VI.
$$\begin{array}{cccc} & + & & \\ - & & + & -\\ & - & & \\ & + & & \end{array}$$

VII.
$$\begin{array}{cccc} & + & & \\ + & & + & +\\ & + & & \\ & + & & \end{array}$$

VIII.
$$\begin{array}{cccc} & - & & \\ - & & - & -\\ & - & & \\ & - & & \end{array}$$

What can be said about these sign patterns? In each of them, the four operation signs in the box are such that there are an even number of addition signs and, of course, an even number of subtraction signs. What happens when a given sign appears only once among the four?

4	+	3	7
−		−	
2	−	1	1
2		2	No

4	−	3	1
−		+	
2	−	1	1
2		4	No

4	−	3	1
−		−	
2	+	1	3
2		2	Yes

4	−	3	1
+		−	
2	−	1	1
6		2	No

4	−	3	1
+		+	
2	+	1	3
6		4	No

4	+	3	7
+		−	
2	+	1	3
6		2	Yes

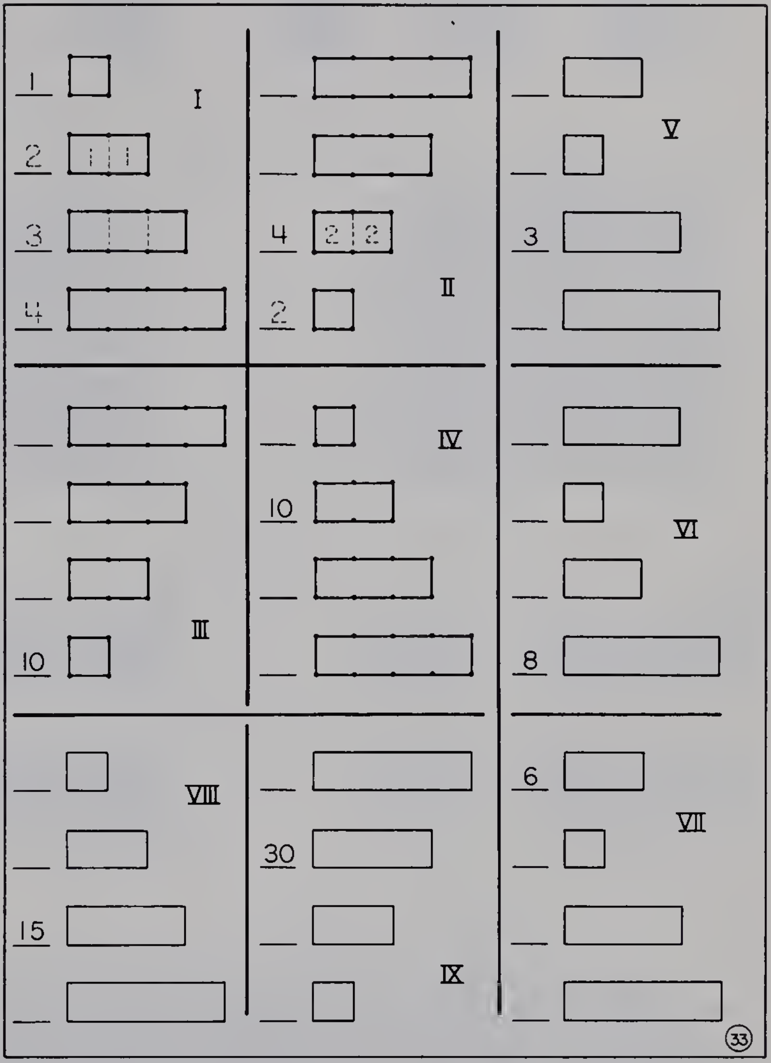
4	+	3	7
+		+	
2	−	1	1
6		4	No

4	+	3	7
−		+	
2	+	1	3
2		4	No

This test shows that there are two more sign patterns which work for the given numbers—4, 3, 2, and 1. The next step in the investigation should be an experiment in which other numbers are tried for those sign patterns that have worked so far. We leave this next step to you or your pupils. (Hint: We are willing to tell you that two of the ten sign patterns that we have found to work for the numbers 4, 3, 2, and 1 do not work generally.)

Drill

Consider using as a drill sheet page 33 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



A.

3 + 4 = 7

4 + 4 = 8

4 + 3 = 7

ADD

B.

$\begin{array}{r}5\\6\\\hline 11\end{array}$ $\begin{array}{r}6\\6\\\hline 12\end{array}$ $\begin{array}{r}6\\5\\\hline 11\end{array}$

C.

$\begin{array}{r}5\\3\\\hline 8\end{array}$ $\begin{array}{r}5\\4\\\hline 9\end{array}$ $\begin{array}{r}5\\5\\\hline 10\end{array}$

D.

$\begin{array}{r}6\\0\\\hline 6\end{array}$ $\begin{array}{r}6\\2\\\hline 8\end{array}$ $\begin{array}{r}6\\4\\\hline 10\end{array}$

E.

2	4	6
2	1	3
4	5	9

F.

1	5	6
1	0	1
2	5	7

G.

1	2	3
3	0	3
4	2	6

H.

0 + 1 = 1

0 + 2 = 2

I.

3 + 1 = 4

3 + 2 = 5

J.

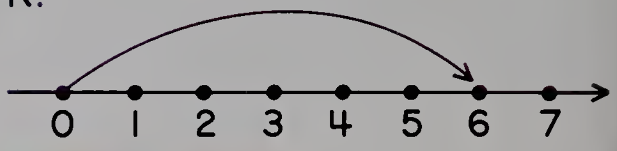
5 + 1 = 6

5 + 2 = 7

L.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	7
2	2	3	4	5	6	7	8
3	3	4	5	6	7	8	9
4	4	5	6	7	8	9	10
5	5	6	7	8	9	10	11
6	6	7	8	9	10	11	12

K.



0 → 6

1 → 7

2 → 8

3 → 9

4 → 10

Notes on Pupil Page 28

Tasks: *Pupils complete the various addition sentences in exercises A-J and K, and record each sum in the table in L.*

Purpose: Mastery and achievement test on addition and subtraction facts for addends not exceeding 6.

Unifying Ideas: Addition and subtraction.

Materials for Teacher: Flash cards.

Materials for Pupils: Cuisenaire rods for those who need them.

The Lesson:

This page and pupil page 29 provide a summary of the addition facts for addends through 10. Use the results to determine which pupils need special help. Such help might be given through work with Cuisenaire rods. (See pages 41-44 of the booklet *Games and Enrichment Activities*. For pupils with serious deficiencies in simple addition, see pages 1-5 of the same booklet.) Also, see the suggestions in the Notes on Pupil Page 30.

Flash Cards

Prepare the pupils for pupil page 28 by giving them flash card drill. Addition flash cards should contain problems in finding missing addends as well as missing sums, and the problems should be presented in horizontal format as well as in vertical format:

$$\boxed{5 + \square = 12} \qquad \boxed{2 + 5 = \square} \qquad \boxed{\square + 3 = 7}$$

$$\boxed{\begin{array}{r} 2 \\ + \square \\ \hline 4 \end{array}} \qquad \boxed{\begin{array}{r} 5 \\ + 0 \\ \hline \square \end{array}} \qquad \boxed{\begin{array}{r} \square \\ + 3 \\ \hline 5 \end{array}}$$

Patterns in the Addition Table

When pupils have completed exercise L, put the addition table on the chalkboard and discuss some of its features.

Look at the main diagonal that runs from lower left to upper right. What do you notice about it? Each cell contains 6. As you move upward from cell to cell, one addend increases by 1 and the other decreases by 1. Thus the sum is fixed. This is true for all diagonals parallel to this one.

The main diagonal that runs from upper left to lower right contains even numbers only. These are the doubles of the numbers 0, 1, 2, . . . 6. As you move from cell to cell along this diagonal, each addend increases by 1, so the sum increases by 2. This is true for all diagonals parallel to this main diagonal. Notice that these subdiagonals are reflected in the main diagonal. This fact illustrates the commutative property of addition.

The table shows all the ways in which the sentence $\square + \triangle = 6$ can be completed. It also shows all the ways to complete the sentences $\square + \triangle = 5$, $\square + \triangle = 4$, etc. Does it show all the ways to complete the sentence $\square + \triangle = 7$? (No)

Headline for the Day: Finish the list:

$(6 + 0) + (6 - 0) = 12$
$(6 + 1) + (6 - 1) = 12$
$(6 + 2) + (6 - 2) = 12$

A. 7 + 0 = 7

B. 8 + 0 = 8

C. 10 + 0 = 10

D. 7 + 3 = 10

E. 9 + 3 = 12

F. 10 + 3 = 13

G. 2 + 7 = 9

H. 4 + 7 = 11

I. 6 + 7 = 13

J. 1 + 8 = 9

K. 3 + 8 = 11

L. 7 + 8 = 15

ADD

M. 8 + 4 = 12

N. 10 + 5 = 15

O. 9 + 4 = 13

P. 6 + 8 = 14

Q. 4 + 10 = 14

R.

+	0	1	2	3	4	5	6	7	8	9	10
0	0	1	2	3	4	5	6	7 (A)	8 (B)		10 (C)
1	1	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	7				
2	2	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	8				
3	3	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	9	10 (D)		12 (E)	13 (F)
4	4	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	10		12 (M)	13 (O)	
5	5	[illegible]	[illegible]	[illegible]	[illegible]	10	11				15 (N)
6	6	7	8	9	10	11	12				
7			9 (G)		11 (H)		13 (I)				
8		9 (J)		11 (K)			14 (P)	15 (L)			
9											
10					14 (Q)						

Notes on Pupil Page 29

Tasks: Pupils complete the various addition sentences in exercises A-Q and record the sums in the addition table in exercise R. Pupils may complete the entire table if they wish to.

Purpose: Mastery and achievement test on addition and subtraction facts for pairs of addends, one of which exceeds 6 and neither of which exceeds 10.

Unifying Ideas: Structure; addition and subtraction; functions and relations.

Materials for Teacher: Flash cards; Ditto masters.

Materials for Pupils: Dittoed sheets.

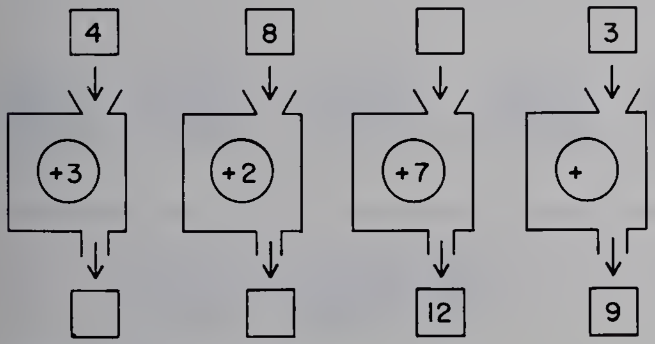
The Lesson:

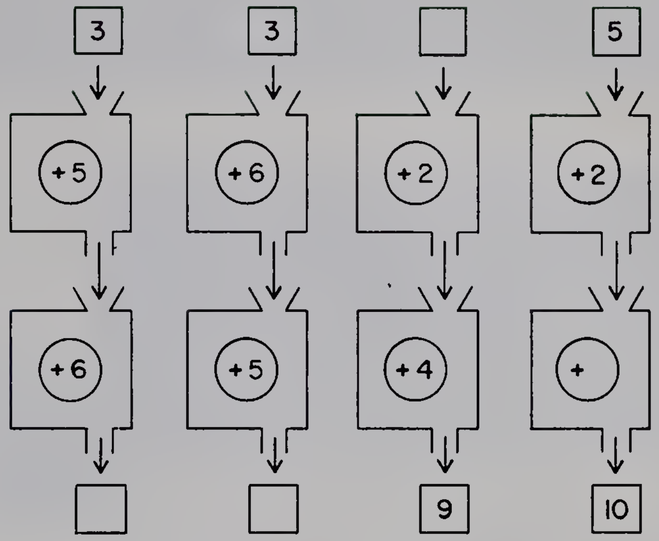
Note that in entering sums in the table in exercise R we have used the number listed on the top edge of the table as the first addend and the number listed on the left edge as the second addend. This choice of order is arbitrary since addition is commutative.

Flash card drill is appropriate as preparation for this page. (See the Notes on Pupil Pages 28 and 30.) For follow-up work see pages 44-62 of the booklet *Games and Enrichment Activities.*

Machine Notation

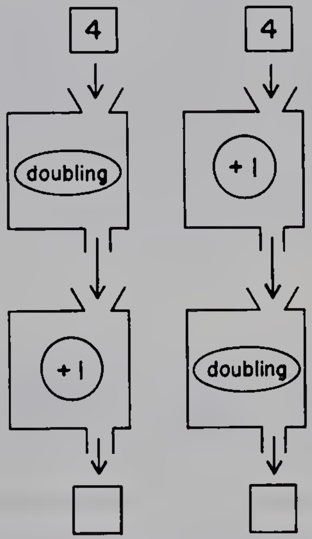
An interesting way to provide drill in addition and subtraction is to use machine notation. Here are some sample exercises which may be put on the chalkboard or on Dittoed sheets:





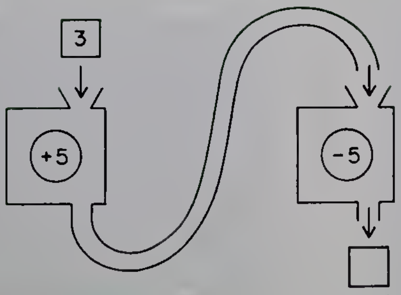
Mr. Marvin Karlin reports that a primary schoolboy in Jefferson County, Colorado, became so excited by the idea of machines that he came to school with a large box having slots cut in the top and at the bottom. He sat in the box having equipped himself with a pad, pencil, and flashlight. If he produced a wrong answer, the questioner could kick the box.

Machine notation can be used to ask deep questions. For example, does it make any difference if we double first and add 1 instead of first adding 1 and then doubling?



(See the Notes on Pupil Page 118 for a related discussion on functions.)

Headline for the Day:



A.

34 35 36 37 38 39 40 41 42

B.

JUNEAU IS THE CAPITAL OF THE STATE OF ALASKA.

How many letters in the sentence above?

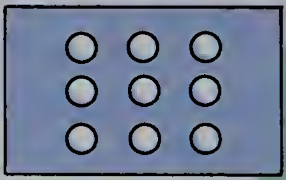
36 letters

C.

13	14	15	16	17
23	24	25	26	27
33	34	35	36	37
43	44	45	46	47
53	54	55	56	57

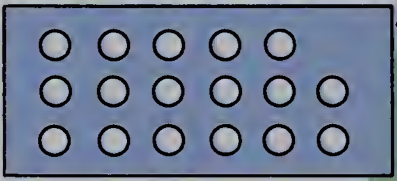
D.

Bill had this many marbles.



9 marbles

His brother Carl had this many marbles.



17 marbles

Together, Bill and Carl had 26 marbles.

Carl had 8 marbles more than Bill had.

E.

27, 25, 23, 21, 19, 17, 15, 13, 11, 9

F.

□ + △ = ⬡

□	3	6	13	17	19
△	8	4	6	3	5
⬡	11	10	19	20	24

G.

▱ − ▽ = ⧗

▱	9	15	15	16	20
▽	7	10	15	8	13
⧗	2	5	0	8	7

Notes on Pupil Page 30

Tasks: *Pupils supply the missing numbers.*

Purpose: To test pupils' understanding of, and skills in, counting, addition, and subtraction.

Unifying Ideas: Number and counting; addition and subtraction.

Materials for Teacher: Flash cards; demonstration dot cards for flannel board display as described below.

Materials for Pupils: Counters or Cuisenaire rods for those who need them.

The Lesson:

Here is a suggested distribution of points:

A—1 point B—1 point C—2-points D—4-points E—1 point F—4 points G—4 points

Flash Cards

Flash card activities can be useful for diagnostic and remedial work. Flash cards are a time-tested device to help develop the ability to recall without hesitation the 100 basic facts of addition and the 100 of subtraction.

The use of flash cards as the only device to accomplish this job is never wise. Their use as punishment is almost always self-defeating, since a resisting mind is not a receptive mind. Flash cards are most effectively used in a gamelike atmosphere. In a race against time the learner competes with himself.

A valuable use of a set of flash cards is to "diagnose"—to locate the particular facts that are most troublesome. Therefore, one set ought to include all 200 facts.

Flash cards may be made of 3-inch by 5-inch cards. On one side a fact is referred to through an open sentence: $5 + 3 = \square$. On the back the complete statement is given: $5 + 3 = 8$.

In preparing a set, you may find the following checklist helpful:

		Second Number									
		0	1	2	3	4	5	6	7	8	9
First Number	0	a✓	b✓	c✓	d✓						
	1										
	2										
	3										
	4										
	5										
	6										
	7										
	8										
	9										

a✓ $0 + 0 =$ ___

b✓\ $0 + 1 =$ ___

c✓ $0 + 2 =$ ___

d✓ $0 + 3 =$ ___

Each cell indicates one of the basic facts of addition for which a flash card is to be made. To prepare flash cards for the total list of facts, first complete the table by adding the first number and second number, moving across the cells horizontally. Then prepare a card for each cell, checking it off after you have completed the card.

For each addition fact, such as $3 + 5 = 8$, there is a subtraction fact: $8 - 5 = 3$.

Used diagnostically, an appropriate group of cards (with facts pupils have already met and considered) are mixed up thoroughly, and the game begins.

As the pupil responds, separate the cards into two piles—those he answers without hesitation and those he does not call up automatically.

The second group delimits the pupil's problem. As he goes through these again, separate this group into two groups—those on which he only hesitates and those which he gets wrong or must figure out. On the former, the pupil needs a little practice; on the others

Headline for the Day: $5 + 3 - 3 + 2 - 2 + 1 - 1 = \square$

he needs hard work. To help him master the second group, consider the following procedures.

First be sure that he draws a diagram, manipulates counters, and reasons to a correct conclusion.

Make supplementary cards for each of the facts, using many different forms of each fact. Suppose that he has trouble with 3 + 9 = □. The following forms are recommended:

3 + 9 = □	12 − 9 = □
3 + □ = 12	12 − □ = 3
□ + 9 = 12	□ − 9 = 3
9 + 3 = □	12 − 3 = □
9 + □ = 12	12 − □ = 9
□ + 3 = 12	□ − 3 = 9

and

$\begin{array}{r}3\\+9\\\hline\end{array}$	$\begin{array}{r}3\\+\\\hline 12\end{array}$	$\begin{array}{r}\\+9\\\hline 12\end{array}$	$\begin{array}{r}9\\+3\\\hline\end{array}$	$\begin{array}{r}9\\+\\\hline 12\end{array}$	$\begin{array}{r}\\+3\\\hline 12\end{array}$
$\begin{array}{r}12\\-9\\\hline\end{array}$	$\begin{array}{r}12\\-\\\hline 3\end{array}$	$\begin{array}{r}\\-9\\\hline 3\end{array}$	$\begin{array}{r}12\\-3\\\hline\end{array}$	$\begin{array}{r}12\\-\\\hline 9\end{array}$	$\begin{array}{r}\\-3\\\hline 9\end{array}$

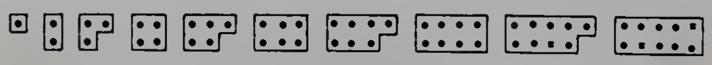
This great variety of arrangements causes the pupil to think through the question in many related ways. In each case, he reinforces his memory. Furthermore, he is deepening his understanding of important relationships among the facts of arithmetic. Using many forms of a question helps keep the practice lively instead of dull and monotonous.

Periodically, those groups of the 200 basic addition and subtraction facts with which a pupil is having trouble ought to be presented in diagnostic fashion, working each time to eliminate more and more facts that have already been committed to memory.

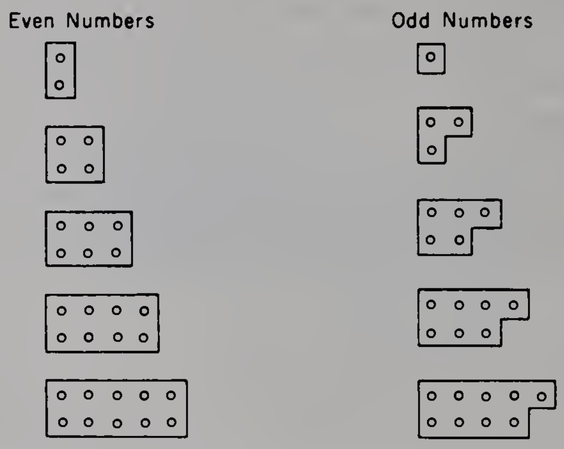
Remember: this drill is most effective when it is interspersed often with dissimilar activities and when a gamelike aspect is maintained.

Even and Odd Numbers

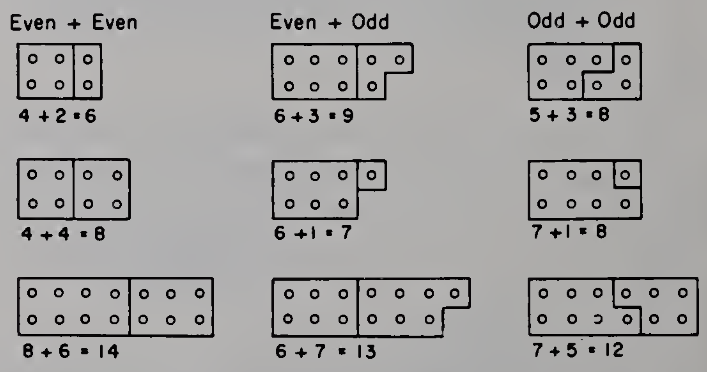
Work with even and odd numbers can provide drill in painless fashion. Make dot cards like those below:



Next, mark off two columns on a flannel board, and have the pupils tell you which cards to put in each column. Finished, the columns will look like this:



Now, ask the pupils to help you make combinations of "even and even," "even and odd," and "odd and odd." Draw the combinations on the chalkboard as follows:



These pictures help to demonstrate that:

even + even → even
even + odd → odd
odd + odd → even

Use the same pictures to suggest subtraction:

even — even	odd — odd odd — even	even — odd
6 − 4 = 2	9 − 3 = 6	8 − 5 = 3
6 − 2 = 4	9 − 6 = 3	8 − 3 = 5
8 − 4 = 4	7 − 1 = 6	8 − 1 = 7
	7 − 6 = 1	8 − 7 = 1
14 − 8 = 6	13 − 7 = 6	12 − 7 = 5
14 − 6 = 8	13 − 6 = 7	12 − 5 = 7

This demonstrates that:

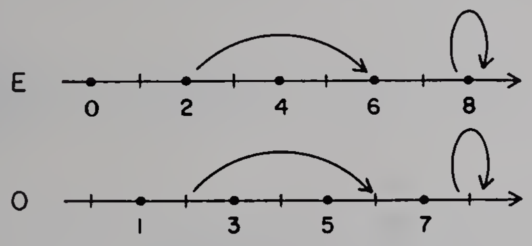
even — even → even
even — odd → odd
odd — odd → even
odd — even → odd

Even and Odd Numbers and the Number Line

Another activity with even and odd numbers involves the number line. Draw a pair of number lines on the board, one with even-number labels and one with odd-number labels. Write mathematical sentences and indicate each on both number lines.

Begin with pairs of even addends:

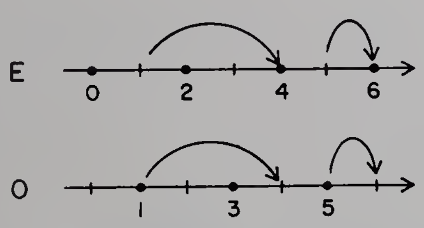
$2 + 4 = 6$ $\quad$ $8 + 0 = 8$



The starting and landing points are both labeled on the E line and neither is labeled on the O line.

Let's see what happens with a pair of odds:

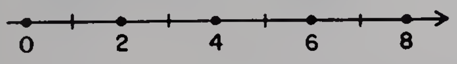
$1 + 3 = 4$ $\quad$ $5 + 1 = 6$



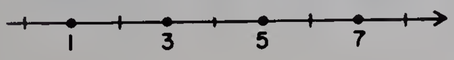
In the two examples we start on line E at an unlabeled point and end on a labeled point. The reverse is true on line O—we start at a labeled point and end on an unlabeled point.

Under each of the number lines, have pupils write number sentences that meet specified conditions:

Start and land on labeled points.

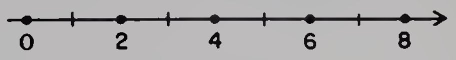


$2 + 4 = 6$
$8 - 6 = 2$
$2 + 2 = 4$
etc.

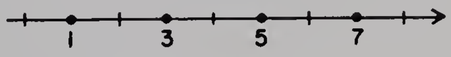


$1 + 2 = 3$
$7 - 6 = 1$
$3 + 4 = 7$
etc.

Start on a labeled point and land on an unlabeled point.



$2 + 1 = 3$
$8 - 7 = 1$
etc.



$3 + 5 = 8$
$7 - 1 = 6$
etc.

Facts From Other Facts

Pupils who have trouble memorizing the basic addition and subtraction facts need help in seeing how these facts are related to each other. For example, if one knows the "doubles"

$$1 + 1 = 2 \quad 2 + 2 = 4 \quad 3 + 3 = 6 \quad \text{etc.}$$

he can easily learn the "near doubles"

$$\begin{array}{llll} 0 + 1 = 1 & 1 + 2 = 3 & 2 + 3 = 5 & \\ 1 + 2 = 3 & 2 + 3 = 5 & 3 + 4 = 7 & \text{etc.} \end{array}$$

Here is another way to derive facts from other facts. Suppose that we write pairs of true statements one under the other, like this:

$$\begin{array}{ccc} \text{(a)} & \text{(b)} & \\ 2 + 3 = 5 & 7 + 1 = 8 & \\ 6 + 4 = 10 & 3 + 3 = 6 & \text{etc.} \end{array}$$

Under each column of two numbers, write the sum of those numbers. Example (a), then, looks like this:

$$\begin{array}{ccccc} 2 & + & 3 & = & 5 \\ 6 & + & 4 & = & 10 \\ \hline 8 & & 7 & & 15 \end{array}$$

What do you notice about the results? You can write:

$$8 + 7 = 15$$

Will you get an addition fact in this way from every pair of addition facts? (Yes)

Experiment with pairs of subtraction facts:

$$\begin{array}{ccccc} 8 & - & 3 & = & 5 \\ 4 & - & 1 & = & 3 \\ \hline 12 & & 4 & & 8 \end{array} \qquad \begin{array}{ccccc} 13 & - & 3 & = & 10 \\ 10 & - & 1 & = & 9 \\ \hline 23 & & 4 & & 19 \end{array} \quad \text{etc.}$$

Is it true that $12 - 4 = 8$, $23 - 4 = 19$, etc.?

The number of letters in each set is —

N{ } = 0   N{ R } = 1   N{ C X } = 2

N{ and } = 3   N{ plus } = 4   N{ equal } = 5

N{ Vacation } = 8   N{ is over } = 6

N{ Vacation is over. } = 14

N{ Ned ran } = 6   N{ to school } = 8

N{ Ned ran to school. } = 14

How many blocks in each set?

N{ ■ ▪ } = 2   N{ ▪ ▪ ■ } = 3   N{ ■ ▪ ▪ ▪ ■ } = 5

N{ ■ ▪ ■ ■ } = 4   N{ ■ ■ ■ ■ ■ ■ } = 6

N{ ■ ▪ ■ ■ ■ ■ ■ ■ ■ ■ } = 10

Notes on Pupil Page 31

Tasks: *In the top six lines pupils record the number of letters displayed between braces; in the bottom three lines pupils record the number of blocks displayed between braces.*

Purpose: (a) To review the definition of addition in terms of the union of disjoint sets (or in terms of combining nonoverlapping displays); (b) to introduce the notation $N\{\ldots\}$.

Unifying Ideas: Sets; number and counting; addition and subtraction.

Materials for Teacher: Flannel board; demonstration alphabet letters; braces; equality signs; hexagonal, square, triangular, and circular shapes cut from felt.

Materials for Pupils: Counters or Cuisenaire rods.

The Lesson:

Pupil pages 31-34 comprise a unit which provides a foundation for the work on pupil pages 35-43 and 46-50. The unit is continued on pupil pages 51-52.

The notation $N\{\ldots\}$ tells the pupil to determine the number of objects displayed between the braces.

The rest of the page illustrates the fact that the number of objects in a display formed by combining two nonoverlapping displays is the sum of the numbers of objects in the two displays. (This mirrors the definition of addition in terms of the union of disjoint sets.) For example, when a pupil has determined the counts, 8 and 6, in the third line, we hope that he will determine the count in the fourth line by adding 8 and 6. (If he decides to count the period in the fourth line, give him a pat on the back.)

Note that in the last three lines the drawings have been carefully made so as to show clearly that the objects in the combined display are the objects in the component displays.

The displays on pupil page 31 can be easily reproduced on the flannel board (or the chalkboard). Prepare pupils for this page by having them work with concrete displays.

Drill

Here is a suggestion for a drill sheet:

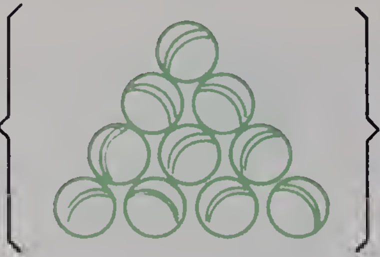
Add.

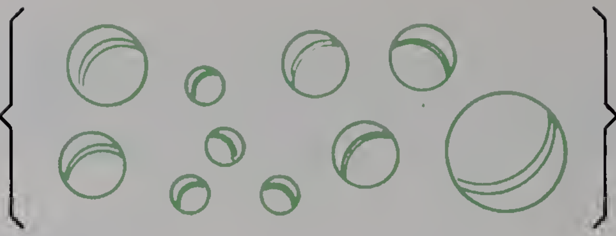
$\begin{array}{r} 4 \\ +2 \\ \hline \end{array}$	$\begin{array}{r} 3 \\ +2 \\ \hline \end{array}$	$\begin{array}{r} 2 \\ +5 \\ \hline \end{array}$	$\begin{array}{r} 4 \\ +4 \\ \hline \end{array}$	$\begin{array}{r} 5 \\ +2 \\ \hline \end{array}$
$\begin{array}{r} 2 \\ +3 \\ \hline \end{array}$	$\begin{array}{r} 6 \\ +1 \\ \hline \end{array}$	$\begin{array}{r} 6 \\ +0 \\ \hline \end{array}$	$\begin{array}{r} 6 \\ +2 \\ \hline \end{array}$	$\begin{array}{r} 1 \\ +5 \\ \hline \end{array}$
$\begin{array}{r} 1 \\ +1 \\ \hline \end{array}$	$\begin{array}{r} 1 \\ +3 \\ \hline \end{array}$	$\begin{array}{r} 4 \\ +3 \\ \hline \end{array}$	$\begin{array}{r} 3 \\ +5 \\ \hline \end{array}$	$\begin{array}{r} 2 \\ +1 \\ \hline \end{array}$
$\begin{array}{r} 7 \\ +0 \\ \hline \end{array}$	$\begin{array}{r} 1 \\ +2 \\ \hline \end{array}$	$\begin{array}{r} 2 \\ +6 \\ \hline \end{array}$	$\begin{array}{r} 5 \\ +3 \\ \hline \end{array}$	$\begin{array}{r} 2 \\ +4 \\ \hline \end{array}$
$\begin{array}{r} 4 \\ +1 \\ \hline \end{array}$	$\begin{array}{r} 5 \\ +1 \\ \hline \end{array}$	$\begin{array}{r} 1 \\ +4 \\ \hline \end{array}$	$\begin{array}{r} 3 \\ +4 \\ \hline \end{array}$	$\begin{array}{r} 3 \\ +3 \\ \hline \end{array}$

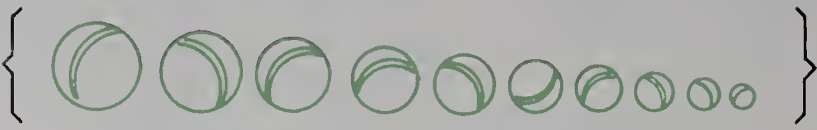
Headline for the Day: $13 = \square + 3$ Begin your story this way: I need some balloons for a party. . . .

How many in each set?

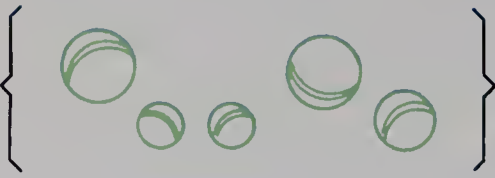
A.

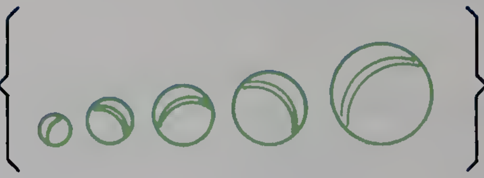
N  = 10

N  = 10

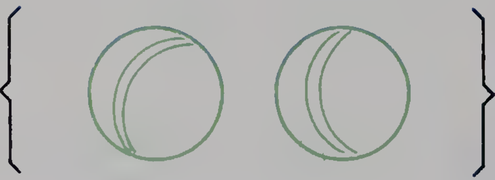
N  = 10

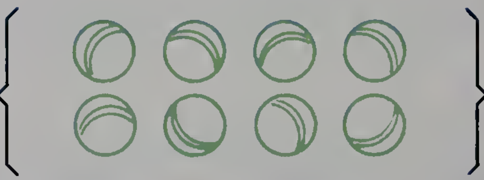
B.

N  = 5

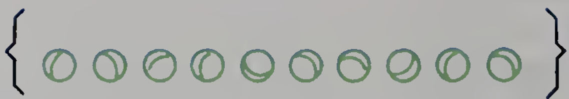
N  = 5

C.

N  = 2

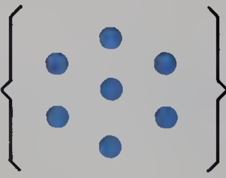
N  = 8

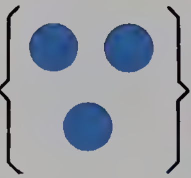
D.

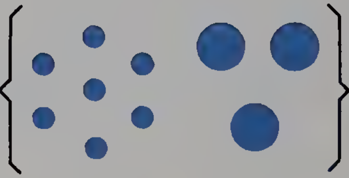
N  = 10

How many in each set?

E.

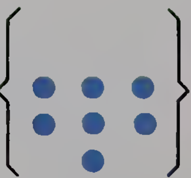
N  = 7

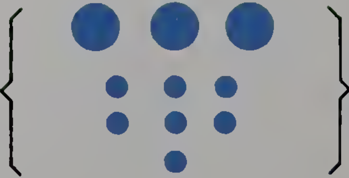
N  = 3

N  = 10

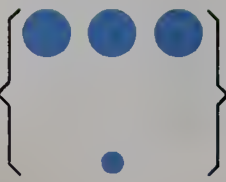
F.

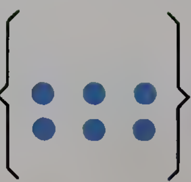
N = 3

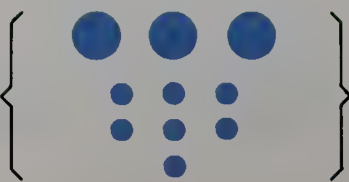
N  = 7

N  = 10

G.

N  = 4

N  = 6

N  = 10

Notes on Pupil Page 32

Tasks: *Pupils record the number of objects in each display between braces.*

Purpose: (a) To show pupils that many displays have the same number of objects; (b) to show pupils that counting is facilitated by certain rearrangements of a display; (c) to show pupils that numbers can be partitioned in many ways; (d) to provide pupils with practice in using the definition of addition in terms of the union of disjoint sets.

Unifying Ideas: Sets; number and counting; addition and subtraction.

Materials for Teacher: Flannel board; demonstration alphabet letters; braces; equality signs; hexagonal, square, triangular, and circular shapes cut from felt; 10 pennies obtained as described below.

Materials for Pupils: Counters or Cuisenaire rods.

The Lesson:

Exercise A shows two displays whose objects are arranged in a way that is useful for purposes of counting, and one display whose arrangement is not quite as useful. It is instructive to ask pupils if they could find the number of balls in the scrambled display without counting. We hope they will recall the matching procedure of their "childhood."

Exercise B contrasts a scrambled display with an orderly one. Exercise C shows displays that invite counting by 2's or recognizing a rectangular array.

Exercise D invites you to ask pupils to show that $5 + 5 = 10$ and that $2 + 8 = 10$. We hope they will suggest drawing pairing lines from the picture in D to the pictures in B and in C.

Exercises E, F, and G show clearly that $7 + 3 = 3 + 7 = 4 + 6 = 10$. Because of the arrangements of the displays, it isn't necessary to draw pairing lines.

Hidden Pennies

Prepare the pupils for the various partitions of 10 by playing *Hidden Pennies.* Collect 10 pennies from the pupils, put your hands behind your back, distribute the pennies between your hands, and place your closed hands in front of you. Let a pupil count the pennies in one of your hands, and let the rest of the class predict the number of pennies in your other hand. Be sure to include $0 + 10$.

Try hiding a couple of pennies on your desk top. But be prepared for a minor explosion, since you are playing with the pupils' money.

Drill

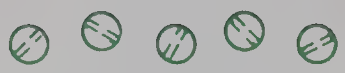
Here is a suggested drill sheet:

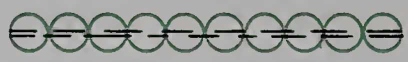
Subtract.

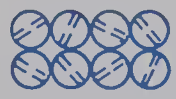
4 −1	6 −3	3 −2	5 −3	6 −1
2 −1	7 −4	8 −1	7 −6	6 −5
4 −3	5 −4	4 −2	5 −2	6 −6
7 −1	6 −0	4 −0	8 −3	6 −2
3 −0	8 −2	8 −6	5 −1	8 −4

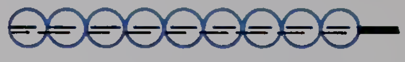
Headline for the Day: There are 25 children in the class. If __ are boys, then . . . are girls. Make a chart.

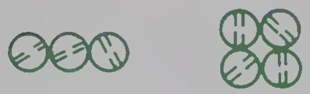
The number of beads is —

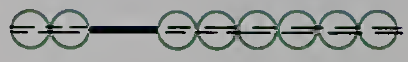
(a) N{  } = 5

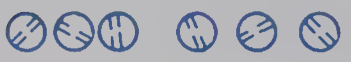
(b) N{  } = 10

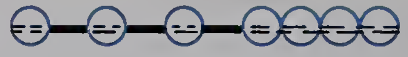
(c) N{  } = 8

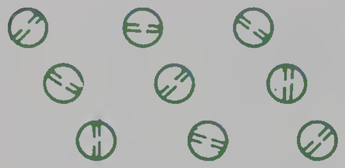
(d) N{  } = 9

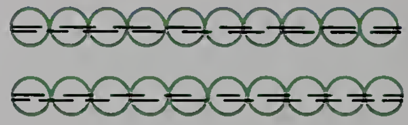
(e) N{  } = 7

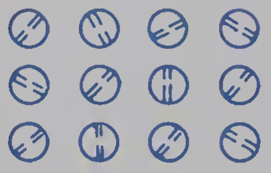
(f) N{  } = 8

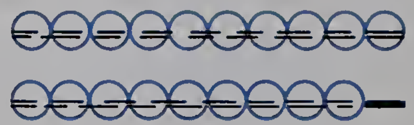
(g) N{  } = 6

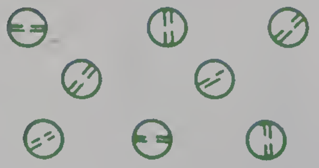
(h) N{  } = 7

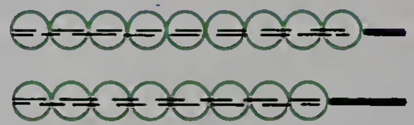
(i) N{  } = 9

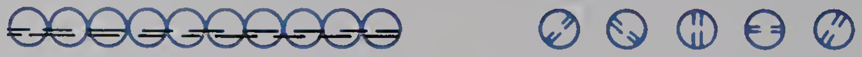
(j) N{  } = 20

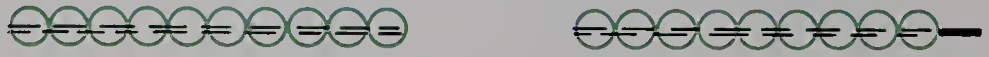
(k) N{  } = 12

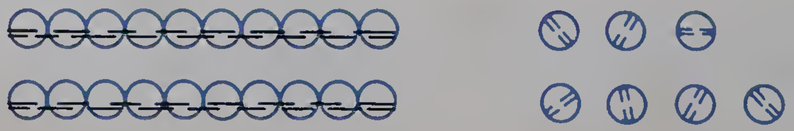
(l) N{  } = 19

(m) N{  } = 8

(n) N{  } = 17

(o) N{  } = 15

(p) N{  } = 19

(q) N{  } = 27

Notes on Pupil Page 33

Tasks: *Pupils record the number of beads in each display.*

Purpose: (a) To provide pupils with practice in skillful counting of objects arranged in a variety of useful patterns; (b) to provide pupils with practice in counting when objects have been grouped; (c) to provide pupils with practice in using decimal notation.

Unifying Ideas: Sets; number and counting; numeration.

Materials for Teacher: Flannel board; demonstration alphabet letters; braces; equality signs; hexagonal, square, triangular, and circular shapes cut from felt; abacus or counting frame; a Counting Board (see below).

Materials for Pupils: Cuisenaire rods.

The Lesson:

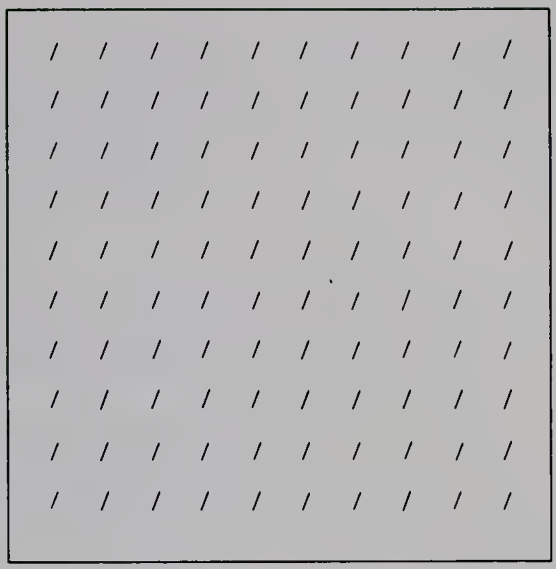
Once pupils have grasped the fact that a full rod of beads contains 10 beads, the exercises on the right and those at the bottom of the page become very easy. Make sure the pupils know that a full rod consists of 10 beads before they work on pupil page 34.

Prepare pupils for this work by having them manipulate an abacus or counting frame that can accommodate 10 beads to a wire.

Cuisenaire rod activities in which pupils count by using the orange rod as "grouper" or "partitioner" are also helpful in preparing pupils for this page.

The Counting Board

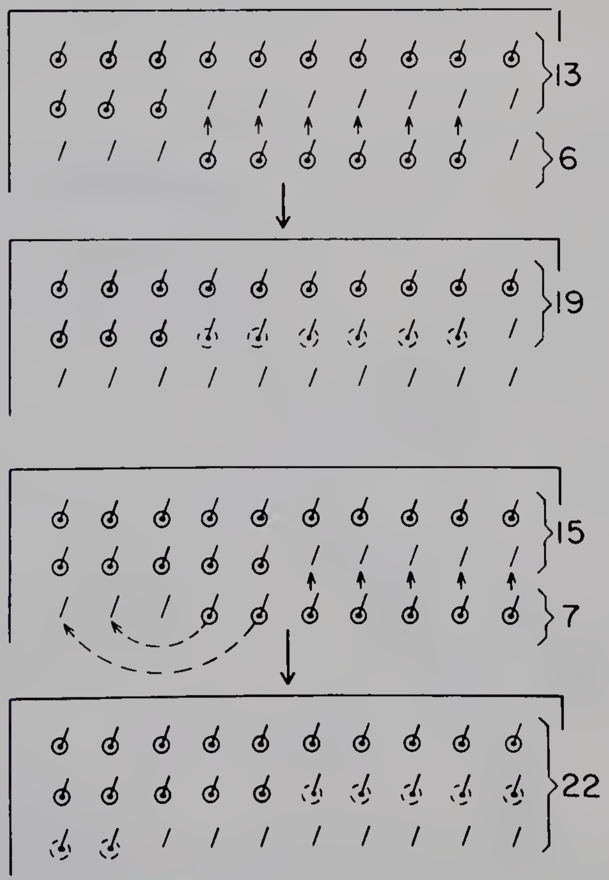
Now might be the time to bring in a Counting Board. This most versatile device can be made from a square piece of wood with a 10-by-10 arrangement of 100 nails driven in part way, and a box of washers which can be obtained at any hardware store. Use finishing nails—the kind without heads—so that the washers will slip over them easily.



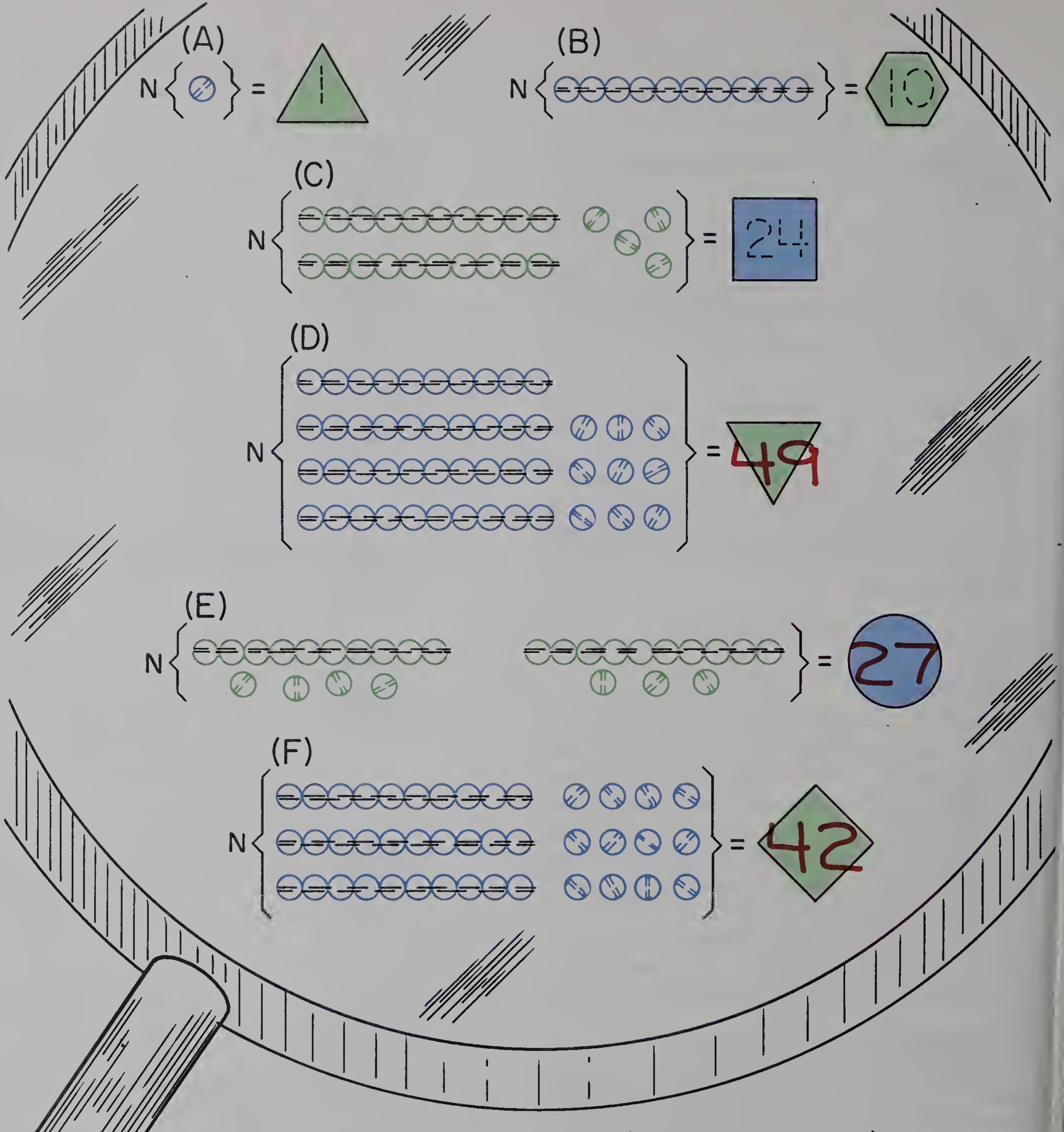
A few of the many uses of a Counting Board are shown in the following examples:

$13 + 6 = 19$ $\quad$ $15 + 7 = 22$

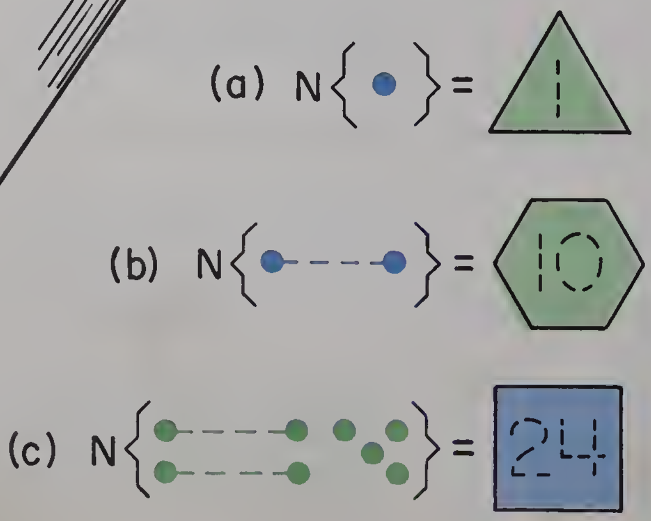
$19 - 6 = 13$ $\quad$ $22 - 7 = 15$

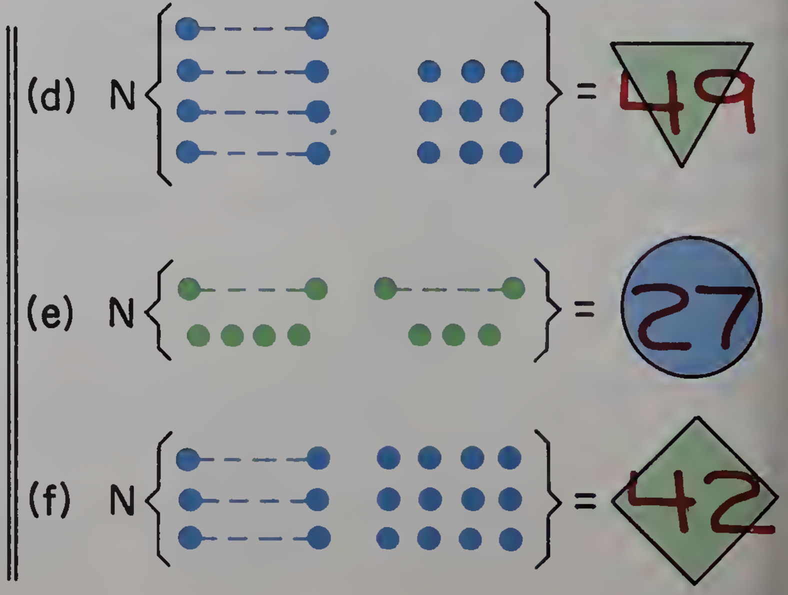


Headline for the Day: Design a flag with 12 stars arranged in columns and rows. In how many ways can such a field of stars be arranged?


(A)
N = 1
(B)
N = 10
(C)
N = 24
(D)
N = 49
(E)
N = 27
(F)
N = 42

In shorthand


(a) N = 1
(b) N = 10
(c) N = 24


(d) N = 49
(e) N = 27
(f) N = 42

Notes on Pupil Page 34

Tasks: Pupils record the number of beads in each display under the magnifying glass. For the pictures at the bottom of the page, pupils must interpret shorthand notation before determining counts.

Purpose: To introduce a shorthand notation as a first step toward understanding a numeration system.

Unifying Ideas: Sets; number and counting; numeration.

Materials for Teacher: None.

Materials for Pupils: Cuisenaire rods.

The Lesson:

Notice that the displays outside the magnifying glass are miniatures of those seen through the glass. Notice the keying: (A) goes with (a), (B) goes with (b), etc.

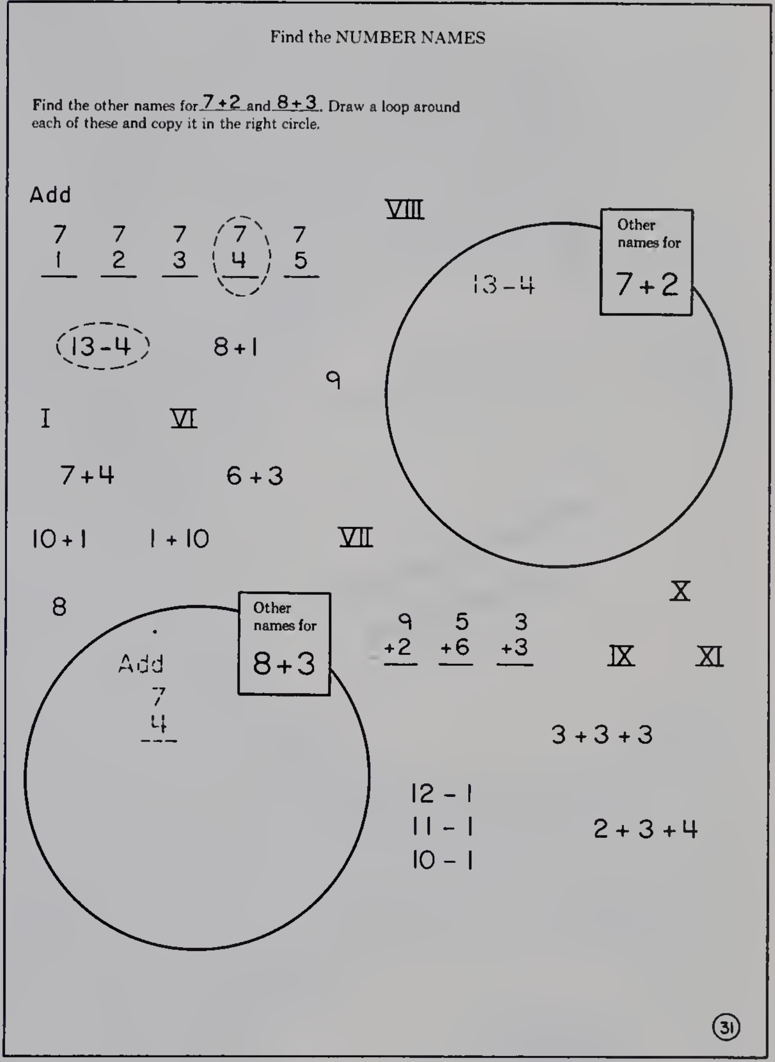
Be certain the pupils recognize that the symbol • - - - - • denotes a full rod of 10 beads, and not 2 beads.

A helpful device for interpreting the shorthand is to let pupils use hand signals to describe displays of beads. For example, to describe (c) a pupil might extend his forefinger, trace two segments in the air, and jab four times. Let pupils practice this language with each other, one pupil giving the signal and the other recording the number in decimal notation. Note that there is no need to "group" the signals. A pupil might trace three segments, jab twice, trace two more segments, and jab once. His partner should record 53.

Pupils might decide to use sounds instead of hand motions—a long boo-oo-oo-oop for a 10-group and a short beep for a 1-group. This will create a noisy but excited classroom.

Drill

Consider using as a drill sheet page 31 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Find the NUMBER NAMES

Find the other names for 7+2 and 8+3. Draw a loop around each of these and copy it in the right circle.

Add

7 7 7 7 7
1 2 3 4 5

13 - 4
8 + 1
9
I
VI
7 + 4
6 + 3
10 + 1
1 + 10
VII
VIII

Other names for 7 + 2

13 - 4

8

Other names for 8 + 3

Add
7
4

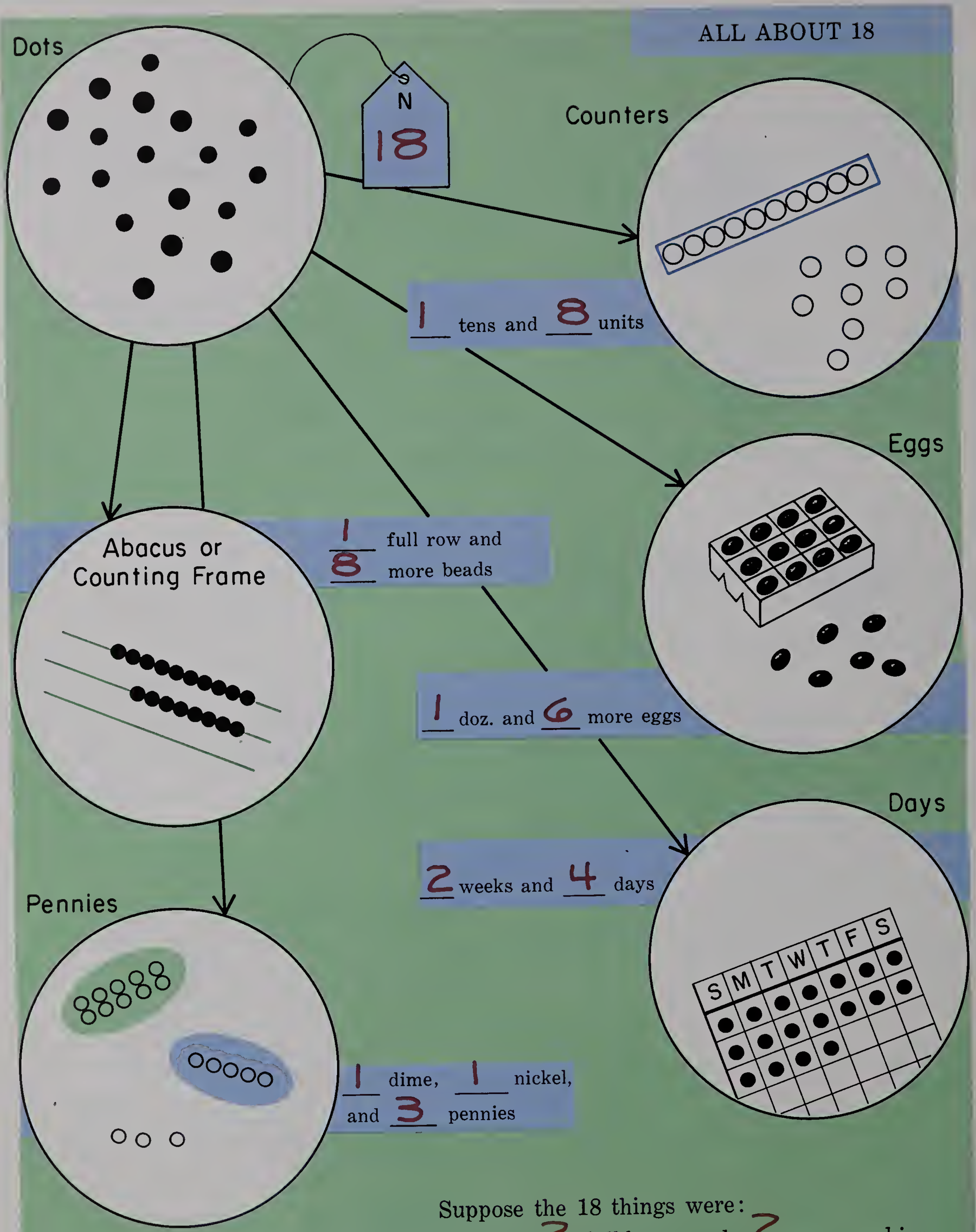
9 5 3
+2 +6 +3

X
IX
XI
3 + 3 + 3
12 - 1
11 - 1
10 - 1
2 + 3 + 4

(31)

Headline for the Day: Extend the pattern: 1, 4, 7, 10, 13

ALL ABOUT 18


Dots
N
18
Counters
Eggs
Abacus or Counting Frame
Days
S M T W T F S
Pennies

1 tens and 8 units

1 full row and
8 more beads

1 doz. and 6 more eggs

2 weeks and 4 days

1 dime, 1 nickel,
and 3 pennies

Suppose the 18 things were:

Cookies packed 8 in a box. 2 full boxes and 2 more cookies.

Baseball players — 9 players to a team.

Ounces — a pound is the same as 16 ounces.

Inches. Wheels on wagons. Wheels on tricycles.

Notes on Pupil Page 35

Tasks: *Pupils express the number of objects (counters, beads, eggs, etc.) displayed in each circle in accordance with the grouping or partitioning pattern indicated in the blue field. The examples at the bottom of the page are to be used in class discussion.*

Purpose: (a) To review and extend pupils' knowledge of units of measurement; (b) to demonstrate that grouping is used in everyday life.

Unifying Ideas: Number and counting; numeration; measurement.

Materials for Teacher: Counters; egg cartons; play money; calendar; counting frame; Ditto masters.

Materials for Pupils: Dittoed sheets.

The Lesson:

This is the first of four *All About . . .* pages. (See pupil pages 40, 43, and 50.) Our purpose here is to show pupils that one often uses a grouping or partitioning system in order to express the number of objects in a display. The system used depends largely upon custom. One uses a grouping system in reporting a number to make it easier to comprehend the numerosity of the display being counted.

Use the demonstration materials listed above in discussing pupil page 35. Then have the pupils fill in the blanks in the blue fields. The examples at the bottom should evoke responses such as:

There are 18 players on 2 baseball teams.
There are 18 ounces in 1 pound and 2 ounces.
There are 18 inches in 1 foot and 6 inches.
18 wheels on 4 wagons with 2 left over.
6 tricycles have a total of 18 wheels.

Fewest Units

With only nickels and pennies available, how can I make the listed amounts using the fewest coins?

Amount	4 ¢	5 ¢	6 ¢	10 ¢	15 ¢	18 ¢
Nickels	0	1				
Pennies	4	0				

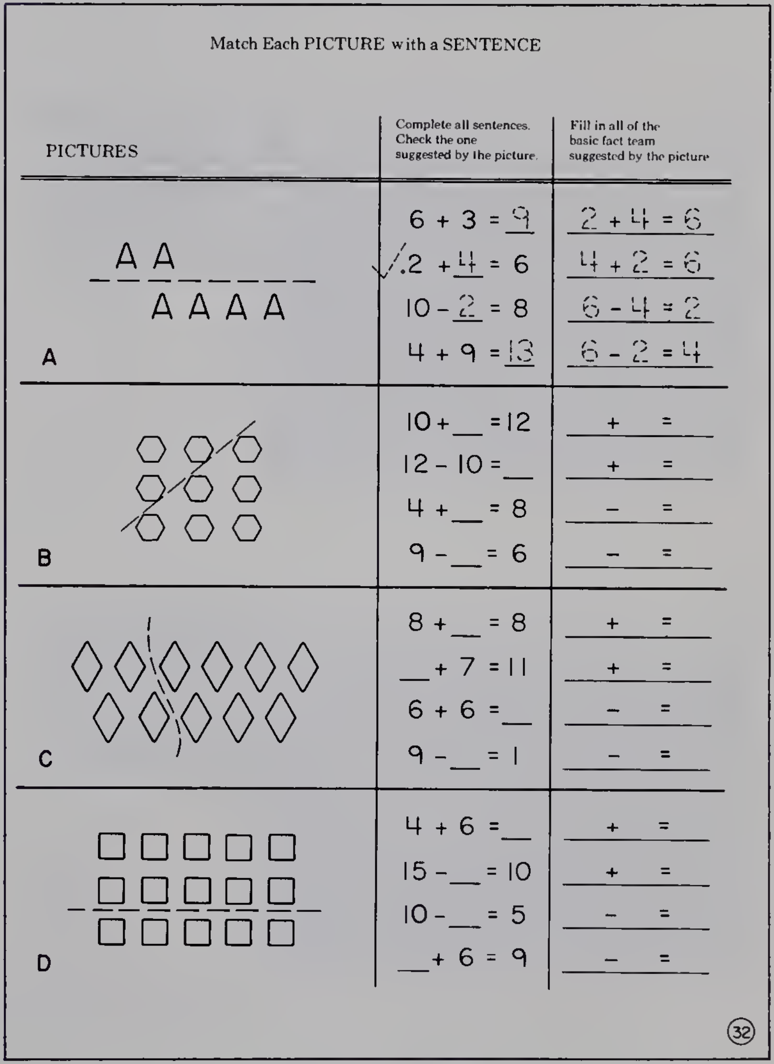
Represent the time periods using the fewest possible units.

Period	3 days	7 days	13 days	14 days	23 days	
Weeks	0	1				3
Days	3	0				4

Make up similar exercises using other measurement units suggested on pupil pages 40 and 43.

Drill

Consider using as a drill sheet page 32 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Match Each PICTURE with a SENTENCE

PICTURES	Complete all sentences. Check the one suggested by the picture.	Fill in all of the basic fact team suggested by the picture
A	6 + 3 = 9 ✓ 2 + 4 = 6 10 − 2 = 8 4 + 9 = 13	2 + 4 = 6 4 + 2 = 6 6 − 4 = 2 6 − 2 = 4
B	10 + __ = 12 12 − 10 = __ 4 + __ = 8 9 − __ = 6	+ = + = − = − =
C	8 + __ = 8 __ + 7 = 11 6 + 6 = __ 9 − __ = 1	+ = + = − = − =
D	4 + 6 = __ 15 − __ = 10 10 − __ = 5 __ + 6 = 9	+ = + = − = − =

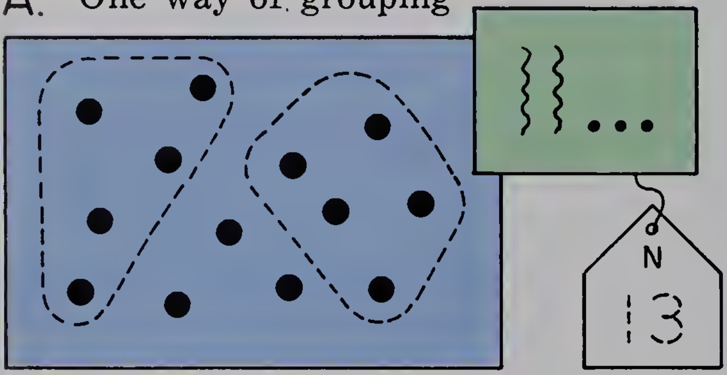
㉜

Headline for the Day: Extend the pattern: B, D, F, H, J

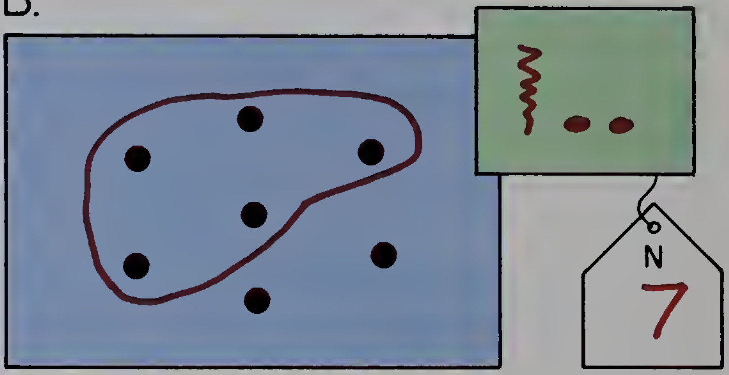
In each example, draw loops around groups of 5 dots with no dot in two groups.

In the green spaces, keep a record of the grouping of dots.
Use
⦃ to mean a group of 5,
and • to mean a dot left over.

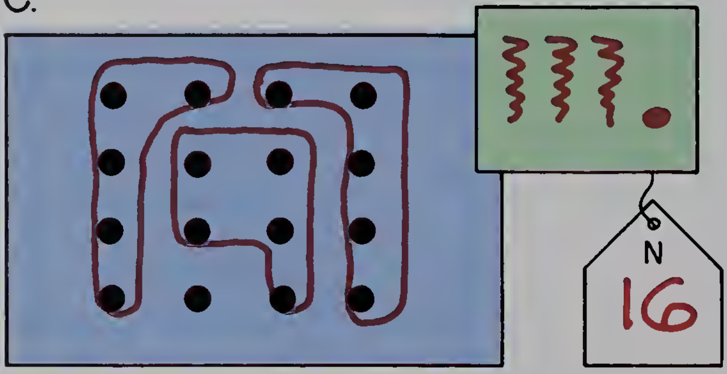
A. One way of grouping



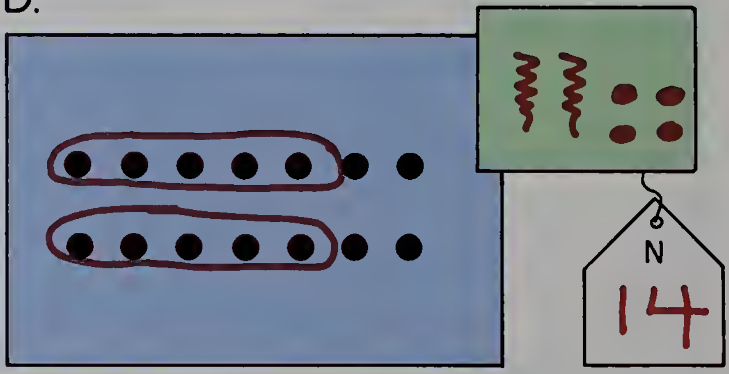
B.



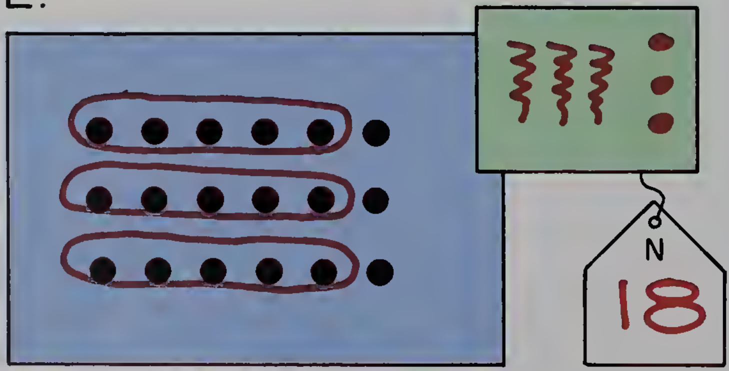
C.



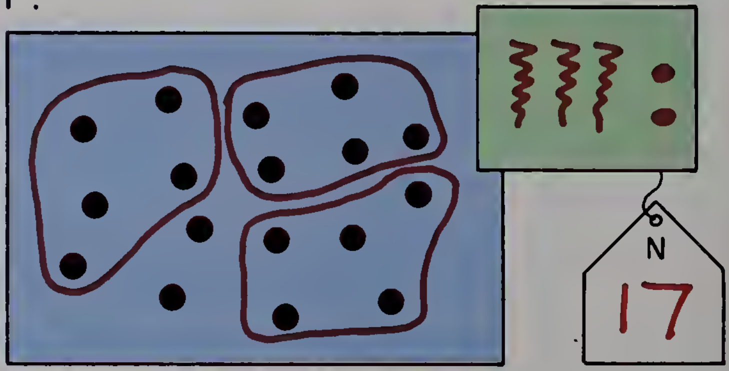
D.



E.



F.



G. What Are My Rules?

k − 10	2	10	5	20	9	4	0	8	17	26
k	12	20	15	30	19	14	10	18	27	36
k + 5	17	25	20	35	24	19	15	23	32	41

Notes on Pupil Page 36

Tasks: Pupils partition each given display into 5-groups and 1-groups and show the partitioning by drawing loops. Then they make a record of the partitioning by writing the special symbols in the green space. Finally, they record on the tag the number of dots in the display. In exercise G pupils demonstrate their discovery of the rules by filling in the empty cells in the table.

Purpose: (a) To build pupils' understanding of numeration systems; (b) to provide pupils with practice in adding 10, subtracting 10, adding 5, and subtracting 5.

Unifying Ideas: Number and counting; numeration; functions and relations.

Materials for Teacher: Flannel board; demonstration counters.

Materials for Pupils: Counters; Cuisenaire rods.

The Lesson:

On this page and on pupil pages 37-39 and 41 we show pupils how a numeration system develops. In a typical exercise, a pupil is shown a display. He is asked to partition the display in a certain way, and he indicates the partitioning by drawing loops. (Note that for any display there are many ways of carrying out the partitioning.)

The next step is to make a record of the partitioning, using a special notation. For pupil page 36 the pupil draws a wiggle for each 5-group and a dot for each 1-group. To record the number of objects in the given display, the pupil could count by 1's, but we hope that he will count by the larger grouping number. (Of course, when the number is reported in decimal notation, this also indicates a certain partitioning.)

In preparation for this page you may wish to use flannel board demonstrations.

Using Cuisenaire Rods

Cuisenaire rods may be used quite effectively as preparation for this page. A pupil scatters 18 white rods on his desk top to represent the situation in the blue field of exercise E. He then exchanges each 5 white rods for a yellow rod. At this point he has 3 yellow rods and 3 white rods on his desk top. This represents the situation in the green field of exercise E. Another pupil looks at the desk-top display and is asked to tell how many white rods were on the desk top before the exchange was made.

Drill

Here is a suggested drill sheet:

$2 + 3 + \square = 7$

$5 + \square + 4 = 10$

$6 + 5 + \square = 11$

$\square + 2 + 3 = 6$

$4 + \square + 5 = 13$

$6 + \square + 1 = 12$

$3 + 9 + \square = 18$

$2 + \square + 5 = 11$

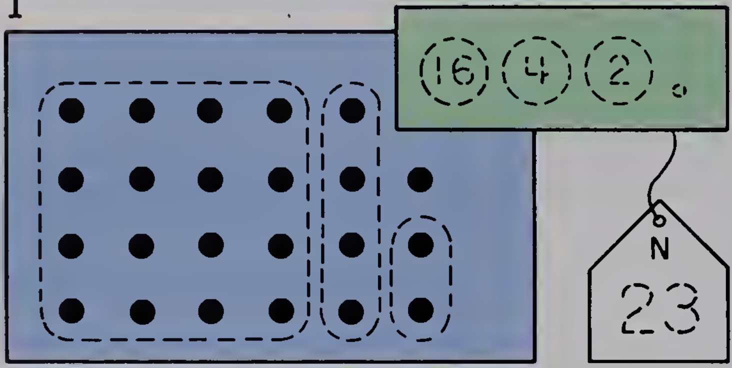
Headline for the Day: How many nickels can you get for a pile of pennies? Make a chart to help.

Put a loop around as many groups of 16 as you can. Put a loop around groups of 8 that are left. Then put a loop around groups of 4 that are left. Put a loop around groups of 2 that are left. No dots can be in more than one group.

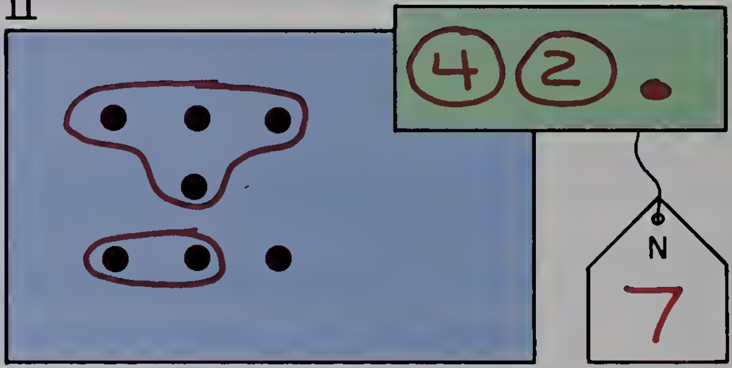
Record results:

⑯ .... a group of 16
⑧ .... a group of 8
④ .... a group of 4
② .... a group of 2
• .... dots not in a group

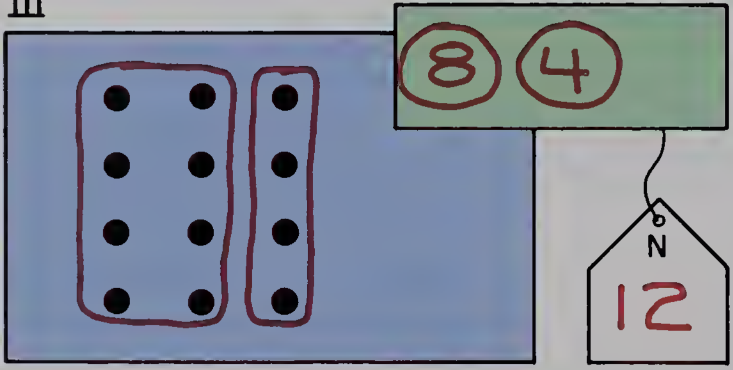
I



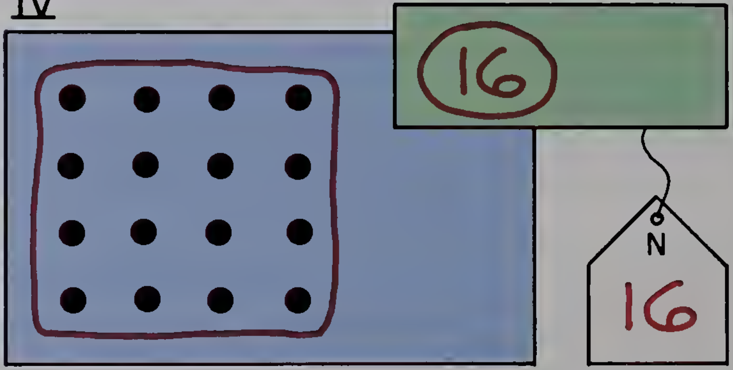
II



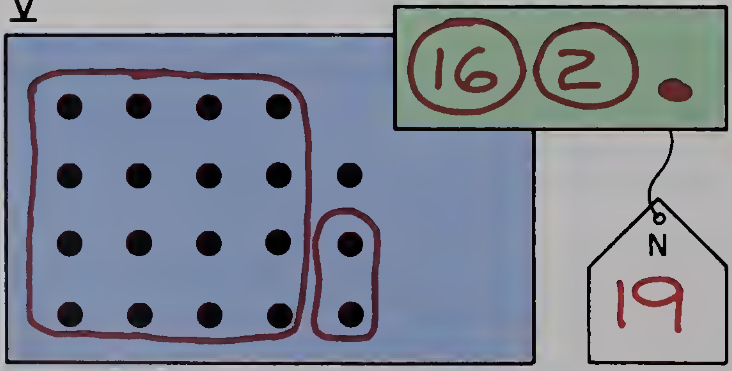
III



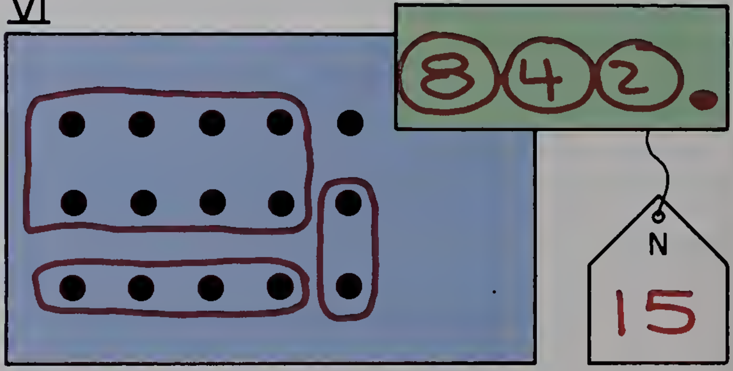
IV



V



VI



VII What Are My Rules?

□	1	4	7	13	8	10	19	20	0	28
□ + 2	3	6	9	15	10	12	21	22	2	30
□ + 12	13	16	19	25	20	22	31	32	12	40

Notes on Pupil Page 37

Tasks: *Pupils partition each display according to the instructions. They record the partitioning by writing special symbols in the green space. They record the count on the white tag. In exercise VII pupils fill the empty cells in the table.*

Purpose: (a) To build pupils' understanding of numeration systems; (b) to provide practice in adding 2, subtracting 2, adding 12, and subtracting 12.

Unifying Ideas: Number and counting; numeration; functions and relations.

Materials for Teacher: Containers of half-pint size, pint size, quart size, half-gallon size, and gallon size.

Materials for Pupils: None.

The Lesson:

The exercises on pupil page 37 are sufficiently like those on pupil page 36 that pupils should need no special preparation for pupil page 37. You may wish to follow up this page with an activity based on binary grouping. Here is such an activity.

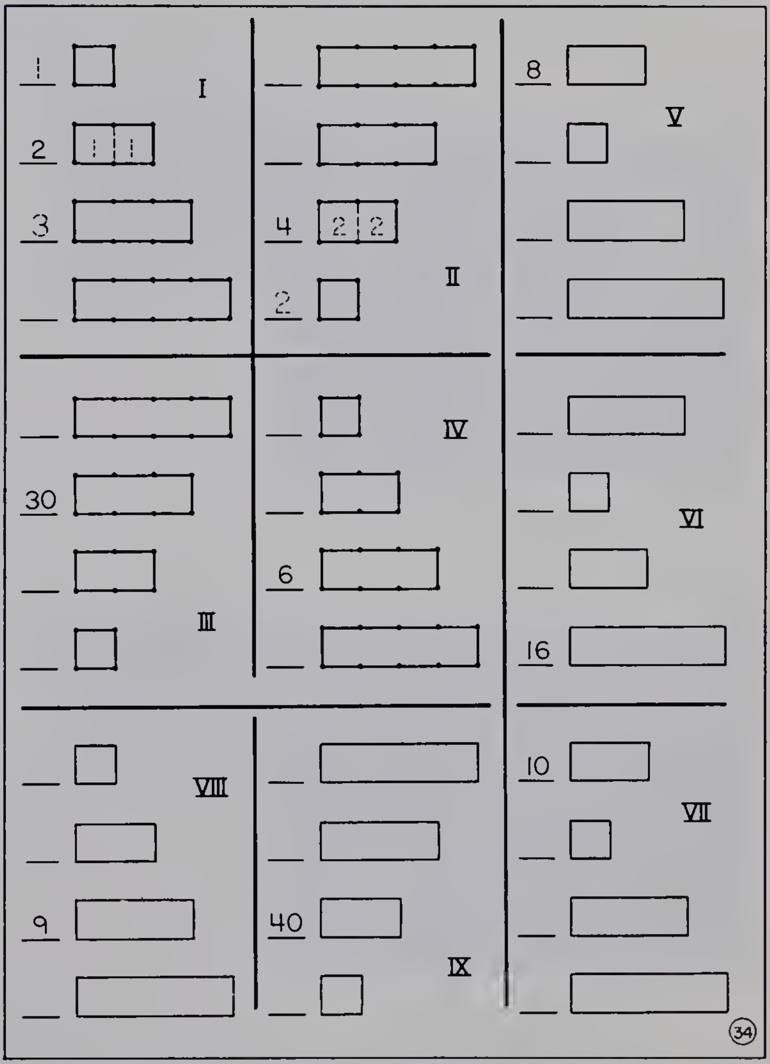
Fewest Containers

Amount of water	Containers				
	gallon	half-gal.	quart	pint	half-pint
1 glass	0	0	0	0	1
2 glasses	0	0	0	1	0
3 glasses	0	0	0	1	1
4 glasses	0	0	1	0	0
5 glasses	0	0	1	0	1
18 glasses	1	0	0	1	0
19 glasses	1	0	0	1	1
etc.					

Experiment with the containers mentioned above to help pupils see that the gallon container holds twice as much as the half-gallon container, that the half-gallon container holds twice as much as the quart, etc. Since the half-pint container holds just enough water to fill 1 glass, the pint container holds just enough to fill 2 glasses, etc. Determine for each of various amounts (measured in glasses) the smallest number of containers required to hold just that amount. The results should be recorded in a table like the one above. How many glasses would it take to fill one of each of the five containers? (16 + 8 + 4 + 2 + 1 = 31)

Drill

Consider using as a drill sheet page 34 of Math Workshop Supplementary Activities, Level B:

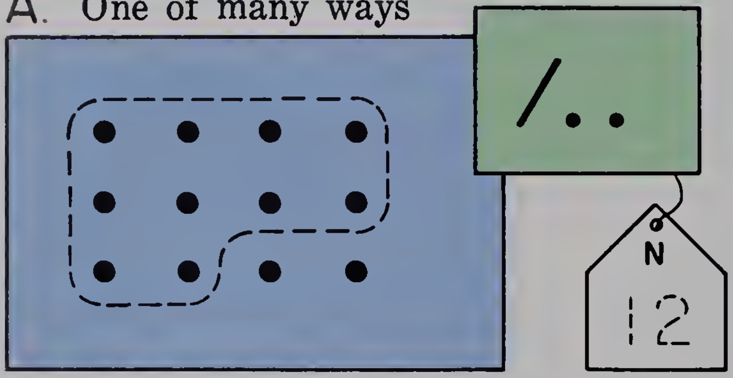


Headline for the Day: Make a table to help you change gallons to quarts.

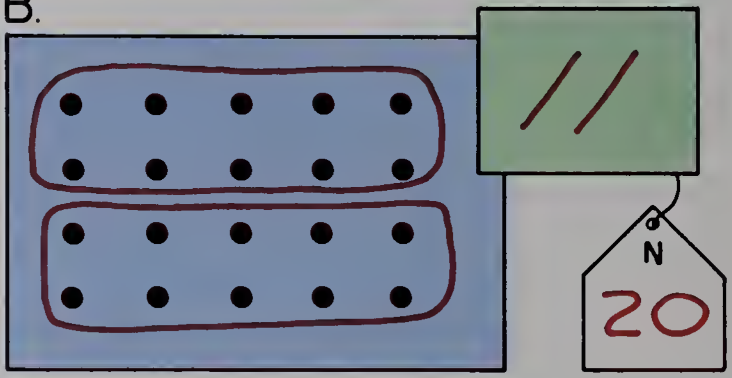
In each example on these two pages draw loops around groups of 10.

In green blocks, use / to indicate groups of 10 and ● to indicate objects not in groups of 10.

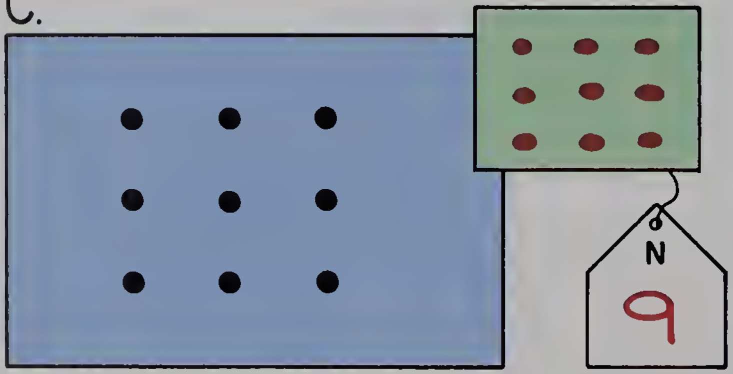
A. One of many ways



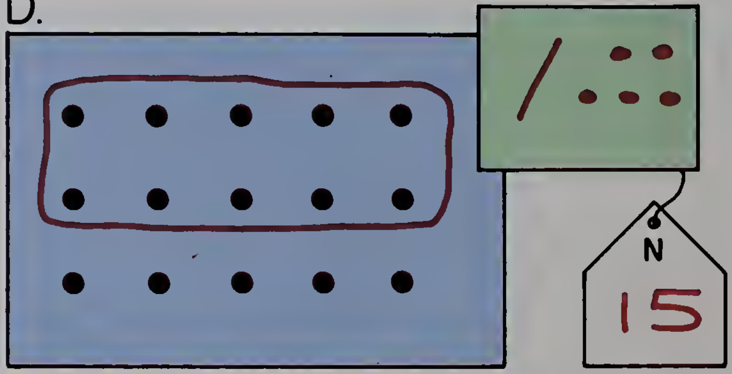
B.



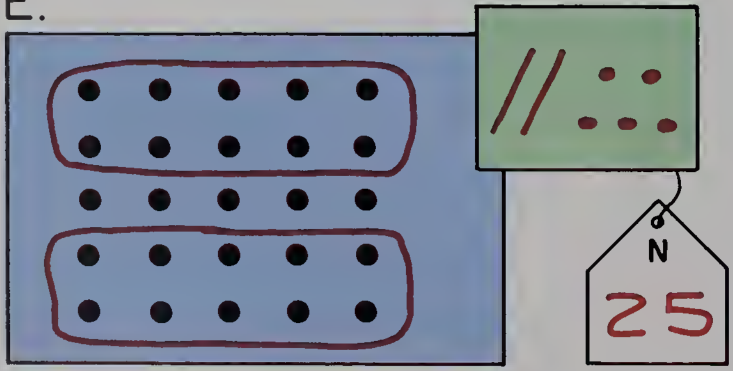
C.



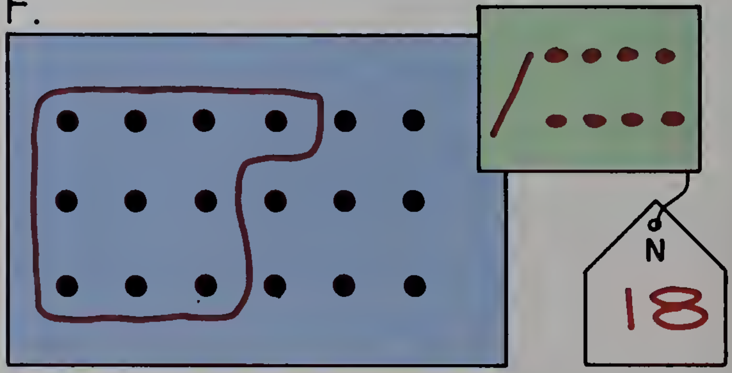
D.



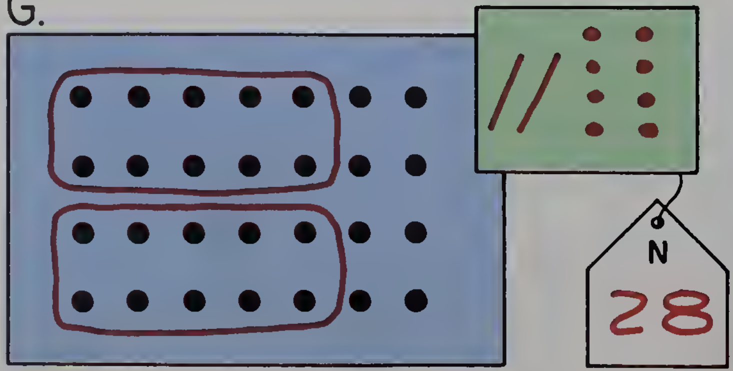
E.



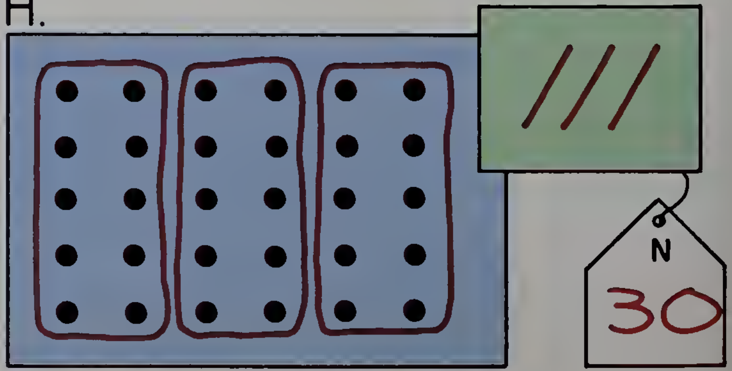
F.



G.



H.



Notes on Pupil Page 38

Tasks: *Pupils partition each given display according to the instructions at the top of the page. They record the partitioning by writing special symbols in the green space. They record the count in standard decimal notation on the white tag.*

Purpose: (a) To build pupils' understanding of numeration systems; (b) to illustrate how easy it is to move from the partitioning record to the number when the number is expressed in decimal notation and the partitioning is in terms of 10-groups and 1-groups.

Unifying Ideas: Number and counting; numeration.

Materials for Teacher: Abacus or counting frame.

Materials for Pupils: Cuisenaire rods.

The Lesson:

On pupil pages 38 and 39 the pupil is asked to partition displays into 10-groups and 1-groups. He then discovers that he can use the partitioning record in a direct way to record the number of objects in the given display. All one needs to do is write the number of 10-groups and, at the right of this, write the number of 1-groups. In case there are no 10-groups, as in exercise C, one merely writes the number of 1-groups.

Abacus

You may wish to follow up the work on pupil pages 38 and 39 by using an abacus in counting. Count the children in the class. Point to a child who then says "1," and move a bead into place on the right-hand rod of the abacus. Repeat until you reach 10. Then remove thet 10 beads and place 1 bead on the rod to the left. Make sure all pupils understand that this bead symbolizes a 10-group. Continue counting.

A bead on an abacus rod may signify a 1-group, a 10-group, a 100-group, etc. This makes the abacus presentation somewhat more abstract than the presentations on the pupil pages or those with Cuisenaire rods. Thus, the abacus presentation must be handled with great care. It should be accompanied by work with Cuisenaire rods along the lines suggested in the Notes on Pupil Page 36.

Drill

Here is a suggested drill sheet:

In any sentence with frames, you must write the same number in all frames of the same size and shape.

$8 = 5 + \square$

$\square + \square = 10$

$10 - 3 = \square$

$10 - 8 = \bigcirc$

$\bigcirc + \bigcirc = 8$

$18 - \square = \square$

$7 = 3 + \square + \square$

$\triangle + \triangle + \triangle = 9$

$10 = 3 + \triangle + \bigcirc$

$10 = \square + 8 + \square$

$10 = \square + \square$

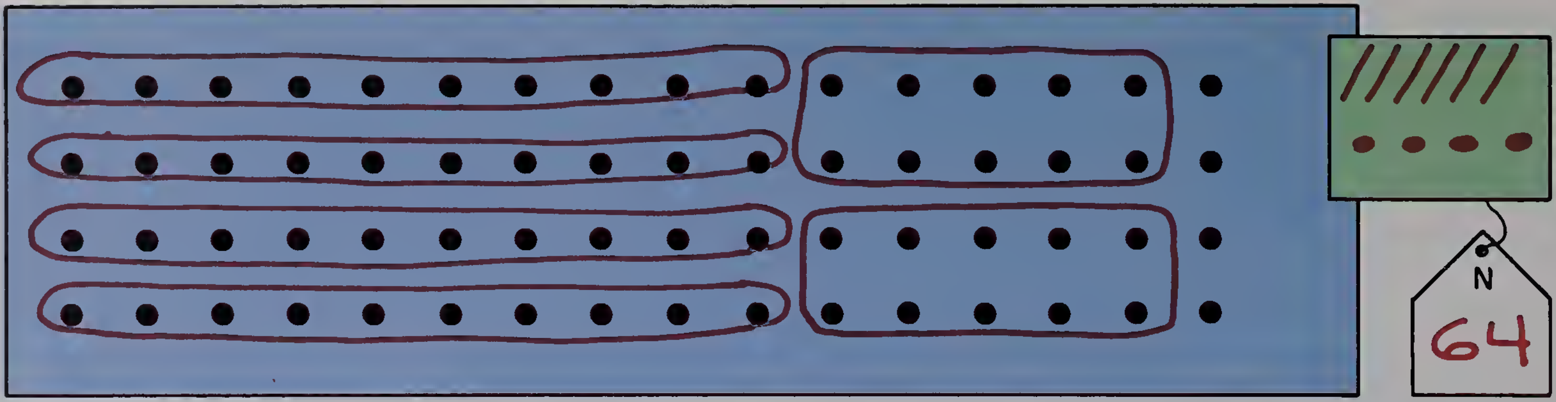
$10 - \square = \square$

$\square + \square + \square + \square = 4$

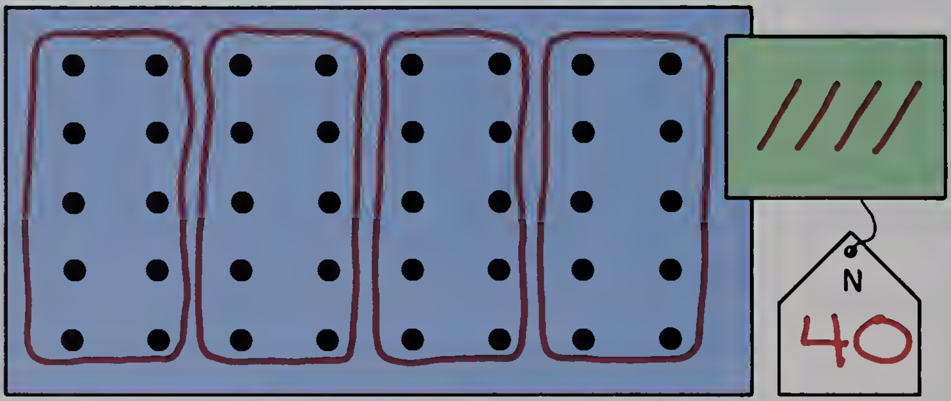
$\triangle + \triangle + \triangle + \triangle = 8$

Headline for the Day: John had 10 dimes. If he spent ____ dimes for a toy, how much money did he have left? (. . . . dimes or ___ cents.)

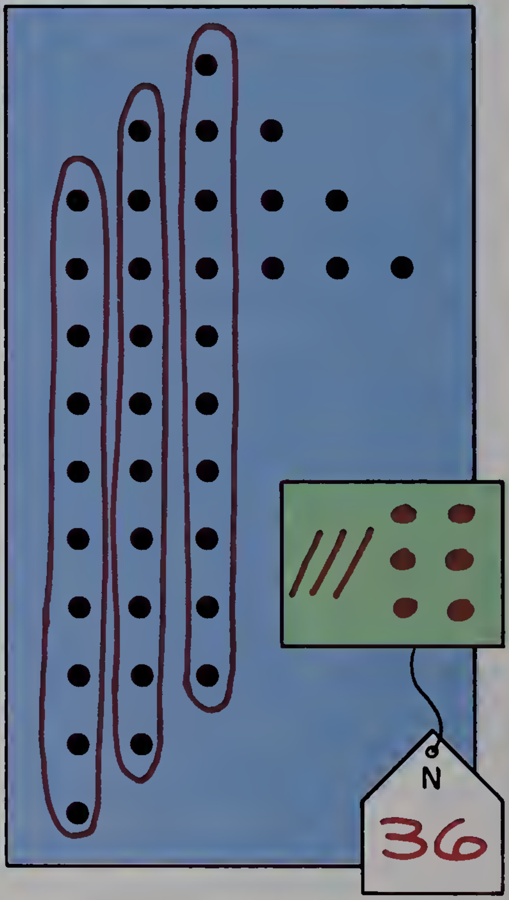
I.



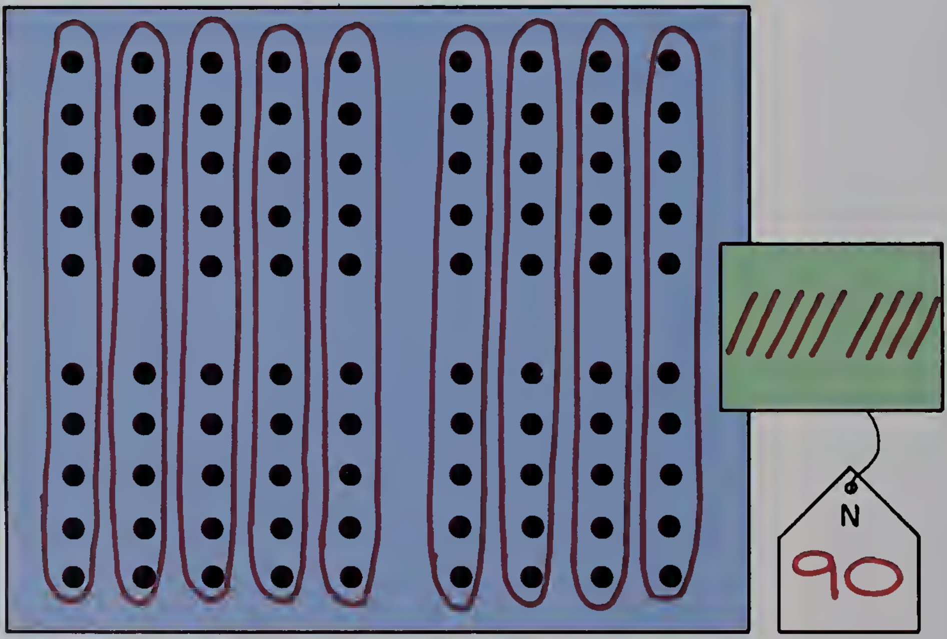
J.



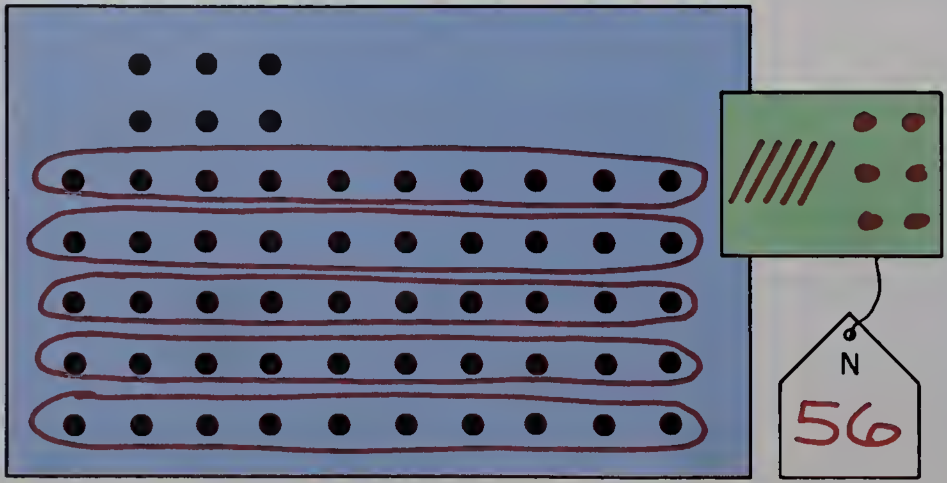
K.



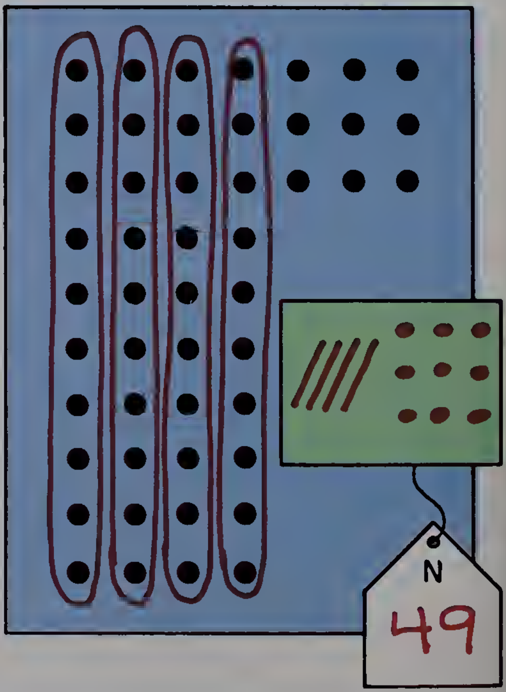
L.



M.



N.



Tasks: *Pupils partition each given display according to the instructions at the top of pupil page 38. They record the partitioning by writing special symbols in the green space. They record the count on the white tag.*

Purpose: (a) To build pupils' understanding of numeration systems; (b) to illustrate how easy it is to move from the partitioning record to the number when the number is expressed in decimal notation and the partitioning is in terms of 10-groups and 1-groups.

Unifying Ideas: Number and counting; numeration.

Materials for Teacher: Abacus or counting frame; Ditto masters.

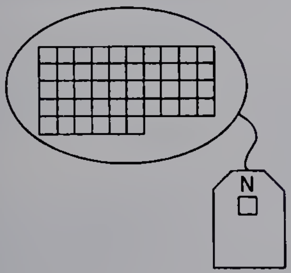
Materials for Pupils: Cuisenaire rods; Dittoed sheets.

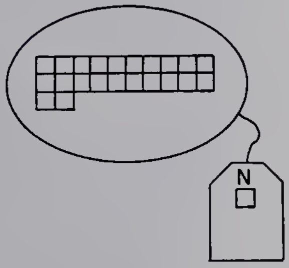
The Lesson:

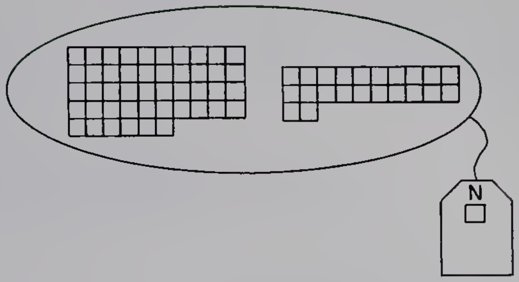
The remarks in the Notes on Pupil Page 38 apply here also.

Follow up pupil page 39 with Dittoed exercises like the following:

1)

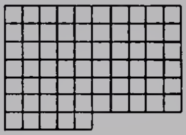






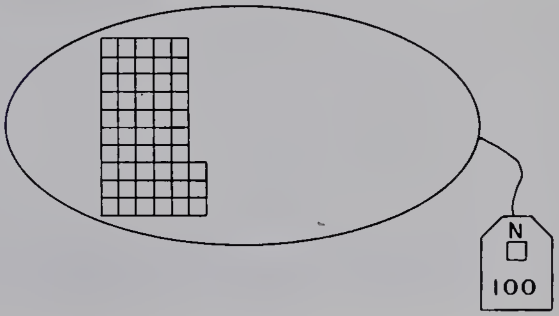
2) $\begin{array}{r} 35 \\ +\ 24 \\ \hline \end{array}$ $\begin{array}{r} 58 \\ +\ 21 \\ \hline \end{array}$ $\begin{array}{r} 17 \\ +\ 42 \\ \hline \end{array}$ $\begin{array}{r} 27 \\ +\ 41 \\ \hline \end{array}$

3) Shade 32 little squares and tell how many are left unshaded.



4) $\begin{array}{r} 59 \\ -\ 21 \\ \hline \end{array}$ $\begin{array}{r} 38 \\ -\ 22 \\ \hline \end{array}$ $\begin{array}{r} 27 \\ -\ 21 \\ \hline \end{array}$ $\begin{array}{r} 85 \\ -\ 35 \\ \hline \end{array}$

5) Draw more little squares to make 100 little squares in all.

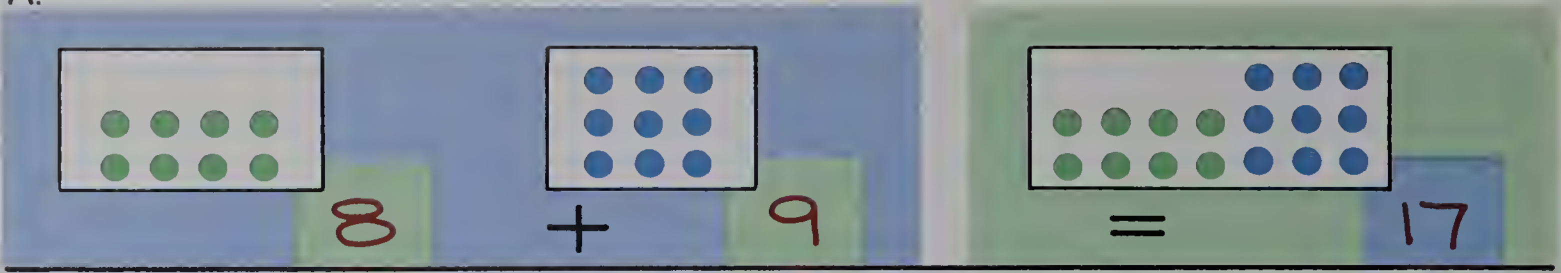


How many did you need to draw?

6) $\begin{array}{r} 63 \\ +\ \square \\ \hline 70 \end{array}$ $\begin{array}{r} 63 \\ +\ \square \\ \hline 80 \end{array}$ $\begin{array}{r} 63 \\ +\ \square \\ \hline 90 \end{array}$ $\begin{array}{r} 63 \\ +\ \square \\ \hline 100 \end{array}$

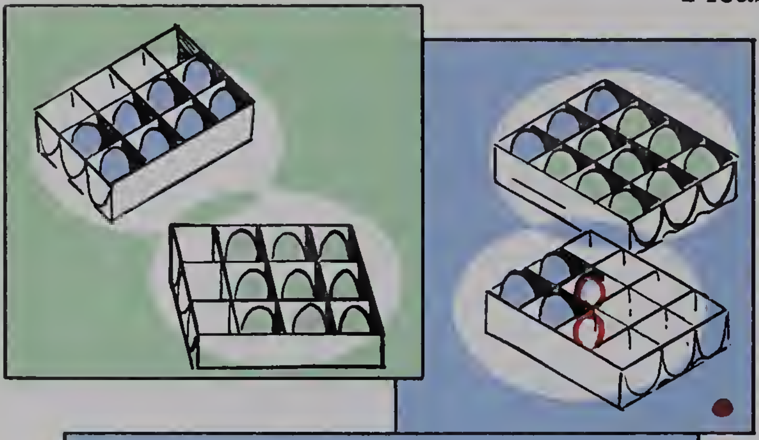
Headline for the Day: In how many ways can you make this sentence true using only even numbers (including 0)? ____ + ... = 10

A.



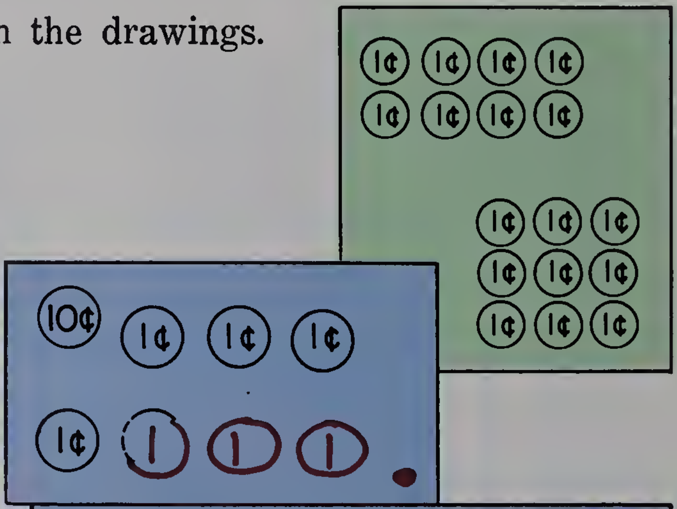
All about 8 + 9 = 17

B.



Please finish the drawings.

C.


1¢ 1¢ 1¢ 1¢
1¢ 1¢ 1¢ 1¢
1¢ 1¢ 1¢
1¢ 1¢ 1¢
1¢ 1¢ 1¢
10¢ 1¢ 1¢ 1¢
1¢ 1 1 1

1 dozen and 5 more eggs

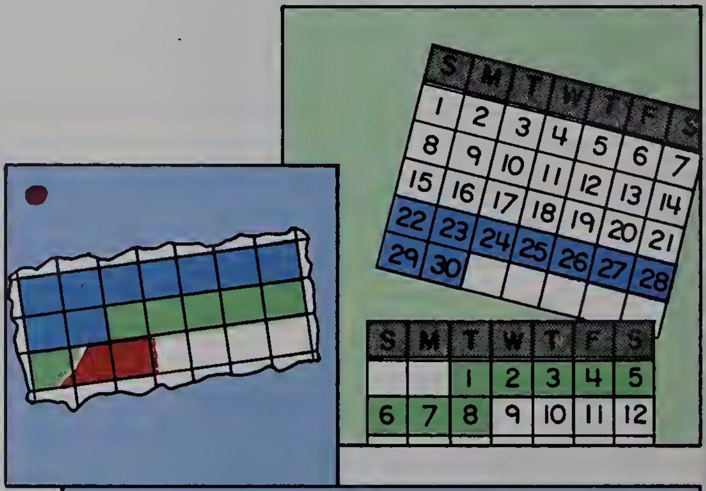
1 dime and 7 pennies

D.

An 8-ounce package and a 9-ounce package are weighed together. There are 16 ounces in a pound.

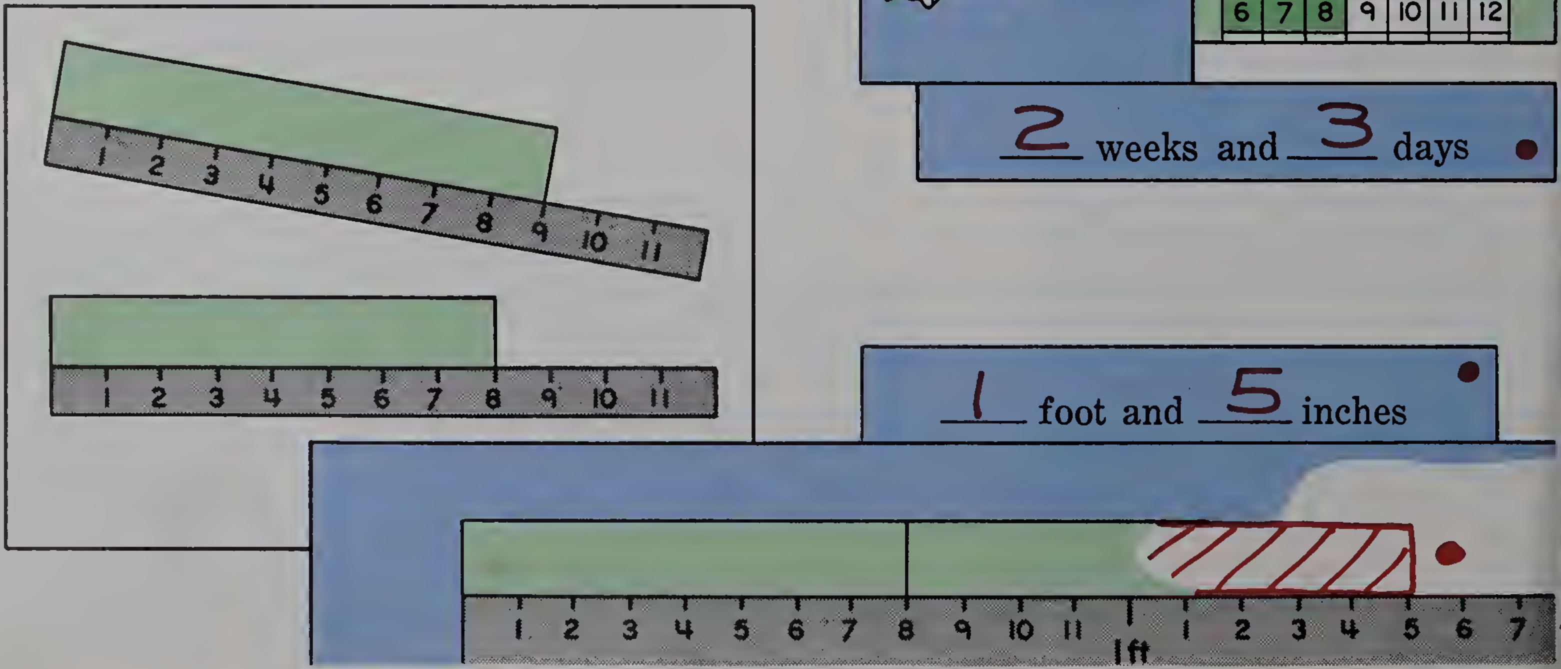
1 pound and 1 ounce

E.


S M T W T F S
1 2 3 4 5 6 7
8 9 10 11 12 13 14
15 16 17 18 19 20 21
22 23 24 25 26 27 28
29 30
S M T W T F S
1 2 3 4 5
6 7 8 9 10 11 12

2 weeks and 3 days

F.


1 2 3 4 5 6 7 8 9 10 11
1 2 3 4 5 6 7 8 9 10 11
1 2 3 4 5 6 7 8 9 10 11 1ft 1 2 3 4 5 6 7

1 foot and 5 inches

Notes on Pupil Page 40

Tasks: *In exercise A pupils record the number of dots displayed on each of the three white fields. In exercise B they complete the picture in the blue field to show a total of 17 eggs and then express this total in terms of dozens and extra eggs. In exercise C they complete the picture in the blue field to show coins totaling 17 cents and then express this total in terms of dimes and extra pennies. In exercise D pupils express a weight of 17 ounces in terms of pounds and extra ounces. In exercise E they complete the drawing in the blue field to show a total of 17 days and then express this total period in terms of weeks and extra days. In exercise F they complete the drawing at the very bottom of the page to show a length of 17 inches and then express this length in terms of feet and extra inches.*

Purpose: (a) To test pupils' knowledge of units of measurement; (b) to demonstrate that grouping or partitioning is used in everyday life.

Unifying Ideas: Number and counting; numeration; measurement.

Materials for Teacher: Flannel board; counters; egg cartons; demonstration-size calendar; 12-inch ruler; yardstick; play money; Ditto masters.

Materials for Pupils: Dittoed sheets.

The Lesson:

This page tests pupils' competence in the activity started on pupil page 35. Suggested point distribution: A–1, B–2, C–2, D–1, E–2, F–2.

Note the various ways in which displays of 17 objects are partitioned.

In exercise B the egg carton itself serves as a basis for the partitioning.

In exercise C the separation is carried out in a more subtle way—the dime represents a 10-group.

In exercise D the partitioning is to be carried out on a purely mental level. (If one had a scale with a circular dial that listed ounces up to 16, it would be possible to see the partitioning. The pointer would go once around and then point to the 1 on the dial.)

In exercise E each row of the calendar page is a 7-group.

In exercise F the foot markings on the yardstick show the 12-groups.

Consider the following Dittoed exercises as follow-up work for the test on pupil page 40.

1) 5 days and 3 days
= 5 days and (2 days and ___ day)
= (5 days and 2 days) and ___ day
= ___ week and ___ day

2) 12 ounces and 11 ounces
= 12 ounces and (4 ounces and ___ ounces)
= (12 ounces and . . . ounces) and 7 ounces
= ___ pound and 7 ounces

3) 15 inches and 14 inches
= (___ foot and 3 inches) and (___ foot and . . . inches)
= (___ foot and ___ foot) and (3 inches and . . . inches)
= ___ feet and ___ inches

4) 8 + 6
= 8 + (2 + ___)
= (8 + 2) + ___
= 10 + ___
= . . .

5) 27 + 7
= 27 + (___ + 4)
= (27 + ___) + 4
= . . . + 4
= ___

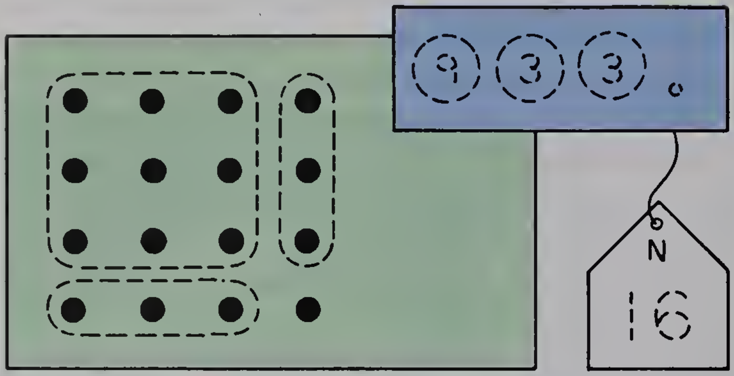
Headline for the Day: Make a table to help you change feet to inches.

Draw loops around groups of 9. Then draw loops around groups of 3 not already in a group of 9. No dot can be in two groups.

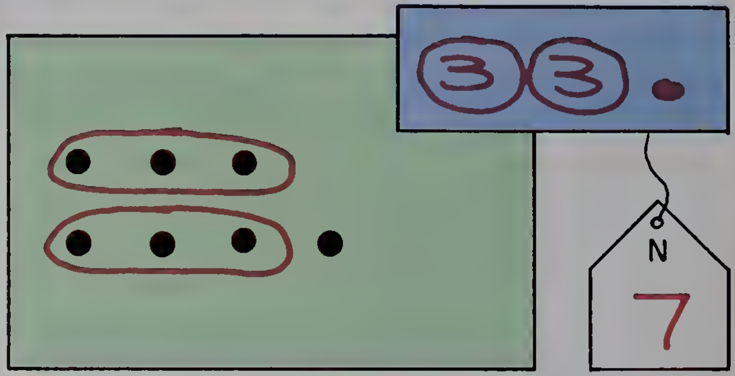
Record results:

⑨ . . . . a group of 9
③ . . . . a group of 3
. . . . .dots not in a group

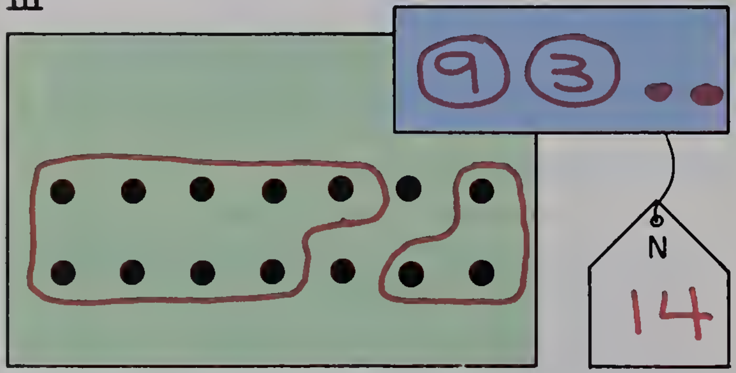
I One way of grouping



II



III



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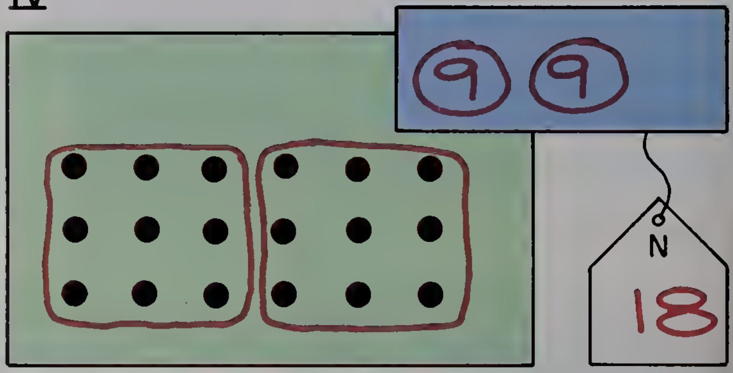
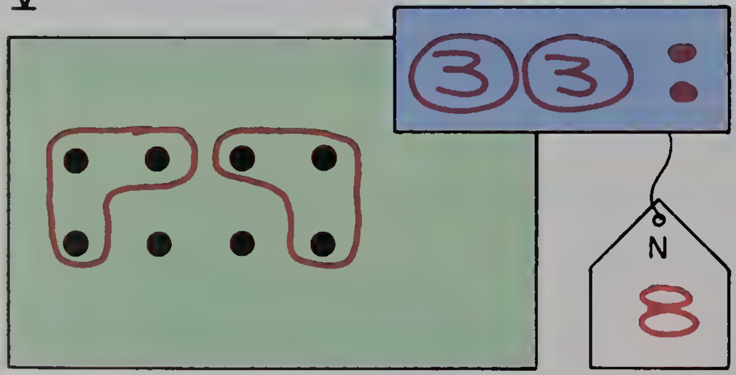
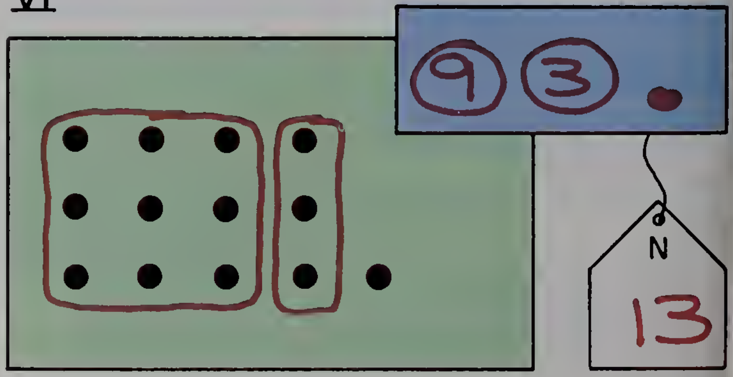


V



VI



What Are My Rules?

VII

△ - 3	12	17	20	26	27	18	29	10	23	37
△	15	20	23	29	30	21	32	13	26	40
△ - 13	2	7	10	16	17	8	19	0	13	27

Notes on Pupil Page 41

Tasks: *Pupils partition each given display according to the instructions. They record the partitioning by writing special symbols in the green space. They record the count on the white tag. In exercise VII pupils demonstrate their discovery of the rules by filling in the empty cells in the table.*

Purpose: (a) To build pupils' understanding of numeration systems; (b) to provide pupils with practice in adding 3, subtracting 3, adding 13, and subtracting 13.

Unifying Ideas: Number and counting; numeration; functions and relations.

Materials for Teacher: Ditto masters.

Materials for Pupils: Dittoed sheets.

The Lesson:

This page may be assigned without special preparation.

Note that exercise VII is planned so that pupils will become aware of the fact that subtracting 13 amounts to subtracting 3 and then subtracting 10. It is easy to make up *What Are My Rules?* games to illustrate similar principles as follows:

△	5	9	13	17	23	27	33	37	58
△ + 2	7	11							
△ + 12	17	21							

a	19	27	32	39	45	52	61	73
$a - 4$	15	23	28					
$a - 14$	5	13						

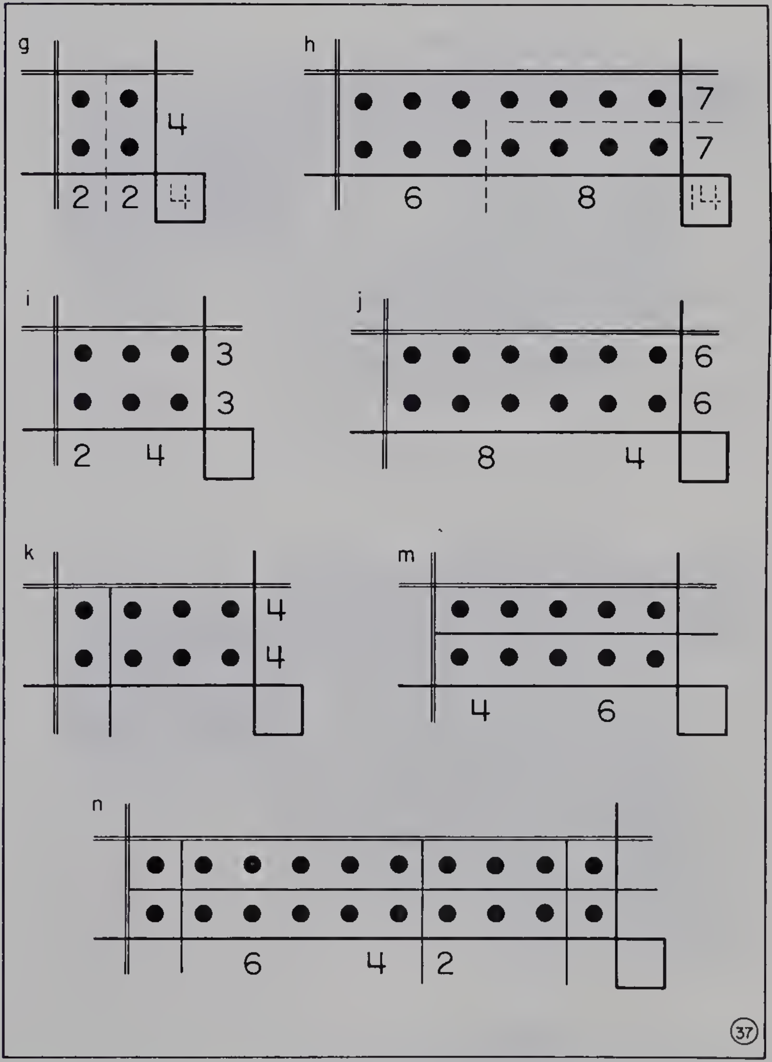
□	18	27	39	48	57	66	77	84	94
□ − 16	2	11							

x	2	4	5	16	24	32	37	87
$x + 10$	12	14						
$x + 13$	15	17						

y	1	2	21	42	52	61	73	84	94
$y + 17$	18	19	38						

Drill

Consider using as a drill sheet page 37 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:

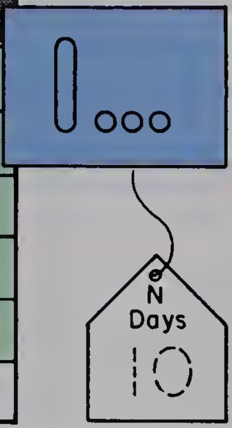


Headline for the Day: Make a table to help you change yards to feet.

The symbol ▯ means 7 days, the symbol ○ means 1 day. Record in the blue space the total number of days colored green. Use the least number of symbols possible in each case.

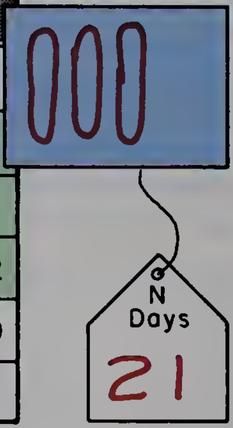
A. JANUARY

S	M	T	W	T	F	S
					🟩 1	🟩 2
🟩 3	4	5	6	7	8	🟩 9
🟩 10	11	12	13	14	15	🟩 16
🟩 17	18	19	20	21	22	🟩 23
🟩 24	25	26	27	28	29	🟩 30
31						



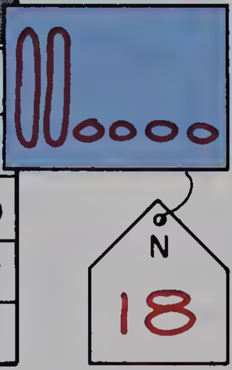
B. MAY

S	M	T	W	T	F	S
						1
🟩 2	🟩 3	🟩 4	🟩 5	🟩 6	🟩 7	🟩 8
🟩 9	🟩 10	🟩 11	🟩 12	🟩 13	🟩 14	🟩 15
🟩 16	🟩 17	🟩 18	🟩 19	🟩 20	🟩 21	🟩 22
23	24	25	26	27	28	29
30	31					



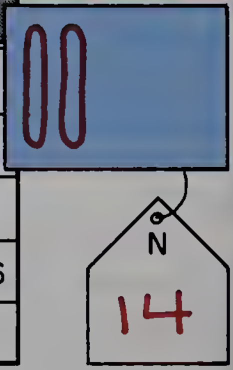
C. FEBRUARY

S	M	T	W	T	F	S
	🟩 1	🟩 2	🟩 3	🟩 4	🟩 5	6
7	🟩 8	🟩 9	🟩 10	🟩 11	12	13
14	🟩 15	🟩 16	🟩 17	🟩 18	🟩 19	20
21	22	🟩 23	🟩 24	🟩 25	🟩 26	27
28						



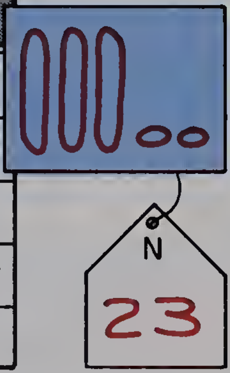
D. JUNE

S	M	T	W	T	F	S
		🟩 1	🟩 2	🟩 3	🟩 4	5
6	🟩 7	🟩 8	🟩 9	🟩 10	🟩 11	12
13	🟩 14	🟩 15	🟩 16	🟩 17	🟩 18	19
20	21	22	23	24	25	26
27	28	29	30			



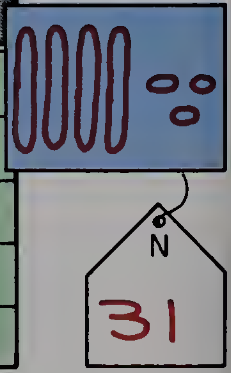
E. MARCH

S	M	T	W	T	F	S
	🟩 1	🟩 2	🟩 3	🟩 4	🟩 5	6
7	🟩 8	🟩 9	🟩 10	🟩 11	🟩 12	13
14	🟩 15	🟩 16	🟩 17	🟩 18	🟩 19	20
21	🟩 22	🟩 23	🟩 24	🟩 25	🟩 26	27
28	🟩 29	🟩 30	🟩 31			



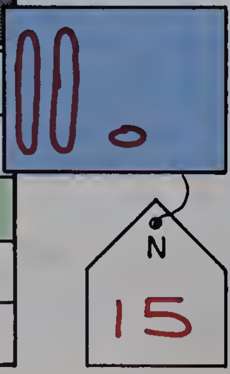
F. JULY

S	M	T	W	T	F	S
				🟩 1	🟩 2	🟩 3
🟩 4	🟩 5	🟩 6	🟩 7	🟩 8	🟩 9	🟩 10
🟩 11	🟩 12	🟩 13	🟩 14	🟩 15	🟩 16	🟩 17
🟩 18	🟩 19	🟩 20	🟩 21	🟩 22	🟩 23	🟩 24
🟩 25	🟩 26	🟩 27	🟩 28	🟩 29	🟩 30	🟩 31



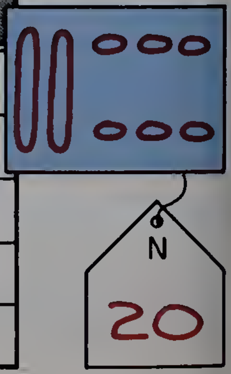
G. APRIL

S	M	T	W	T	F	S
				🟩 1	2	🟩 3
4	🟩 5	6	🟩 7	8	🟩 9	10
🟩 11	12	🟩 13	14	🟩 15	16	🟩 17
18	🟩 19	20	🟩 21	22	🟩 23	24
🟩 25	26	🟩 27	28	🟩 29	30	



H. AUGUST

S	M	T	W	T	F	S
1	🟩 2	🟩 3	🟩 4	🟩 5	🟩 6	7
8	🟩 9	🟩 10	🟩 11	🟩 12	🟩 13	14
15	🟩 16	🟩 17	🟩 18	🟩 19	🟩 20	21
22	🟩 23	🟩 24	🟩 25	🟩 26	🟩 27	28
29	30	31				



Notes on Pupil Page 42

Tasks: *Pupils write special symbols in the blue field to indicate the number of 7-groups and 1-groups into which the display of green-marked days can be partitioned. Pupils then record on the white tag the number of green-marked days.*

Purpose: (a) To extend pupils' understanding of numeration; (b) to show pupils that certain arrangements of displays can simplify the job of partitioning (exercises B and F are very easy).

Unifying Ideas: Number and counting; numeration; measurement.

Materials for Teacher: Demonstration-size calendar.

Materials for Pupils: None.

The Lesson:

When pupils complete the page, ask them to check their answers by counting the unmarked days in each exercise. Let the pupils tell you how this will help them in checking their answers. (Exercise G is an interesting one for checking. How many pupils are able to predict that there must be as many marked as unmarked days?)

We hope that pupils will use their knowledge of rectangular arrays in doing exercises C, D, E, and H. In C the number of marked days is 2 less than 4-by-5; in D the number is 1 less than 3-by-5; in E the number is 3 more than 4-by-5; in H the number is 4-by-5.

A Calendar Game

Consider using the calendars on pupil page 42 to play the *One Week Ago-One Week From Now* game:

One Week Ago	*Present Date*	*One Week From Now*
Jan. 2	Jan. 9	Jan. 16
	Jan. 14	
	Jan. 26	
	Feb. 10	
	Feb. 25	
	etc.	

Drill

Here is a suggestion for a drill sheet:

15 − 9 = ___
11 − 2 = ___
14 − 9 = ___
15 − 6 = ___
12 − 5 = ___
14 − 6 = ___
15 − 8 = ___
13 − 6 = ___
11 − 3 = ___
17 − 8 = ___

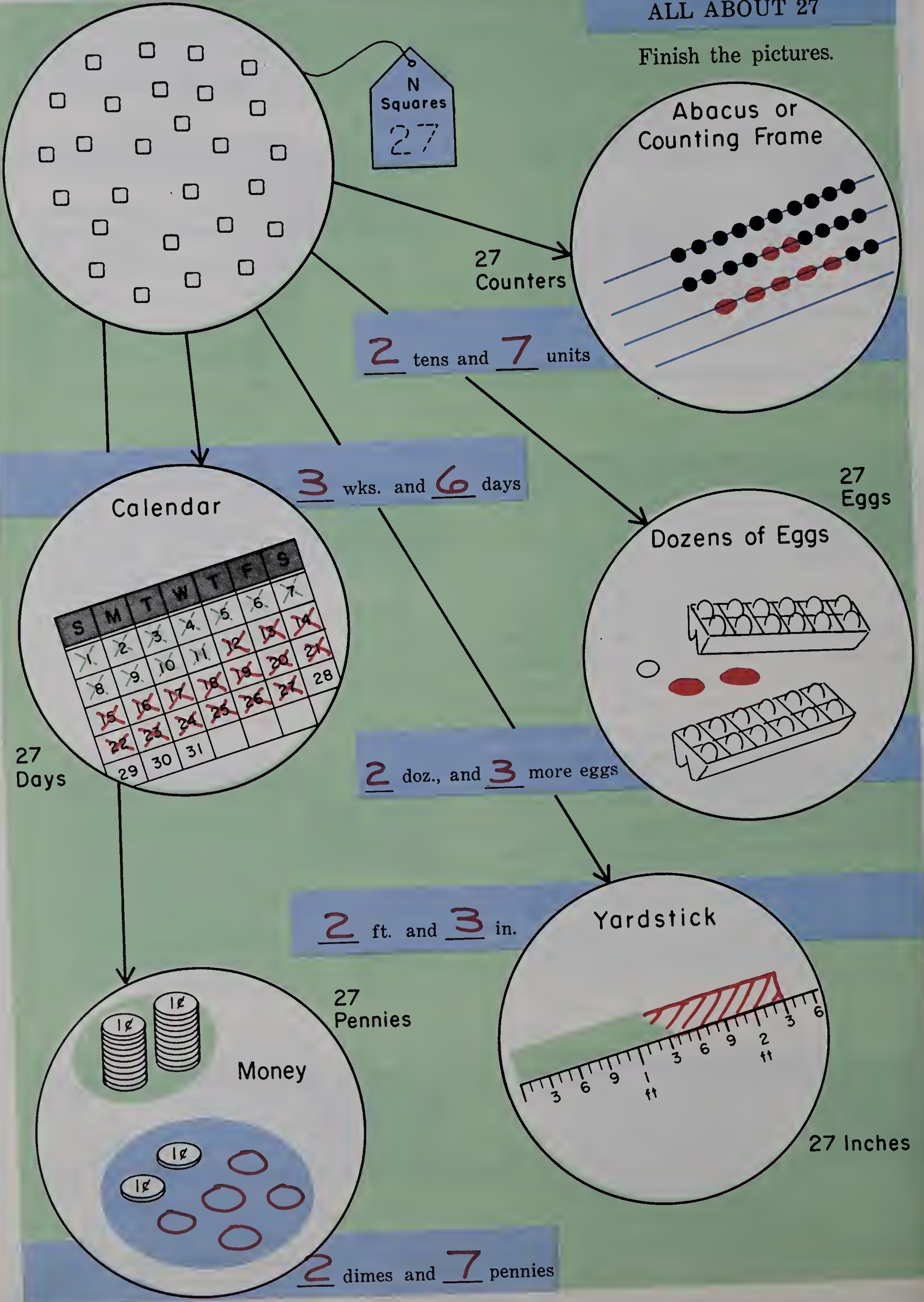
13 − 7 = ___
17 − 9 = ___
10 − 8 = ___
16 − 8 = ___
16 − 9 = ___
11 − 8 = ___
10 − 5 = ___
10 − 7 = ___
12 − 7 = ___
12 − 4 = ___

12 − 9 = ___
11 − 5 = ___
12 − 8 = ___
10 − 3 = ___
13 − 4 = ___
11 − 7 = ___
15 − 7 = ___
11 − 4 = ___
13 − 9 = ___
13 − 8 = ___

Headline for the Day: I have a dollar in coins. What combinations do I have if I have 4 coins? 7 coins? 30 coins? 51 coins? etc.

ALL ABOUT 27

Finish the pictures.


N Squares
27
Abacus or Counting Frame
27 Counters
Calendar
S M T W T F S
1 2 3 4 5 6 7
8 9 10 11 12 13 14
15 16 17 18 19 20 21
22 23 24 25 26 27 28
29 30 31
27 Days
Dozens of Eggs
27 Eggs
Yardstick
3 6 9 1 ft
3 6 9 2 ft
3 6
27 Inches
Money
1¢
27 Pennies

2 tens and 7 units

3 wks. and 6 days

2 doz., and 3 more eggs

2 ft. and 3 in.

2 dimes and 7 pennies

Tasks: Pupils complete the drawings and record totals.

Purpose: To extend pupils' understanding of numeration and units of measurement.

Unifying Ideas: Number and counting; numeration; measurement.

Materials for Teacher: Flannel board; counting frame; demonstration calendar; egg cartons; yardstick; play money; Ditto masters.

Materials for Pupils: Dittoed sheets.

The Lesson:

This page shows pupils some of the conventional ways of reporting the number of objects in a 27-group. If the objects are counters or beads on an abacus, one might say that there are two 10-groups and seven 1-groups. If the objects are days, one might report that there are 3 weeks and 6 days in 27 days, etc.

Measurement

It has been said that in arithmetic we search for answers to but a pair of simple questions: How many? and How much? When confronted with How many? we count; when we want to know How much? we measure.

This argument leads to the further fact that when we count we are exact, but when we measure we can never be exact. The latter is necessarily true because measurements are made by fallible men with man-made instruments, and so can be only approximate.

more

Headline for the Day: Write number sentences for the following:
I traded a half-dollar for two quarters. I spent 16¢; 10¢ went for an ice cream cone.

Every scientist and observer of the real world knows well this simple truth and is appropriately humble.

But school arithmetic is basically concerned only with the answer to How many? The techniques and problems of making measurements, fashioning instruments, and observing results take us beyond school arithmetic.

Yet it must be pointed out immediately that school arithmetic plays a fundamental role in measurement. No matter what you measure or what instruments you use, the final report will reflect counting of units. Thus we must be vitally concerned with the applications of arithmetic in the area of measurement.

The *All About . . .* pages are concerned mainly with counting of units of measure and recording the counts.

Unfortunately, we must become familiar with many different systems of measurement and the peculiar arithmetic involved. For example:

50 inches is the same as 1 yard, 1 foot, and 2 inches.

90 minutes is the same as 1½ hours or 1 hour and 30 minutes.

10 days is the same as 1 week and 3 days.

2 pints is the same as 1 quart.

13 eggs is the same as 1 dozen eggs and 1 egg.

In each of these conversions we must know the grouping pattern for a particular system of measurement. As we consider these systems, our main concern should be with the grouping patterns unique to each.

Computing with Measurements

Consider the following computations:

$$\begin{array}{r} 1\ ,\ 3 \\ +\ 1\ ,\ 2 \\ \hline 2\ ,\ 5 \end{array} \qquad \begin{array}{r} 1\ ,\ 6 \\ +\ 1\ ,\ 4 \\ \hline 3\ ,\ 3 \end{array} \qquad \begin{array}{r} 3\ ,\ 0 \\ -\ 1\ ,\ 4 \\ \hline 1\ ,\ 3 \end{array}$$

This would be very strange arithmetic to most people, at least until they were let in on the secret that the computer is thinking about weeks and days:

$$\begin{array}{rl} & \text{1 week and 3 days} \\ + & \text{1 week and 2 days} \\ \hline & \text{2 weeks and 5 days} \end{array}$$

$$\begin{array}{rl} & \text{1 week and 6 days} \\ + & \text{1 week and 4 days} \\ \hline & \text{3 weeks and 3 days} \end{array}$$

$$\begin{array}{rl} & \text{3 weeks and 0 days} \\ - & \text{1 week and 4 days} \\ \hline & \text{1 week and \_\_ days} \end{array}$$

Following are suggestions for Dittoed exercises. In these exercises a shorthand is used. For example, 1, 4 means 1 week and 4 days, or 11 days.

In this weeks-days shorthand and using only the numbers 0, 1, 2, 3, 4, 5, or 6, write the following:

a. 4 days 0, 4
b. 7 days 1, 0
c. 10 days .... ___,___
d. 1 week and 4 days............ ___,___
e. 15 days .... ___,___
f. 1 week and 10 days........... ___,___
g. 20 days .... ___,___
h. 21 days .... ___,___
i. 2 weeks and 9 days............ ___,___

$$\begin{array}{r} 1\ ,\ 3 \\ +\ 0\ ,\ 5 \\ \hline 2\ ,\ 1 \end{array}$$

This last bit of shorthand says that 1 week and 3 days and 0 weeks and 5 days is the same as 2 weeks and 1 day. Complete the following:

j. $\begin{array}{r} 1\ ,\ 3 \\ +\ 1\ ,\ 1 \\ \hline \end{array}$

k. $\begin{array}{r} 0\ ,\ 4 \\ +\ 1\ ,\ 3 \\ \hline \end{array}$

l. $\begin{array}{r} 1\ ,\ 2 \\ +\ 1\ ,\ 6 \\ \hline \end{array}$

m. $\begin{array}{r} 2\ ,\ 4 \\ +\qquad \\ \hline 3\ ,\ 2 \end{array}$

n. $\begin{array}{r} \\ +\ 2\ ,\ 5 \\ \hline 4\ ,\ 3 \end{array}$

o. $\begin{array}{r} 3\ ,\ 1 \\ +\qquad \\ \hline 4\ ,\ 0 \end{array}$

p. $\begin{array}{r} 1\ ,\ 2 \\ 2\ ,\ 1 \\ +\ 1\ ,\ 3 \\ \hline \end{array}$

q. $\begin{array}{r} 0\ ,\ 4 \\ 1\ ,\ 3 \\ +\ 2\ ,\ 2 \\ \hline \end{array}$

This shorthand can also be used for subtraction examples:

r. $$\begin{array}{r} 3 , 5 \\ -\ 1 , 4 \\ \hline \end{array}$$

s. $$\begin{array}{r} 1 , 2 \\ -\ 0 , 4 \\ \hline \end{array}$$

t. $$\begin{array}{r} 2 , 3 \\ - \\ \hline 1 , 0 \end{array}$$

u. $$\begin{array}{r} \\ -\ 2 , 5 \\ \hline 0 , 2 \end{array}$$

Here are examples of both addition and subtraction:

Watch The Signs!

A. $$\begin{array}{r} 3 , 4 \\ -\ 1 , 1 \\ \hline \end{array}$$

B. $$\begin{array}{r} 1 , 1 \\ + \\ \hline 3 , 4 \end{array}$$

C. $$\begin{array}{r} 2 , 3 \\ +\ 0 , 3 \\ \hline \end{array}$$

D. $$\begin{array}{r} 3 , 6 \\ - \\ \hline 0 , 3 \end{array}$$

E. $$\begin{array}{r} 1 , 0 \\ -\ 0 , 2 \\ \hline \end{array}$$

F. $$\begin{array}{r} 2 , 0 \\ -\ 0 , 2 \\ \hline \end{array}$$

G. $$\begin{array}{r} 3 , 1 \\ -\ 0 , 2 \\ \hline \end{array}$$

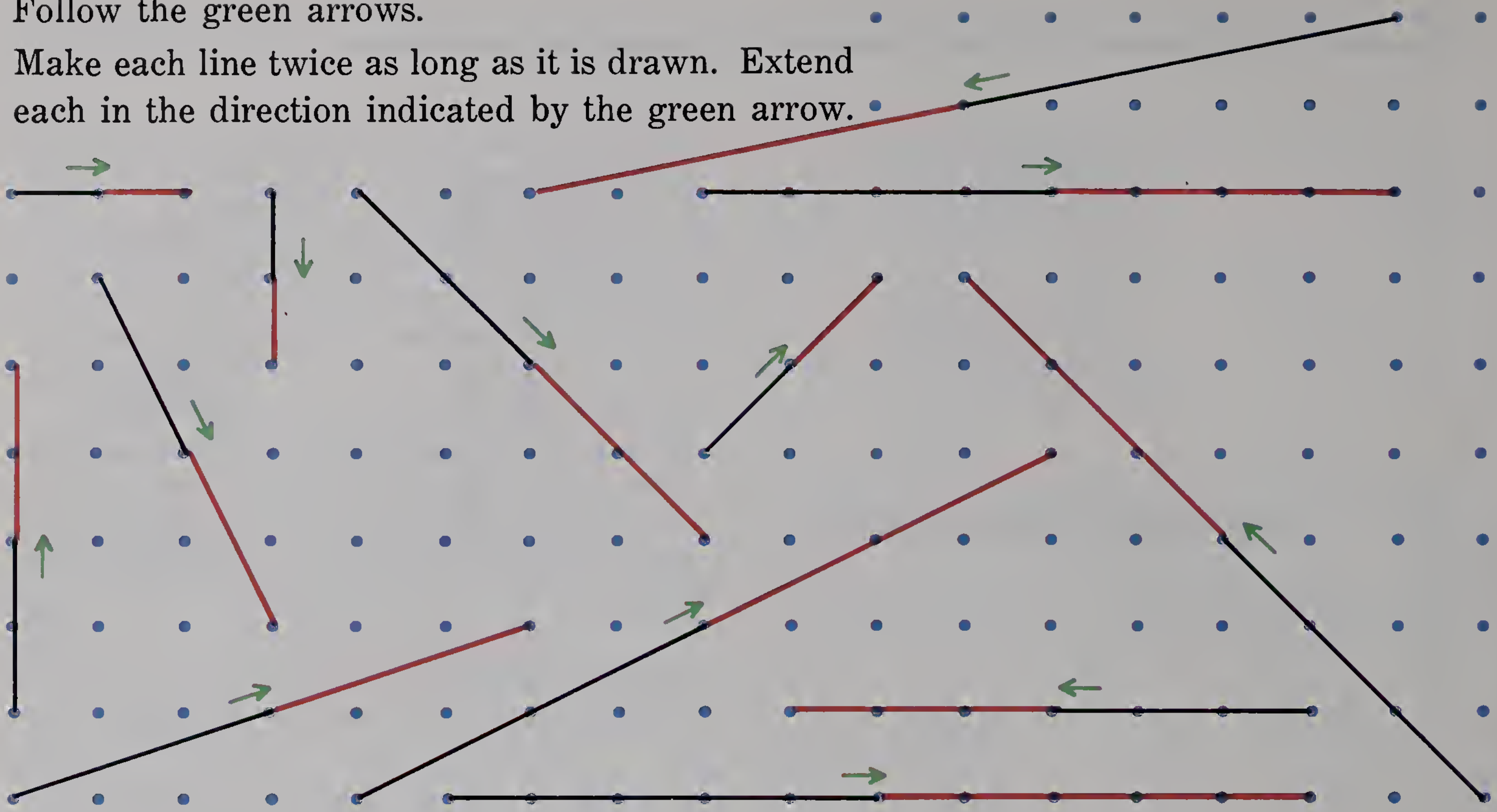
H. $$\begin{array}{r} \\ -\ 2 , 4 \\ \hline 1 , 6 \end{array}$$

I. $$\begin{array}{r} 1 , 6 \\ +\ 1 , 6 \\ \hline \end{array}$$

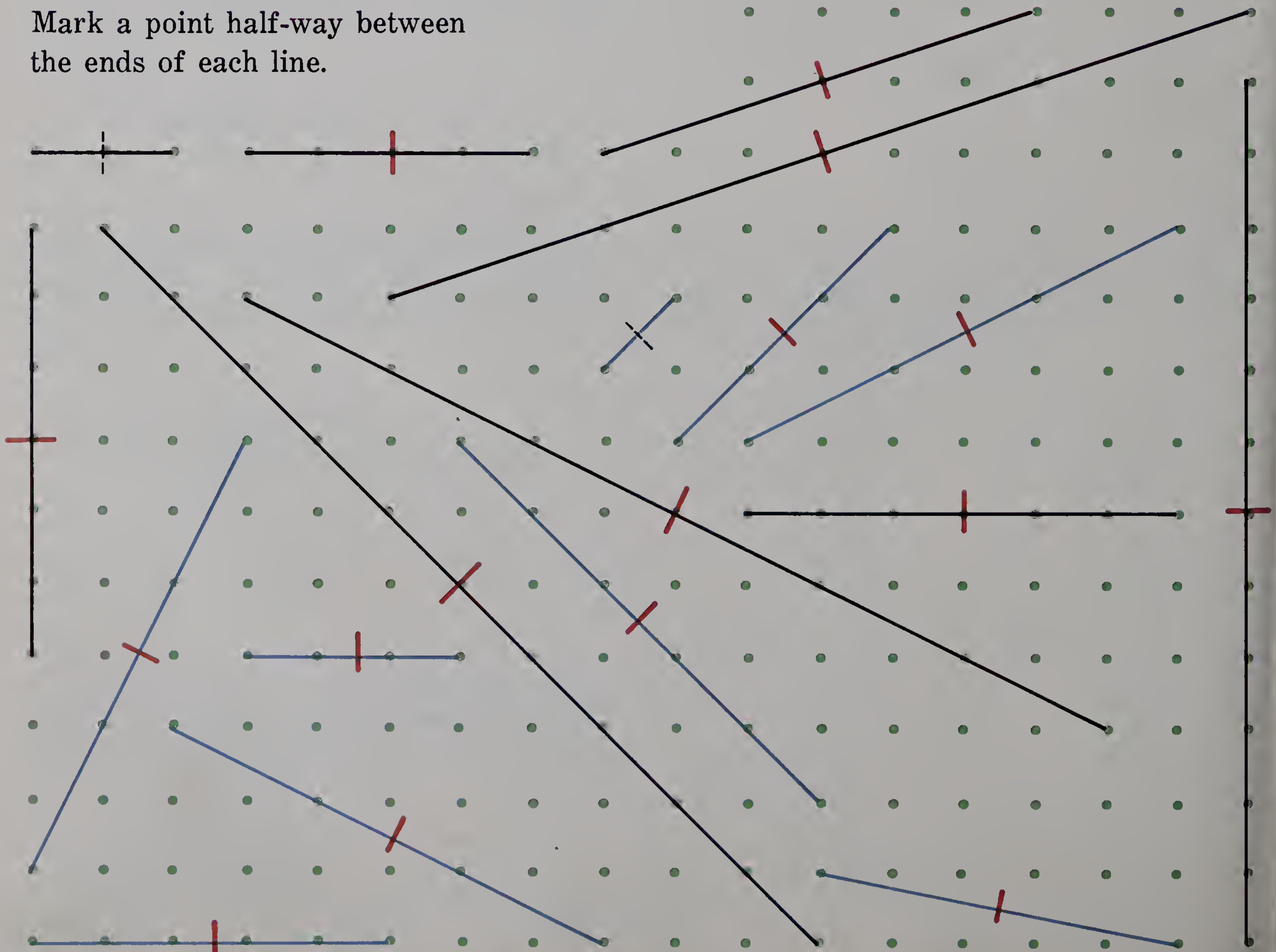
J. $$\begin{array}{r} \\ -\ 1 , 5 \\ \hline 1 , 5 \end{array}$$

Follow the green arrows.

Make each line twice as long as it is drawn. Extend each in the direction indicated by the green arrow.



Mark a point half-way between the ends of each line.



Notes on Pupil Page 44

Tasks: In the upper section pupils extend each given line segment to make one twice as long. In the lower section pupils locate the midpoint of each given line segment.

Purpose: To introduce pupils to measurement of line segments involving doubling and halving.

Unifying Ideas: Geometry; measurement.

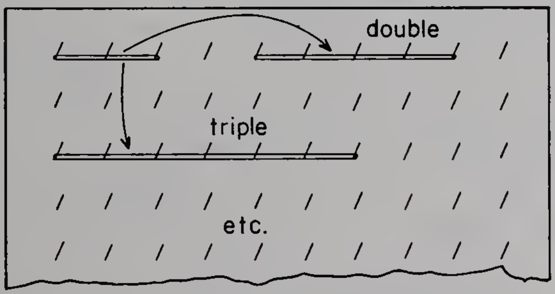
Materials for Teacher: An overhead projector; a transparency with lattice points marked on it as pictured on pupil page 44.

Materials for Pupils: Rulers; Counting Boards or Geoboards.

The Lesson:

Pupils are to carry out these exercises by counting spaces between dots. They may use the scale on a ruler if they wish to, but there is no need to do this.

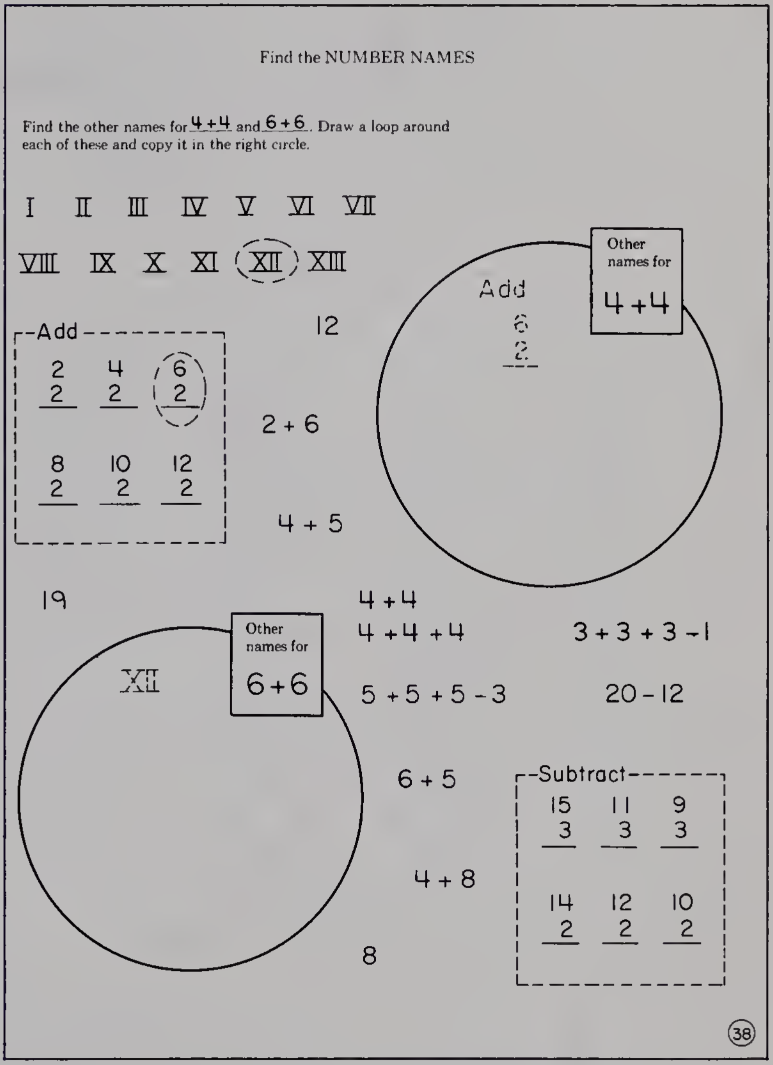
You may wish to prepare pupils for this page by carrying out several of the exercises on the overhead projector. Another preparatory activity involves the use of a Counting Board described in the Notes on Pupil Page 33. By using rubber bands pupils can demonstrate doubling and tripling of line segments as well as halving and otherwise partitioning a line segment into segments of the same length:



See, also, the section on the Geo-board on pages 80-86 of the booklet *Games and Enrichment Activities.*

Drill

Consider using as a drill sheet page 38 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Find the NUMBER NAMES

Find the other names for 4 + 4 and 6 + 6. Draw a loop around each of these and copy it in the right circle.

I II III IV V VI VII

VIII IX X XI XII XIII

Add

$\begin{array}{r}2\\ \underline{2}\end{array}$ $\begin{array}{r}4\\ \underline{2}\end{array}$ $\begin{array}{r}6\\ \underline{2}\end{array}$

$\begin{array}{r}8\\ \underline{2}\end{array}$ $\begin{array}{r}10\\ \underline{2}\end{array}$ $\begin{array}{r}12\\ \underline{2}\end{array}$

12

2 + 6

4 + 5

Other names for 4 + 4

Add $\begin{array}{r}6\\ \underline{2}\end{array}$

19

4 + 4

4 + 4 + 4

3 + 3 + 3 − 1

Other names for 6 + 6

XII

5 + 5 + 5 − 3

20 − 12

6 + 5

4 + 8

8

Subtract

$\begin{array}{r}15\\ \underline{3}\end{array}$ $\begin{array}{r}11\\ \underline{3}\end{array}$ $\begin{array}{r}9\\ \underline{3}\end{array}$

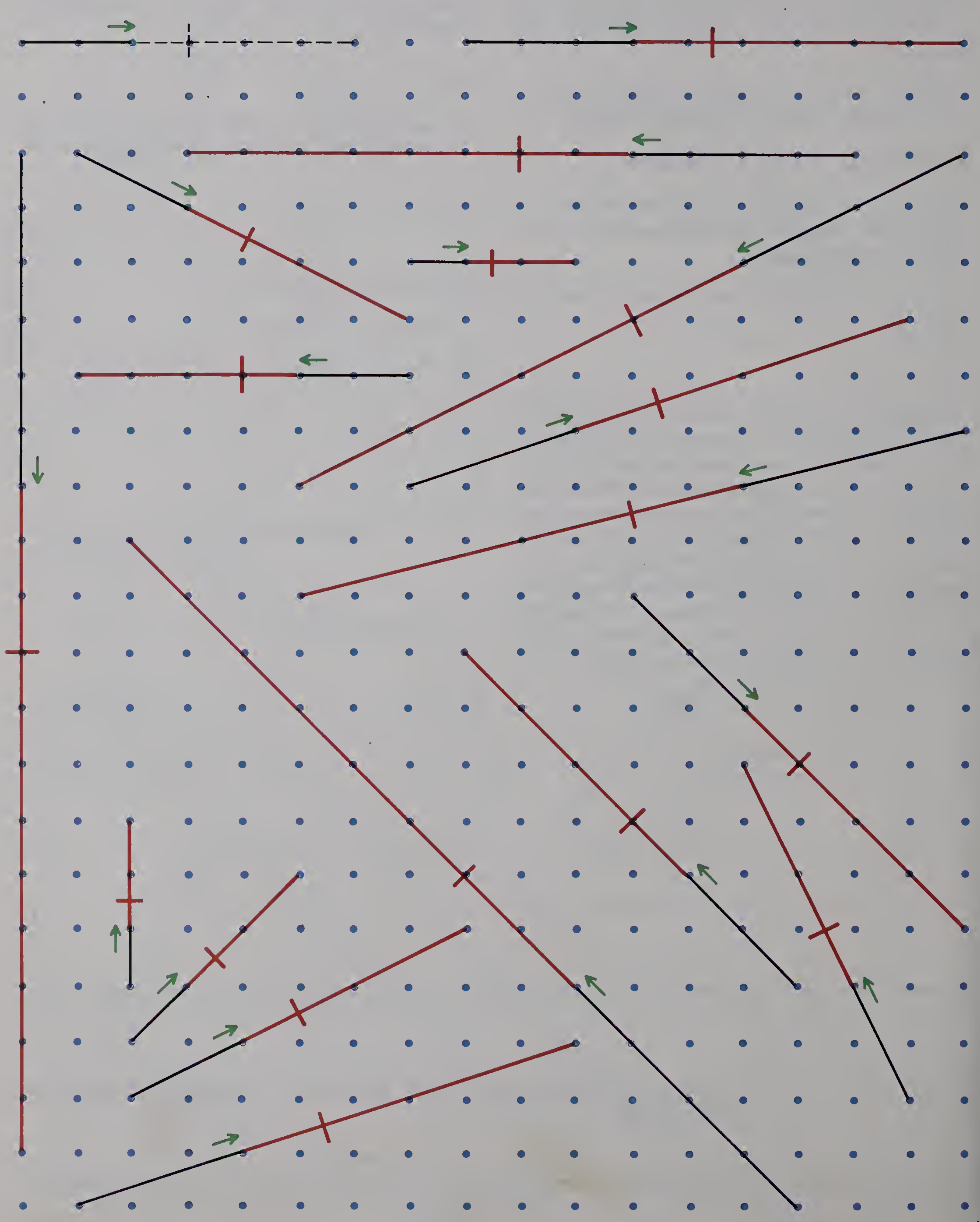
$\begin{array}{r}14\\ \underline{2}\end{array}$ $\begin{array}{r}12\\ \underline{2}\end{array}$ $\begin{array}{r}10\\ \underline{2}\end{array}$

㊳

Headline for the Day: ☐ + 12 = 25 Begin your story with: 25 Cub Scouts were swimming in the river when suddenly. . . .

Problem one: Make each line three times as long as it is drawn. Extend each in the direction shown by the green arrow.

Problem two: Mark a point half-way between the ends of each extended line.



Notes on Pupil Page 45

Tasks: *Pupils extend each given line segment to make one three times as long and then locate the midpoint of each new line segment.*

Purpose: To introduce pupils to measurement of line segments involving tripling and halving.

Unifying Ideas: Geometry; measurement.

Materials for Teacher: An overhead projector; a transparency with lattice points marked on it as pictured on pupil page 45.

Materials for Pupils: Rulers; Counting Boards or Geo-boards.

The Lesson:

The remarks made in "The Lesson" in the Notes on Pupil Page 44 apply here as well.

Stories and Sentences

You may wish to consider including in each day's lesson one or two stories for which pupils are expected to write number sentences or headlines. These stories might be given on a drill sheet along with computational exercises, or they may be given separately. Here are some sample stories along with possible headlines:

1) Carol had 2 headaches and a cold last week.

Sample headlines:

$2+1=3$ $\quad 1+2=\square$ $\quad 2>1$

2) It is now 1 o'clock. School lets out at 3.

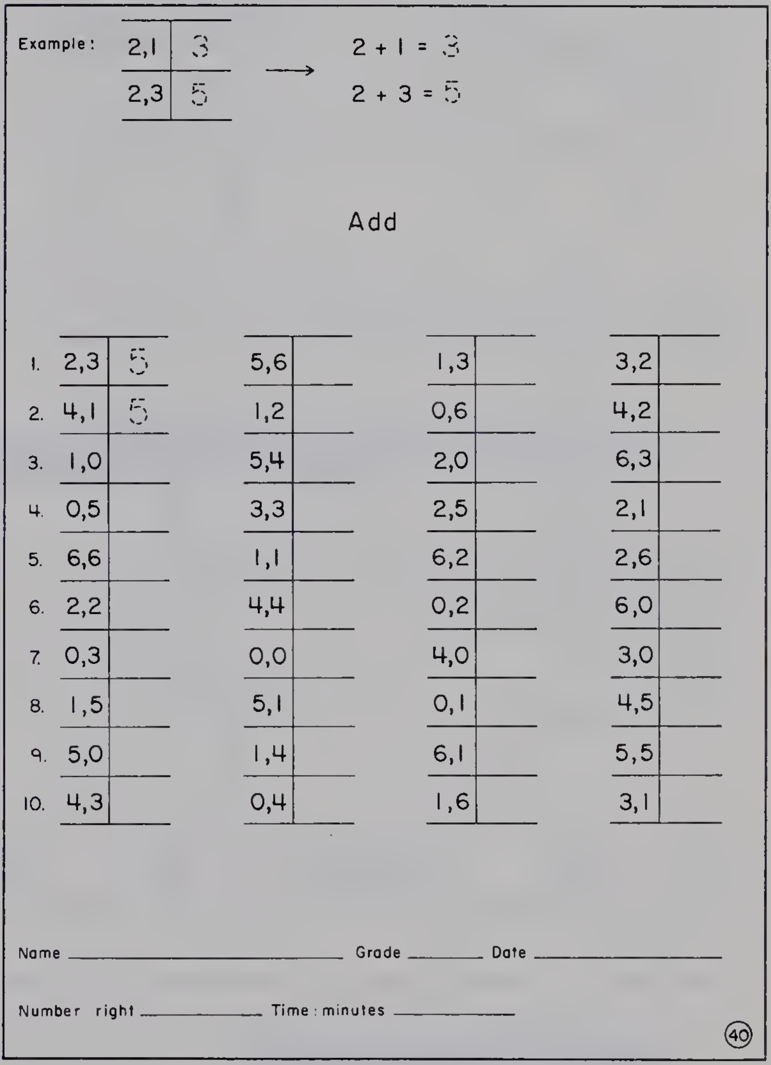
Sample headlines:

$3-1=2$ $\quad 1+\square=3$ $\quad 3=\square+1$

(See "Drill" in the Notes on Pupil Page 46.)

Drill

Consider using as a test page 40 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Example:

2,1	3
2,3	5

→ 2 + 1 = 3

2 + 3 = 5

Add

1.	2,3	5	5,6		1,3		3,2	
2.	4,1	5	1,2		0,6		4,2	
3.	1,0		5,4		2,0		6,3	
4.	0,5		3,3		2,5		2,1	
5.	6,6		1,1		6,2		2,6	
6.	2,2		4,4		0,2		6,0	
7.	0,3		0,0		4,0		3,0	
8.	1,5		5,1		0,1		4,5	
9.	5,0		1,4		6,1		5,5	
10.	4,3		0,4		1,6		3,1	

Name ______ Grade ______ Date ______

Number right ______ Time : minutes ______

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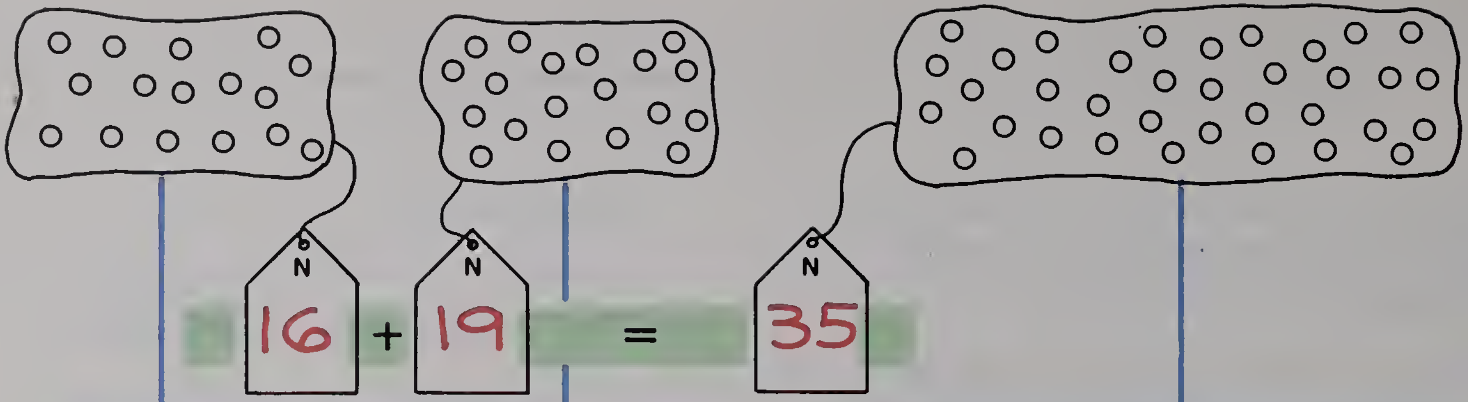
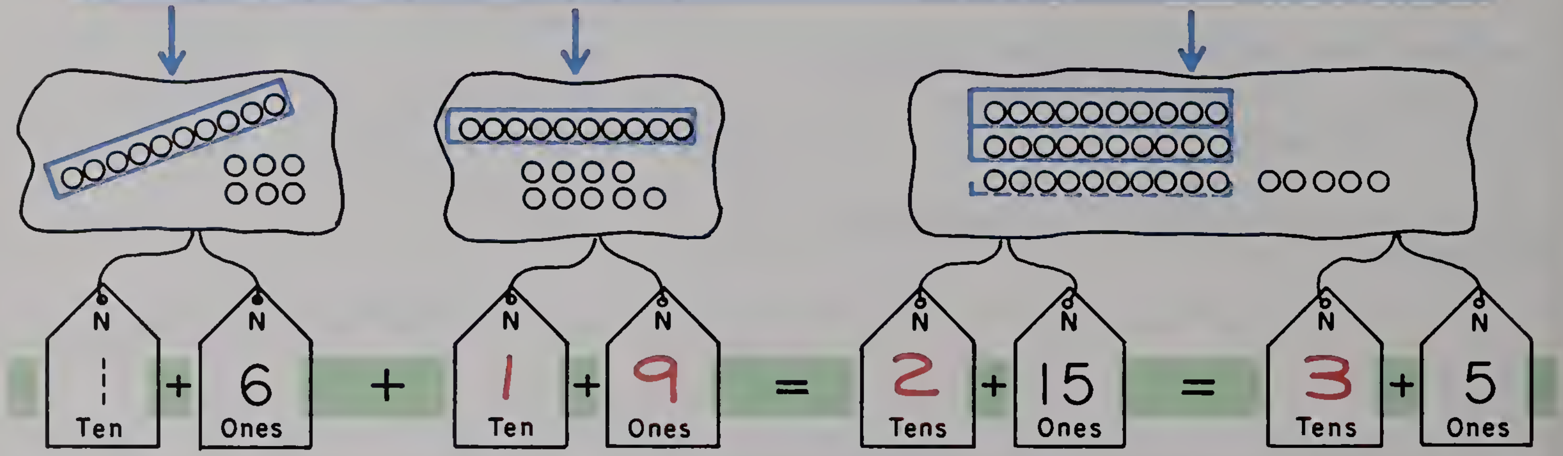
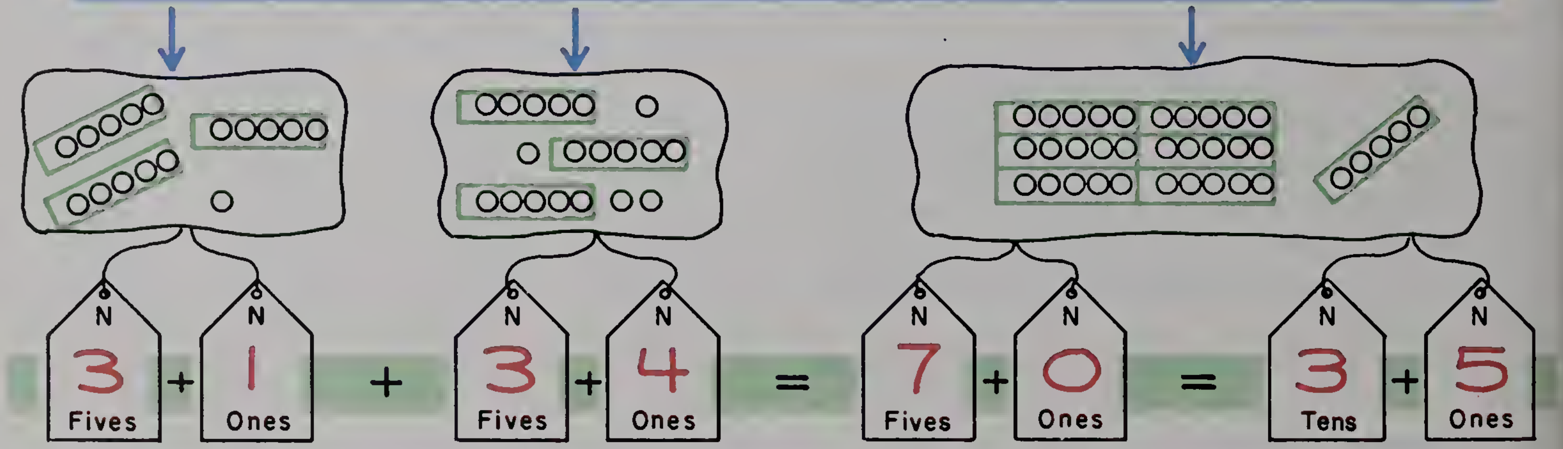
Headline for the Day: Write a number sentence that fits: How many hours between 7 P.M and 10 P.M.?



Each picture below has the same number of balls as the picture above it.


N 1 Ten + N 6 Ones + N 1 Ten + N 9 Ones = N 2 Tens + N 15 Ones = N 3 Tens + N 5 Ones

The balls could have been grouped in 5's and 1's as shown below.


N 3 Fives + N 1 Ones + N 3 Fives + N 4 Ones = N 7 Fives + N 0 Ones = N 3 Tens + N 5 Ones

---

What Are My Rules?

□ + △	7	10	13	15	29	14	24	80	20	26
□	4	8	9	10	20	7	16	60	13	17
△	3	2	4	5	9	7	8	20	7	9
□ − △	1	6	5	5	11	0	8	40	6	8

Tasks: *Pupils express numbers according to the partitioning specified in each section; they complete the table shown at the bottom of the page.*

Purpose: (a) To extend pupils' understanding of numeration systems; (b) to challenge pupils to experiment to find the answers in a difficult *What Are My Rules?* game.

Unifying Ideas: Number and counting; numeration; addition and subtraction; functions and relations.

Materials for Teacher: Flannel board; counters; tags as shown on pupil page 46.

Materials for Pupils: Counters, play money; Cuisenaire rods.

The Lesson:

You can prepare pupils for this page by carrying out activities at the flannel board (or chalkboard) just like those pictured on pupil page 46.

It is preferable that pupils do not respond to the first section until *after* they have completed the next two sections. These sections illustrate graphically the usefulness of partitioning as a step in simplifying the task of expressing the number of objects in a display.

Another way to prepare for the page is to use play money. Ask pupils to report on the count of a pile of pennies in terms of nickels and dimes. Of course, the same activity or related activities can be carried out with Cuisenaire rods. Make two piles of white rods and ask a pupil to tell how many rods there are in each of the piles and in total in terms of orange rods and extra white rods. He should replace each group of ten white rods with an orange rod and then reply: ___orange rods and . . . . . white rods. When he has given this reply, ask him to tell how many white rods were in the pile.

What are My Rules?

The *What Are My Rules?* game is very challenging. Allow pupils to experiment with counters or Cuisenaire rods in order to find the entries for the last two exercises. If they fail to get them, you may wish to help them as follows:

Notice that in the next to the last column the job is to find two numbers whose sum is 20 and whose

more

Headline for the Day: How many quarters can you get for a pile of nickels? Make a chart.

difference is 6. Spread 20 counters on a desk top (or on a flannel board). The job amounts to partitioning a display of 20 counters so that one part has 6 counters more than the other. If we partition the display into two parts with the same number, the difference is 0. Now, if we move one counter from one part to the other, the new parts have a difference of 2. If we move another counter, the new parts have a difference of 4. If we move still another counter, the new parts have a difference of 6. Thus, the partition of 20 that we are looking for is 13 + 7. (See "The Sum and Difference Game" on pages 54-56 of the booklet *Games and Enrichment Activities.*)

Some pupils might notice that you can get the numbers in the second and third rows by halving the sum, halving the difference, adding the halves to obtain the number for the second row, and subtracting the halves to obtain the number for the third row. This "rule" reflects the process described in the preceding paragraph.

Here is another rule for finding the numbers in the second and third rows: Add the difference to the sum and divide by 2 to find the number in the second row; subtract this number from the sum to find the number in the third row. This rule is based on the following argument. If $\square+\triangle=20$ and $\square-\triangle=6$ then $(\square+\triangle)+(\square-\triangle)=20+6=26$. But,

$$(\square+\triangle)+(\square-\triangle)=\square+\square.$$

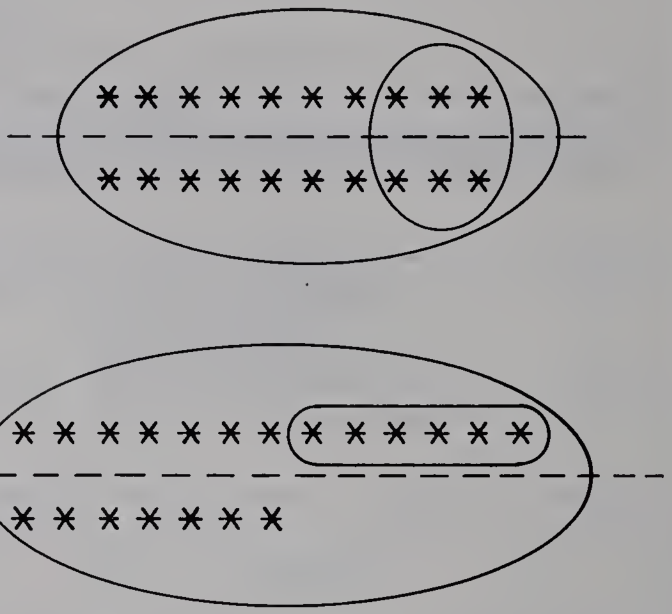
So, $\square+\square=26$. Hence, the number for the second row is 13 (or $26\div2$). Therefore, $13+\triangle=20$. So, the number for the third row is 7.

Of course, pupils can solve this problem by a brute force, trial-and-error approach. Let them make a table to keep track of their trials:

□	△	□−△
20	0	20
19	1	18
18	2	16
17	3	14
16	4	12
	etc.	

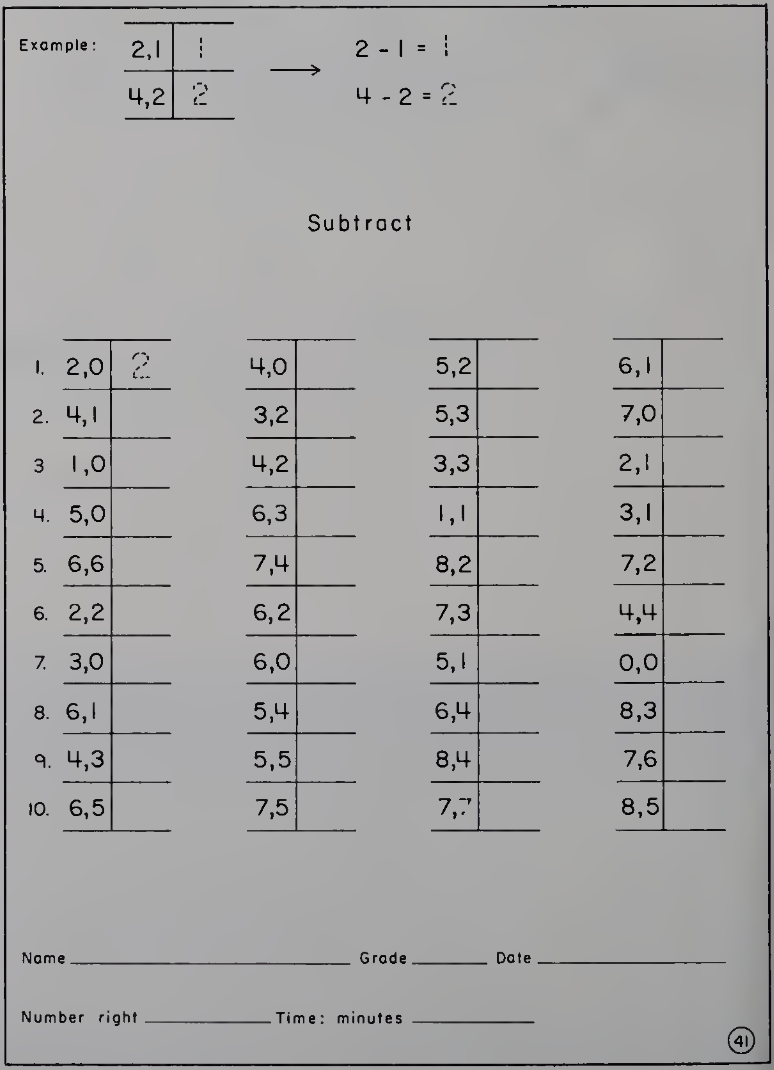
They continue with their trials until they find entries for the first two columns for which the entry for the third column is 6.

The following pictorial explanation was suggested to us by Mr. John Bradford of Jefferson County, Colorado:



Drill

Consider using as a test page 41 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Example:

2,1	1
4,2	2

→ 2 - 1 = 1
4 - 2 = 2

Subtract

1.	2,0	2	4,0		5,2		6,1	
2.	4,1		3,2		5,3		7,0	
3	1,0		4,2		3,3		2,1	
4.	5,0		6,3		1,1		3,1	
5.	6,6		7,4		8,2		7,2	
6.	2,2		6,2		7,3		4,4	
7.	3,0		6,0		5,1		0,0	
8.	6,1		5,4		6,4		8,3	
9.	4,3		5,5		8,4		7,6	
10.	6,5		7,5		7,7		8,5	

Name ____________ Grade ______ Date ____________

Number right ______ Time: minutes ______

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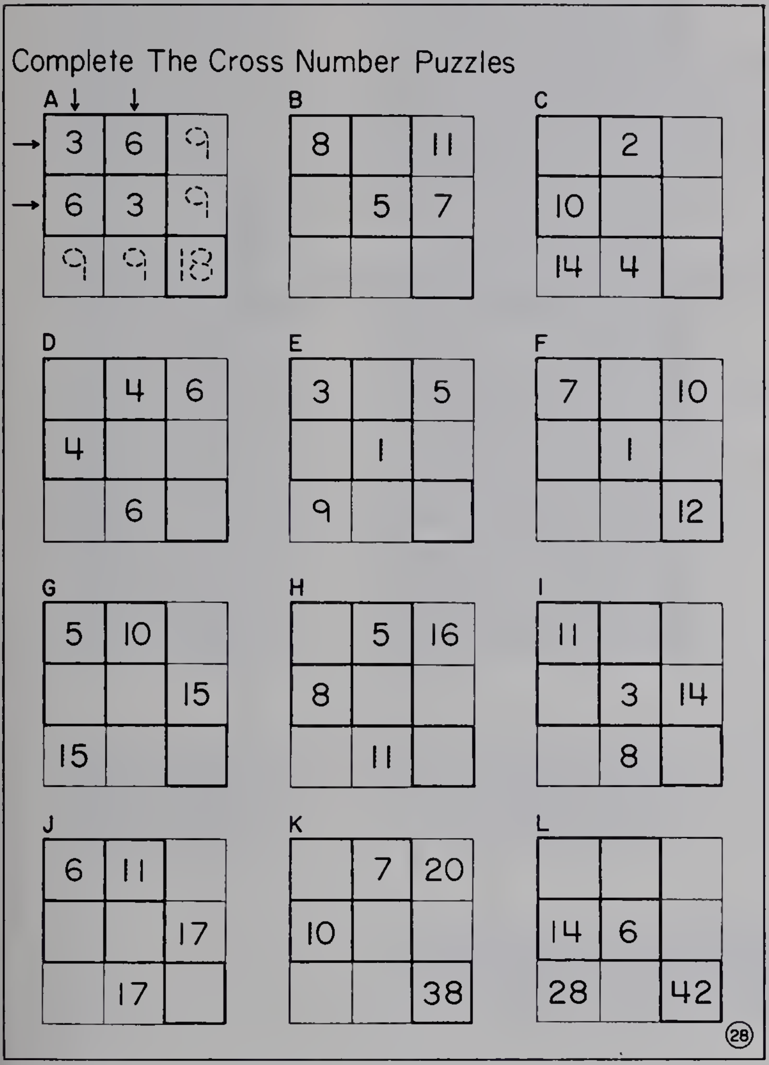
Consider using as a drill sheet page 28 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Complete The Cross Number Puzzles

A

	↓	↓	
→	3	6	9
→	6	3	9
	9	9	18

B

8		11
	5	7

C

	2	
10		
14	4	

D

	4	6
4		
	6	

E

3		5
	1	
9		

F

7		10
	1	
		12

G

5	10	
		15
15		

H

	5	16
8		
	11	

I

11		
	3	14
	8	

J

6	11	
		17
	17	

K

	7	20
10		
		38

L

14	6	
28		42

㉘

Also, here are three stories with samples of correct number sentences or headlines that pupils might write (see "Stories and Sentences" in the Notes on Pupil Page 45):

1) Dolores had 9 fishhooks and gave 4 to Donna.

$9-4=5$ $9=4+\bigcirc$

$\triangle+4=9$ $9-4>4$

2) Dick has 4 toy boats. Jack has 2 toy boats.

$2+\square=4$ $4-2=2$

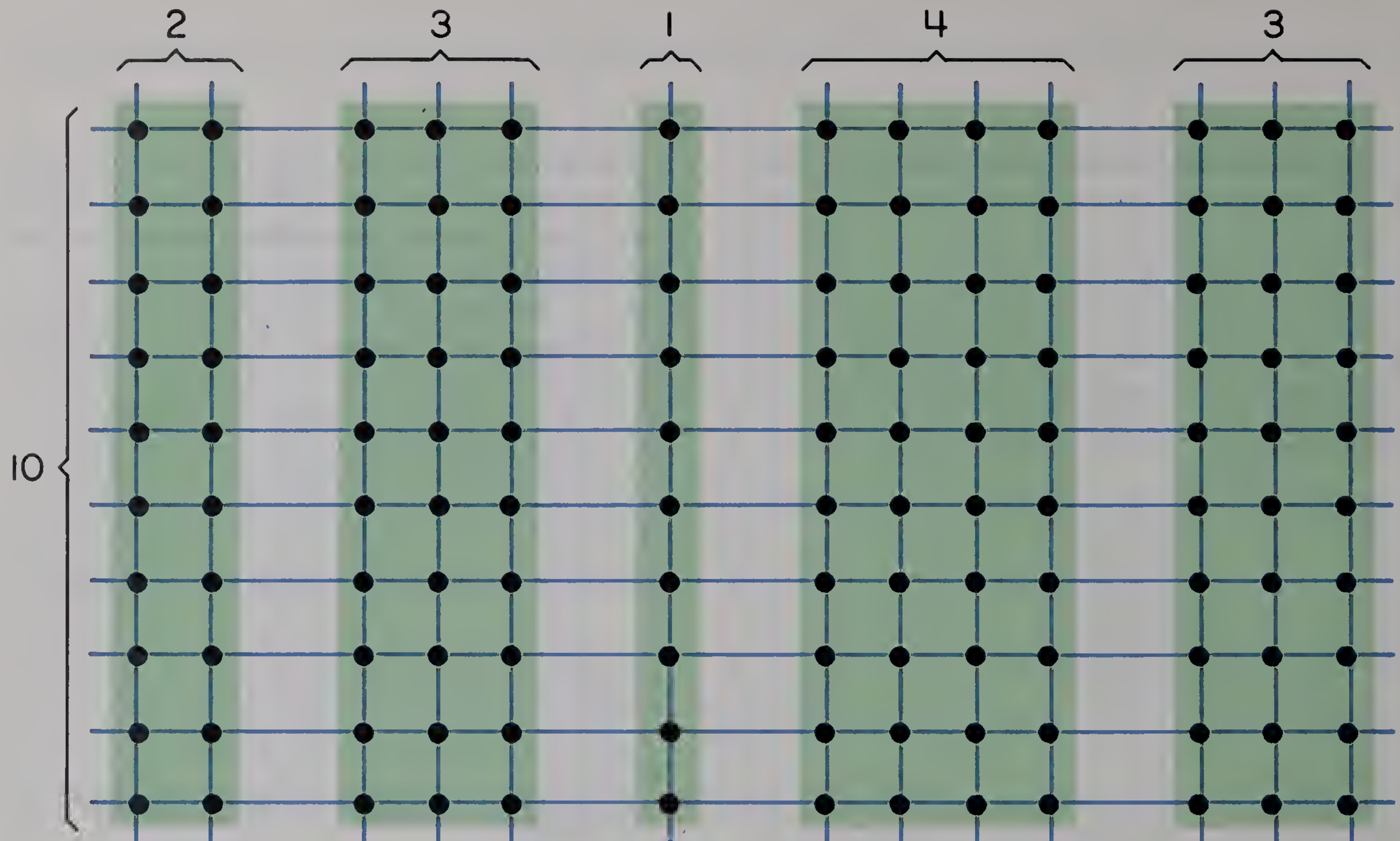
$4>2$ $2<4$ $2=4-\triangle$

3) Max had 3 problems to do. He has done 2 of them.

$2+\triangle=3$ $3-2=1$

$3>2$ $2<3$ $1=3-\square$

Spend some time in class in considering pupils' suggestions for number sentences. Pupils will probably need to see the wide range of sentences that are acceptable to you before they believe that you really wish them to be creative.



20 30 10 40 30

20 + 30 = 50

10 + 40 = 50

30 + 10 = 40

40 + 30 = 70

30 + 10 + 40 = 80

20 + 30 + 10 + 40 = 100

$$\begin{array}{r} 20 \\ +\ 10 \\ \hline 30 \end{array} \qquad \begin{array}{r} 10 \\ +\ 10 \\ \hline 20 \end{array} \qquad \begin{array}{r} 20 \\ +\ 20 \\ \hline 40 \end{array} \qquad \begin{array}{r} 40 \\ -\ 10 \\ \hline 30 \end{array} \qquad \begin{array}{r} 50 \\ -\ 20 \\ \hline 30 \end{array} \qquad \begin{array}{r} 10 \\ +\ 40 \\ \hline 50 \end{array}$$

Notes on Pupil Page 47

Tasks: *Pupils count (by shortcuts) the various displays of crossing points. They also carry out the addition and subtraction problems.*

Purpose: (a) To give pupils practice in working with multiples of 10 to prepare for the decimal numeration system; (b) to prepare pupils for multiplication involving 10 as a factor.

Unifying Ideas: Number and counting; numeration; addition and subtraction; multiplication and division.

Materials for Teacher: Counting frame; Counting Board; Ditto masters.

Materials for Pupils: Dittoed sheets; Cuisenaire rods.

The Lesson:

This is a lesson in working with multiples of 10. Notice that the green fields serve to tie together various 10-groups. Ask pupils to tell you how many crossing points there are altogether. (Count: 20, 50, 60, 100, 130.)

Use a counting frame, a Counting Board (see the Notes on Pupil Page 33), and Cuisenaire rods to demonstrate all of the facts shown on pupil page 47.

Follow-up

When the page has been completed, have pupils complete the following sentences on a Dittoed sheet:

50 + □ = 50
40 + □ = 50
30 + □ = 50
20 + 30 = □
□ + 40 = 50

30 + □ + □ = 50
□ + 30 + 10 = 50
10 + 10 + □ = 50
20 + 20 + 10 = □
□ + 20 + 10 = 50
□ + □ + 10 = 50

20 + □ + □ + □ = 50
50 = 10 + □ + 10 + 10
50 = □ + □ + 20 + 10
□ = 10 + 10 + 10 + 20
□ + □ + □ + □ + □ = 50

Remember that in a sentence with frames, you must write the same number in all frames of the same shape. For example, the first sentence in the second column should be completed as 30 + 10 + 10 = 50, not as 30 + 5 + 15 = 50. (This rule of "same number for frames of the same shape" does *not* mean that frames of different shapes must be filled with different numbers.) For example, the sentence □ + △ = 40 could be completed as 20 + 20 = 40.

Drill

Here is a suggested drill sheet:

In any sentence with frames, you must write the same number in all frames of the same size and shape.

13 = 1 + □
13 = 1 + △ + △
13 = 1 + ○ + ○ + ○
13 = 1 + ⬡ + ⬡ + ⬡ + ⬡
13 = 1 + △ + 1 + △ + 1 + △ + 1
13 = 2 + □ + △
13 = 2 + ○ + ▽

Here are two stories with some sample number sentences.

1)' 8 birds were on a tree. 6 of them flew away.

8 − 6 = 2
8 − 6 = △
6 + △ = 8
8 = 2 + 6

2) Zena saw 4 cows, 2 horses, and 3 chickens at the farm.

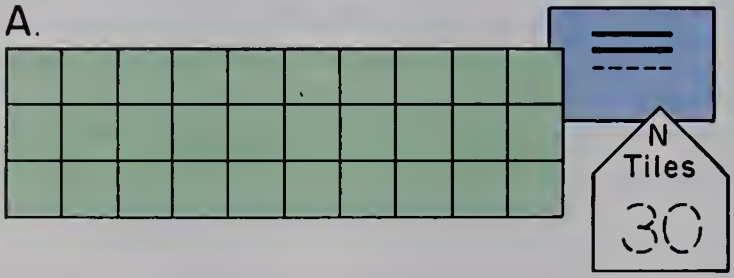
4 + 2 + 3 = □
9 − 3 = 4 + 2

Headline for the Day: Write stories that fit number sentences on page 47.

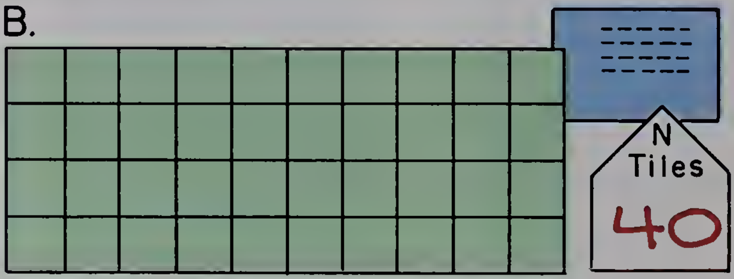
In each row there are 10 tiles.

Use —— to stand for a row of tiles. In each blue space record the number of rows. In each tag record the total number of tiles.

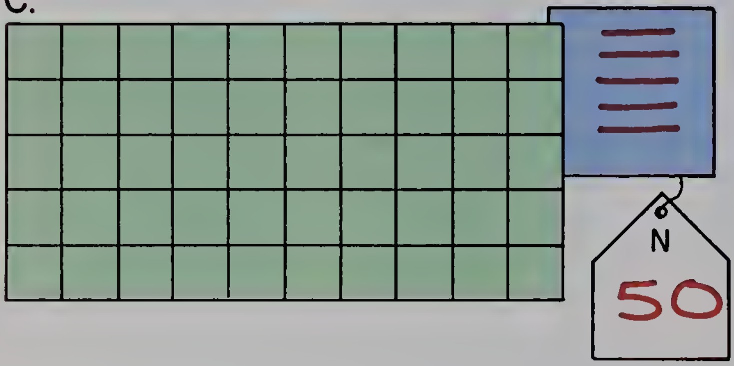
A.



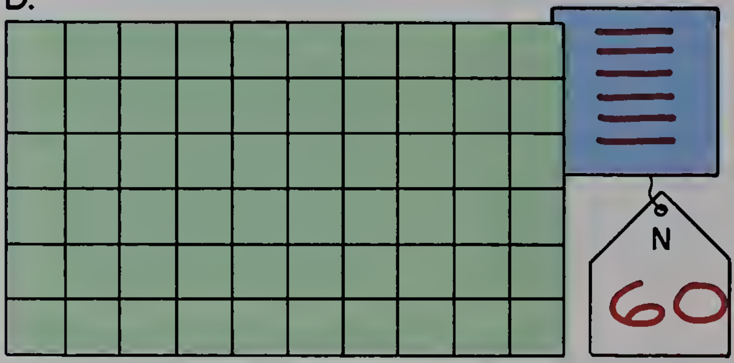
B.



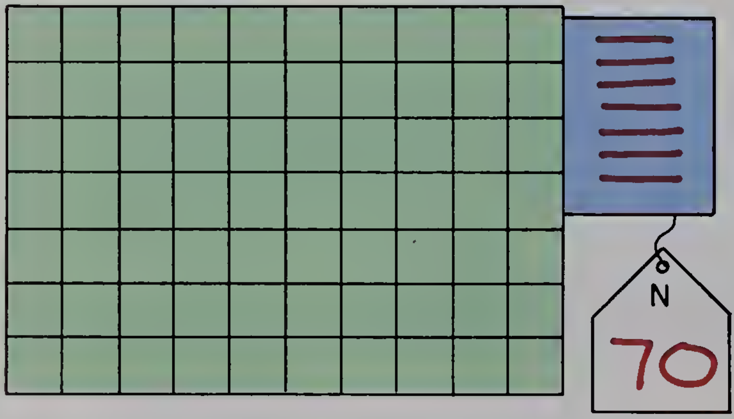
C.



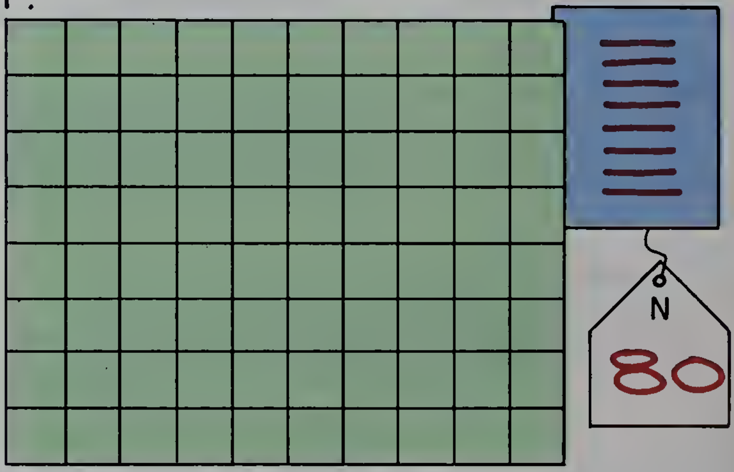
D.



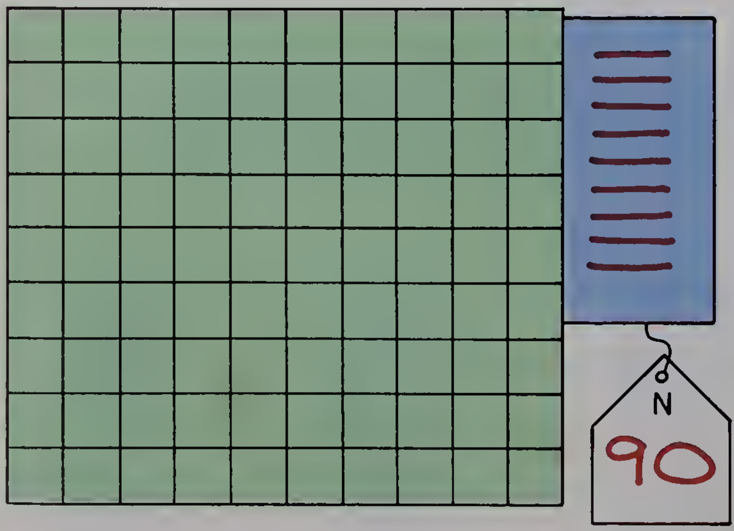
E.



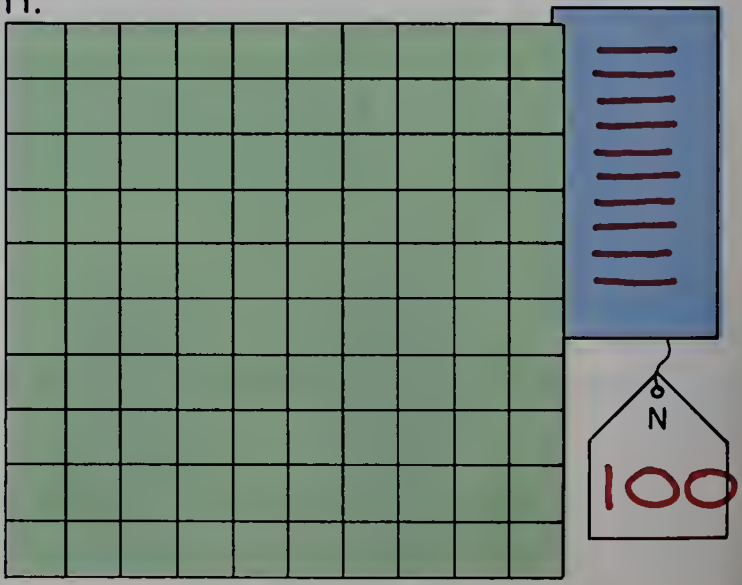
F.



G.



H.



Notes on Pupil Page 48

Tasks: *In each exercise pupils use the special symbols to indicate the number of rows of 10 tiles, and then record on the tag the total number of tiles.*

Purpose: To extend pupils' understanding of decimal numeration.

Unifying Ideas: Number and counting; numeration.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Review the sound-and-body symbolism for 10-groups and 1-groups (see the Notes on Pupil Page 34) and let pupils use it to report the number of rows of 10 tiles each in each exercise.

Ask the pupils to hold up their fingers and to take turns in reporting the total number of fingers in the classroom. Once this has been determined, write the number on the board. Then ask for the total number of toes. Ask for the number of heads, the number of ears, the total number of heads and ears, etc.

The idea of partitioning a display into 10-groups is emphasized heavily on pupil pages 48-55, 57, 58, 60, and 61. Notice that not until pupil page 62 is partitioning using only number symbols studied. Up to that point the pupil can literally count his way to every answer using the representations for 1, 10, and 100 (and for 1, 5, and 25 on pupil page 59).

Drill

Here are two stories along with sample number sentences:

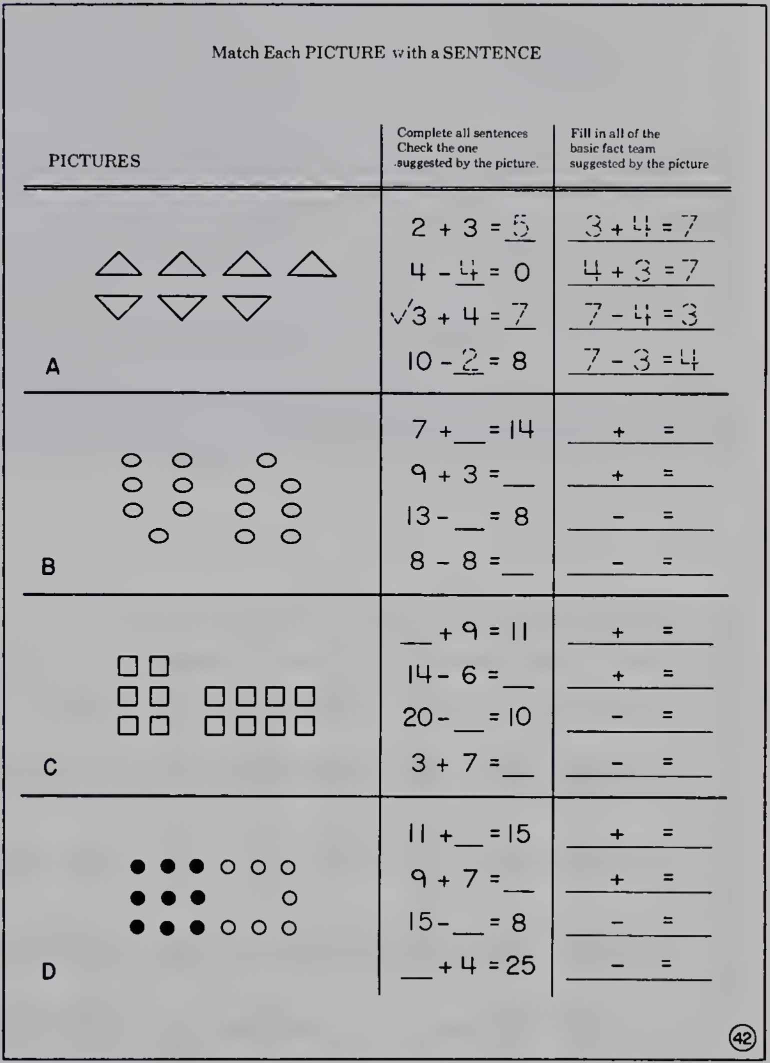
1) Max found 10 fat green bugs and gave 4 to Mort.

$10 - 4 = 6$ $\qquad 10 = 4 + \triangle$
$10 - 4 > 4$ $\qquad \square + 4 = 10$

2) There were 7 candles on Tony's birthday cake. His mother lit 4 of them with one match.

$4 + \square = 7$ $\qquad 7 - 4 = 3$
$4 > 3$ $\qquad 4 + 3 = 7$

Consider using as a drill sheet page 42 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Match Each PICTURE with a SENTENCE

PICTURES	Complete all sentences Check the one suggested by the picture.	Fill in all of the basic fact team suggested by the picture
A	2 + 3 = 5 4 - 4 = 0 ✓3 + 4 = 7 10 - 2 = 8	3 + 4 = 7 4 + 3 = 7 7 - 4 = 3 7 - 3 = 4
B	7 + ___ = 14 9 + 3 = ___ 13 - ___ = 8 8 - 8 = ___	___ + ___ = ___ ___ + ___ = ___ ___ - ___ = ___ ___ - ___ = ___
C	___ + 9 = 11 14 - 6 = ___ 20 - ___ = 10 3 + 7 = ___	___ + ___ = ___ ___ + ___ = ___ ___ - ___ = ___ ___ - ___ = ___
D	11 + ___ = 15 9 + 7 = ___ 15 - ___ = 8 ___ + 4 = 25	___ + ___ = ___ ___ + ___ = ___ ___ - ___ = ___ ___ - ___ = ___

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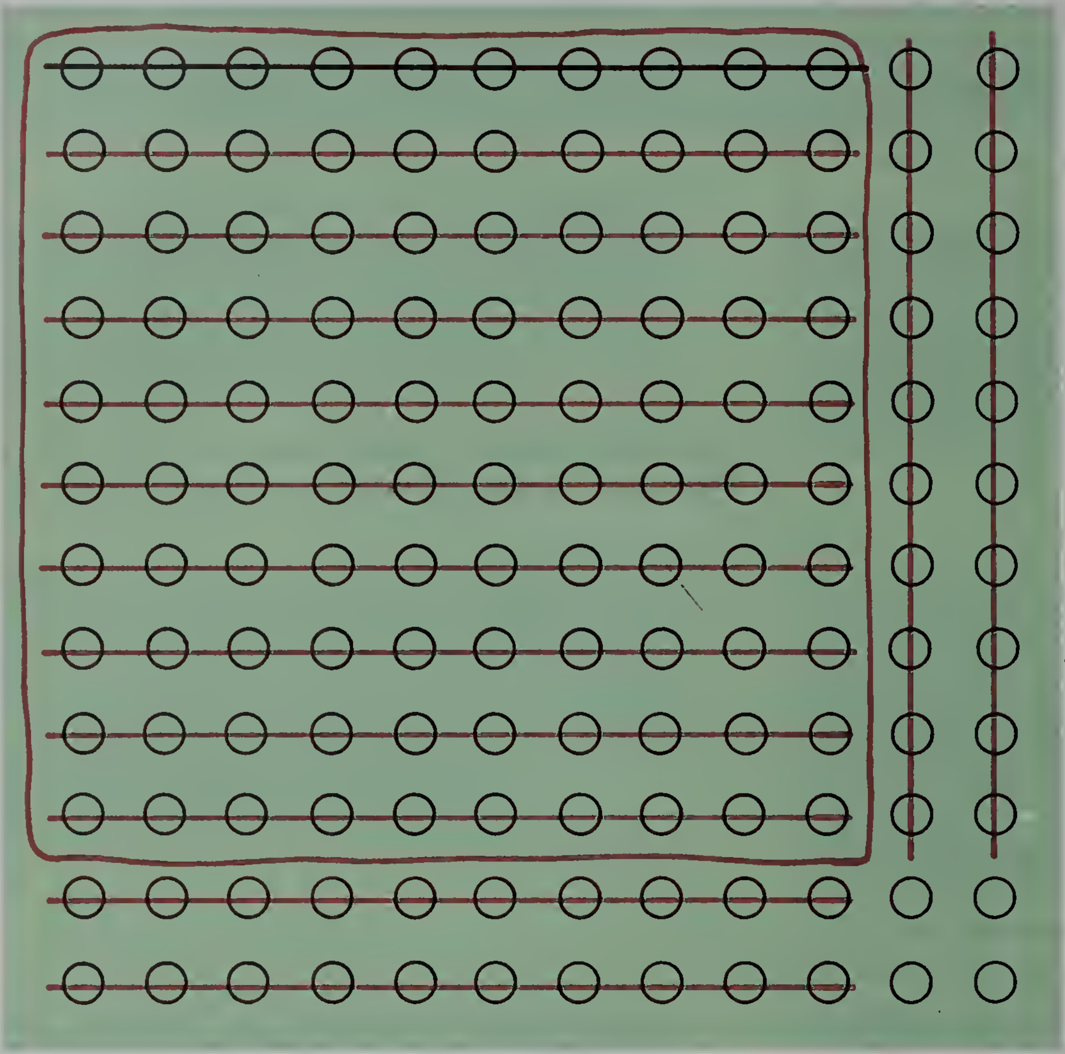
Headline for the Day: Write number sentences for the following:

One squirrel has 1 less than 20 acorns; the other has 1 more than 10. Together they have _____ acorns.

Tom has 10 marbles; Mary has 2 less than Tom.

HOW MANY BEADS IN EACH PICTURE?



How many strings of beads with 10 beads on each string?

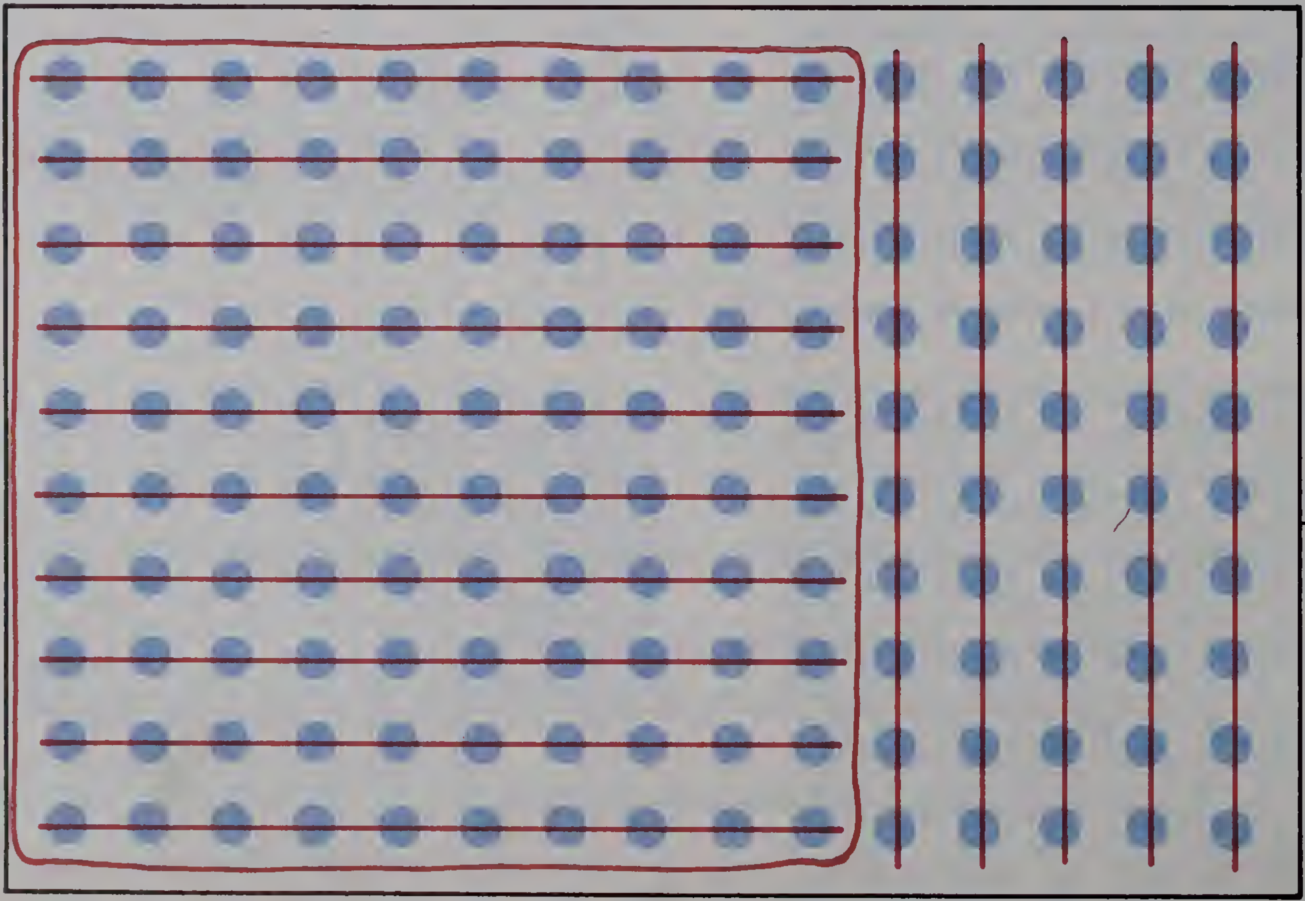
Answer: 14

Draw a loop around 100 beads.

How many beads altogether?

N
Total number
of beads

144



N
Total
number
of beads

150

Notes on Pupil Page 49

Tasks: *Pupils partition the given displays into 10-groups and 1-groups, and then into 100-groups, 10-groups, and 1-groups. This makes it easy to express in decimal notation the total number of objects in each set.*

Purpose: To extend pupils' understanding of the decimal system of numeration.

Unifying Ideas: Number and counting; numeration.

Materials for Teacher: Several thousand sticks; rubber bands.

Materials for Pupils: None.

The Lesson:

Prepare pupils for this page by having them count out specific numbers of sticks. Use numbers like 48, 57, 90, 120, 180, 270, 420, etc. Encourage pupils to prepare for these chores by bundling the sticks into groups of 10 each.

Notice that each of the problems calls for two partitionings. In the first exercise, the first partitioning separates the given display into fourteen 10-groups and four 1-groups. This gives us a display of fourteen objects of one type and four objects of another type. The next step consists of partitioning this new display. The result is a third display consisting of one 100-group, four 10-groups, and four 1-groups.

Drill

Here is a suggestion for a drill sheet:

Watch the Signs!

$6 + 3 = \underline{\quad}$
$3 - 1 = \underline{\quad}$
$7 - 0 = \underline{\quad}$
$9 - 6 = \underline{\quad}$
$4 + 2 = \underline{\quad}$
$5 + 5 = \underline{\quad}$
$4 + 3 = \underline{\quad}$

$8 - 6 = \underline{\quad}$
$4 - 3 = \underline{\quad}$
$2 + 1 = \underline{\quad}$
$8 - 4 = \underline{\quad}$
$7 - 5 = \underline{\quad}$
$10 - 7 = \underline{\quad}$
$10 - 6 = \underline{\quad}$

$5 + 4 = \underline{\quad}$
$6 + 4 = \underline{\quad}$
$7 + 2 = \underline{\quad}$
$1 + 3 = \underline{\quad}$
$7 - 4 = \underline{\quad}$
$9 - 5 = \underline{\quad}$
$6 - 5 = \underline{\quad}$

Here are two stories with sample number sentences:

1) Paul caught 10 fish. He threw 4 small fish back into the lake.

$10 - 4 = \square$
$4 + 6 = 10$

$10 = 4 + \triangle$
$6 > 4$

2) Sue and Danny built a path across the creek. They put 5 rocks in the water. They needed 2 more to get across.

$5 + 2 = 7$
$2 + 5 = 7$

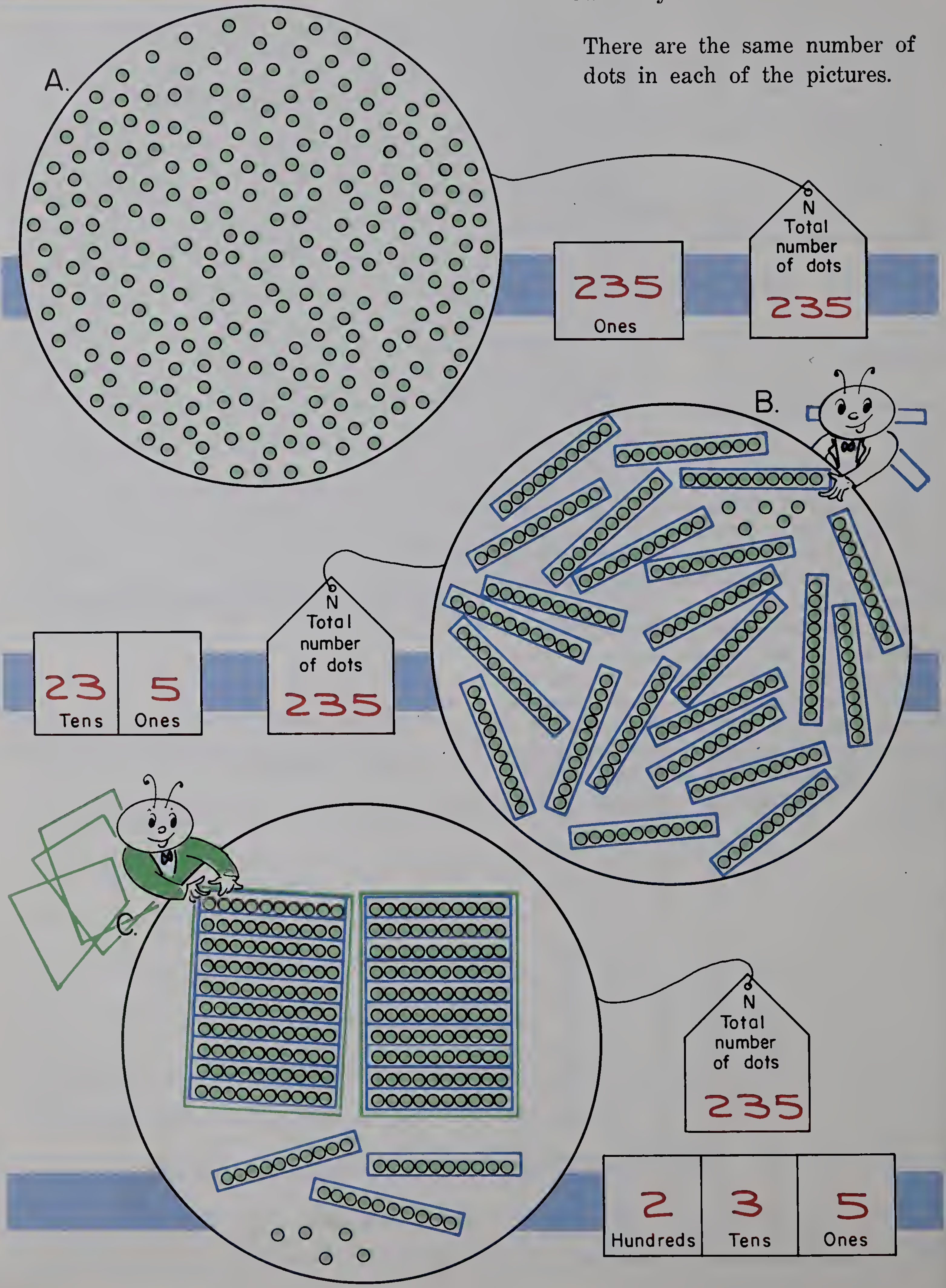
$5 + 2 = \square$
$7 - 2 = 5$

Headline for the Day: Extend the patterns: $20, 40, 60, 80, \underline{\quad}, \underline{\quad}, \underline{\quad}$

$5, 15, 25, 35, \underline{\quad}, \underline{\quad}, \underline{\quad}$

How Many Dots?

There are the same number of dots in each of the pictures.


A.
235
Ones
N
Total
number
of dots
235
B.
N
Total
number
of dots
235
23
Tens
5
Ones
C.
N
Total
number
of dots
235
2
Hundreds
3
Tens
5
Ones

Notes on Pupil Page 50

Tasks: *Pupils record the total number of dots displayed in each circle.*

Purpose: (a) To drive home the idea that counting and expressing a large number of objects can be facilitated by partitioning into 100-groups, 10-groups, and 1-groups; (b) to show pupils that such partitioning is the basis for constructing the decimal number symbols.

Unifying Ideas: Number and counting; numeration.

Materials for Teacher: Several hundred sticks; rubber bands; Ditto masters.

Materials for Pupils: Graph paper; Dittoed sheets.

The Lesson:

This *All About . . .* page might be entitled "What Positional Notation Is About."

No one but the artist and the proofreader has counted the dots in the first circle. Notice in the second circle the extent to which partitioning makes the job of counting and reporting more manageable. But it is the third circle that really impresses one with the contribution of partitioning as a shortcut to counting.

This page should be thoroughly discussed with the pupils. It is much too hard to count all the dots in the first circle. So Numo makes the job easier for you by putting the dots in narrow boxes, 10 to a box. Even now it may be hard to count. So Numo piles the narrow boxes one on top of the other, 10 to a stack. Now it is very easy to count the dots—100, 200, 210, 220, 230, 231, 232, 233, 234, 235.

Let the pupils group themselves so that it will be easy for you to count the total number of fingers in the classroom (groups of 10 pupils). Let them group themselves so that it will be easy for you to count the total number of fingers and toes in the classroom (groups of 5 pupils).

Demonstrate the partitioning into 10-groups and the partitioning into 100-groups by asking a team of pupils to count a pile of several hundred sticks. Encourage the pupils to stack 10 bundles of 10 sticks each into a pyramid—4 bundles on the bottom, 3 in the next layer, 2 in the next, and 1 at the top.

Distribute graph paper and ask pupils to draw rectangles consisting of 50 little squares, 80 little squares, 170 little squares, 230 little squares, etc.

Drill

Here is a suggestion for a drill sheet:

3	5	2	3	5
4	4	8	3	2
5	2	2	3	3
—	—	—	—	—
4	3	2	4	3
2	2	3	3	4
3	4	4	2	2
—	—	—	—	—

Here are two stories with sample number sentences:

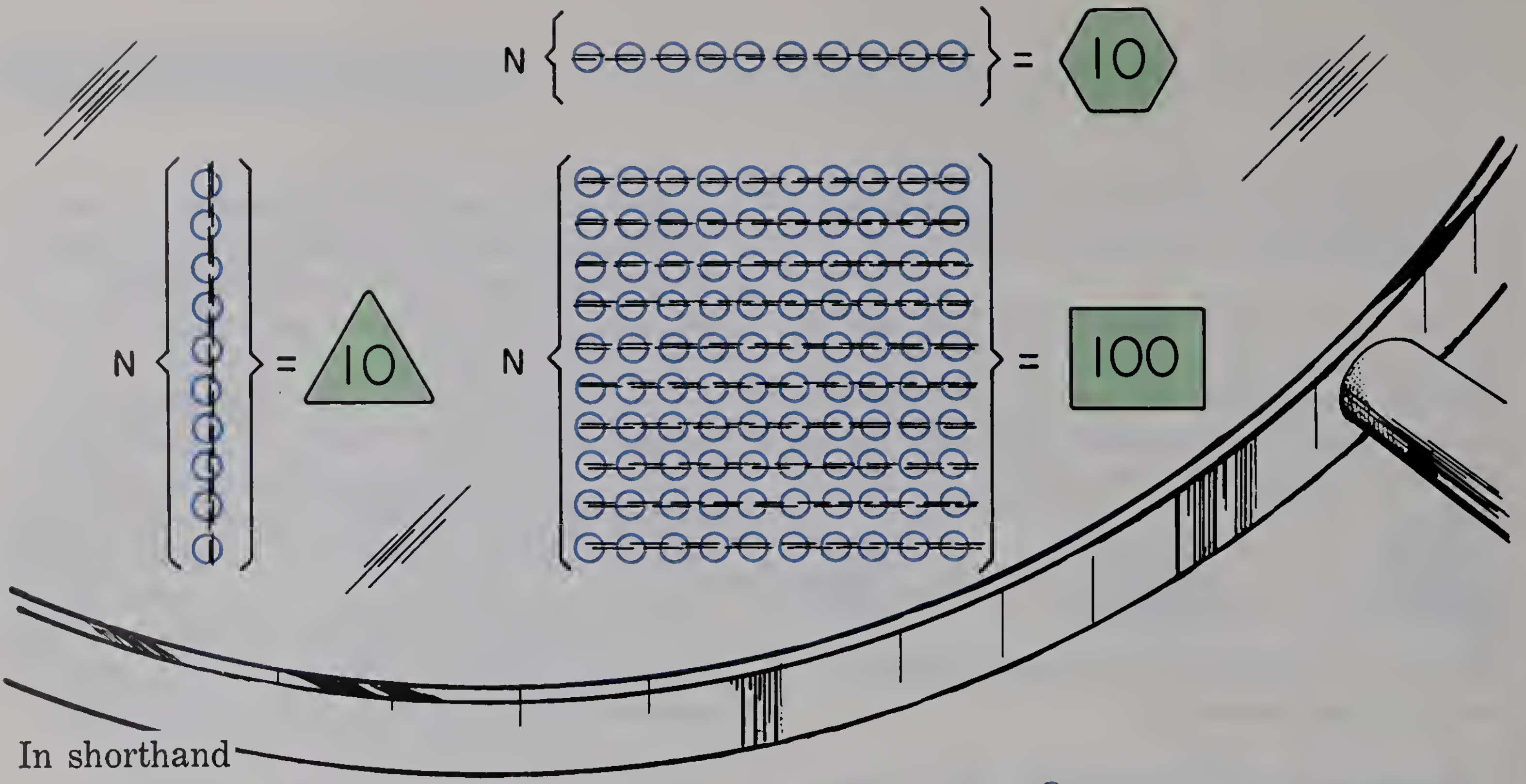
1) Billy has 7 cents. Mary has 10 cents.

$7 < 10$ $7 + \square = 10$ $10-7=3$

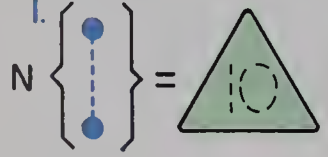
2) Dave has 5 toy cars. John has 4 toy cars.

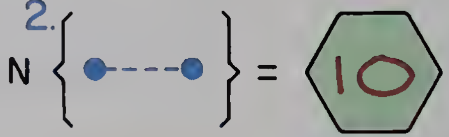
$5 > 4$ $5 = 4 + \square$ $1+4=5$

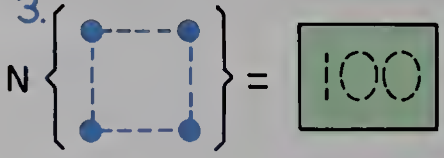
Headline for the Day: $1.00 is ___ pennies, or ..... nickels, or ___ dimes, or __ quarters, or ... half-dollars.

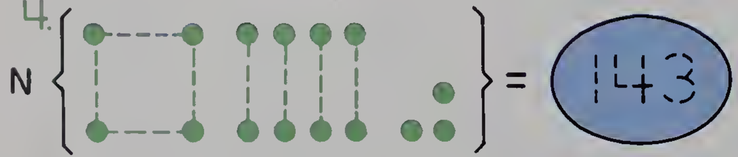

N
= 10
N
= 10
N
= 100

In shorthand

1.


2.


3.


4.


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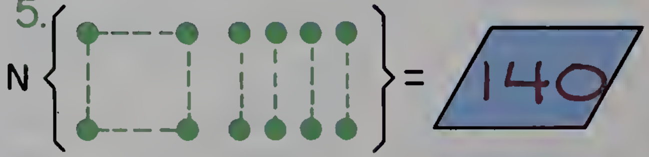

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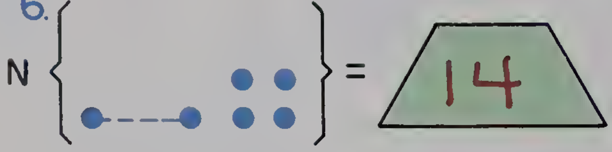

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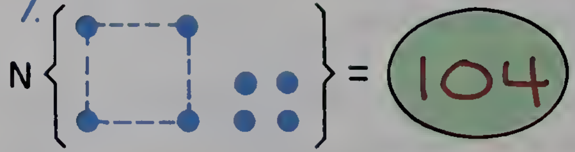

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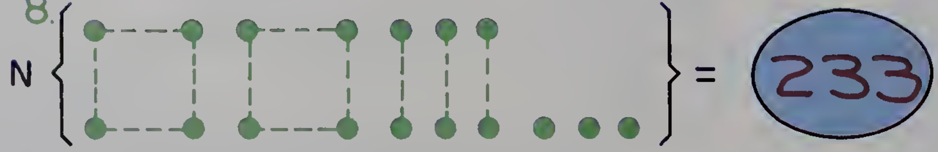
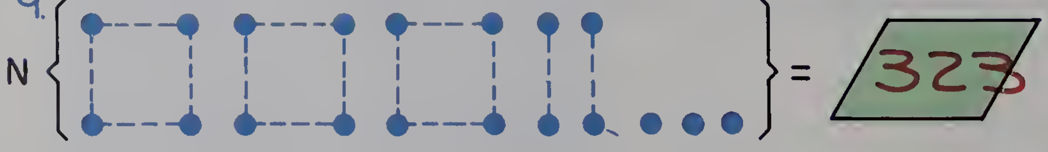


9.


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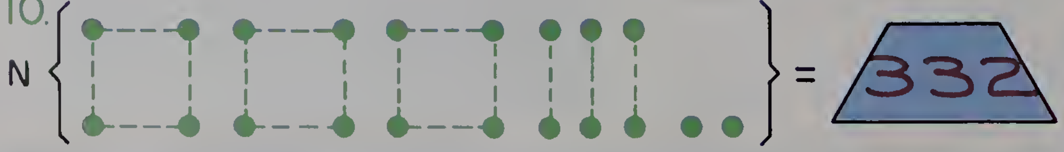


11. What's My Rule?

6, 16, 26, 36, 46, 56, 66, 76, etc.

Notes on Pupil Page 51

Tasks: *Pupils record the number of beads displayed between the braces (using the special symbols suggested in exercises 1-3) and follow their proposed rule in filling in the blanks in exercise 11.*

Purpose: (a) To extend pupils' understanding of the decimal system of numeration; (b) to provide pupils with practice in adding 10.

Unifying Ideas: Sets; number and counting; numeration; functions and relations.

Materials for Teacher: Ditto masters.

Materials for Pupils: Dittoed sheets.

The Lesson:

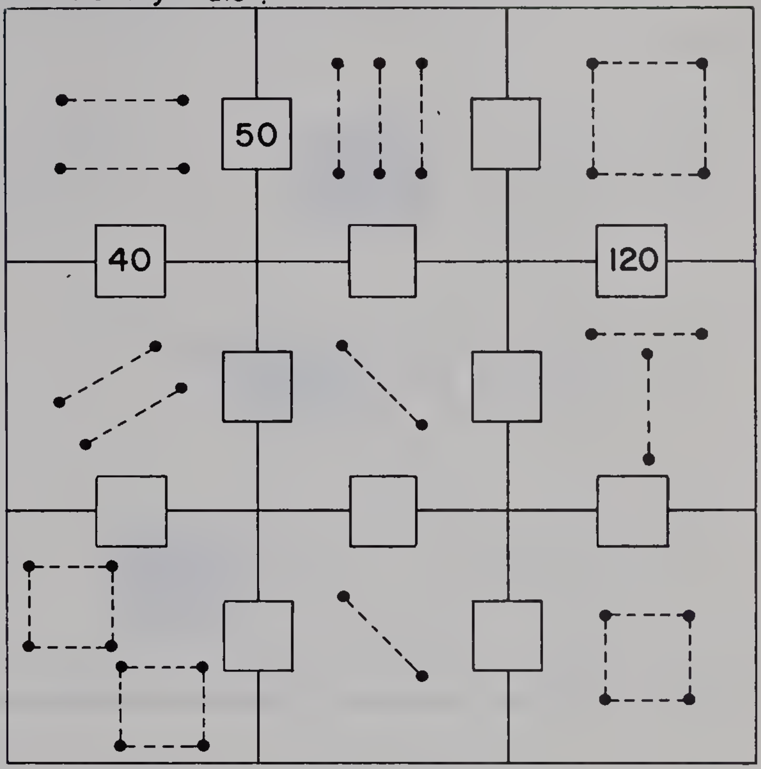
This page and pupil page 52 take up where pupil page 34 left off in developing a shorthand notation to indicate the decimal partitioning of a set. The notation used here is natural and lends itself readily to sound-and-body symbols suggested in the Notes on Pupil Page 34. Introduce a hand motion (tracing out a square) along with a sound (boo-oo-oo-oom) for 100. Let pupils use these sound-and-body symbols to express the partitioning of a set, and let other pupils "read" the symbols and state the number.

Note in exercises 5-7 the attempt to clarify for pupils the role of place value by use of the numbers 140, 14, and 104. Also note how this matter is handled in exercises 8-10.

Drill

Here is a suggested drill sheet:

What's My Rule ?



Here are two stories and sample number sentences:

1) Alphonse saw 4 blue birds, 3 red birds, and 4 yellow birds.

$4 + 3 + 4 = 11$ $\qquad$ $4 + 3 + 4 = \square$

2) Ramona had 2 friends over to spend the night with her. Each of her friends brought 5 dolls. Ramona had 4 dolls.

$2 + 1 = 3$ $\qquad$ $2 + 1 = \square$
$5 + 5 + 4 = \square$ $\qquad$ $10 + 4 = 14$

Headline for the Day: Write a number sentence that fits: How many hours between 7 P.M. and midnight?

We can simplify our shorthand.

□ represents 100 (beads or other objects counted)

| or — represents 10 (beads or other objects counted)

• represents 1 (bead or other objects counted)

So:

a N{ □ } = 100

b N{ | } = 10

c N{ • } = 1

d N{ □ □ } = 200

e N{ — — — — } = 40

f N{ □ | | • • • • • } = 125

g N{ □ □ • • • • • • • • } = 208

---

h N{ • } + N{ • • • } = 4

i N{ □ } + N{ | | | • • } = 132

j N{ • • • • • • } + N{ • • • • • } = 11

k N{ □ • • • • • • } + N{ • • • • } = 110

l N{ □ • • • • • • • • • • } = 110

m N{ | • • • • • • — • • • • • } = 31

n N{ — — } + N{ • • • • • } = 25

o N{ | | | } + N{ □ } = 130

p N{ | | | — — — } + N{ | } = 70

q N{ | | | | | • } + N{ — • • • } = 64

r N{ | | | | | • | • • • } = 64

s N{ | | | | | | | — — — } = 100

Notes on Pupil Page 52

Tasks: *In exercises* a-g *pupils record the number of beads displayed between the braces using the special shorthand suggested at the top of the page. In exercises* h-s *pupils carry out the indicated addition problems or counting problems.*

Purpose: (a) To extend pupils' understanding of the decimal system of numeration; (b) to give pupils experience in manipulating special symbols.

Unifying Ideas: Sets; number and counting; numeration; addition and subtraction.

Materials for Teacher: Ditto masters.

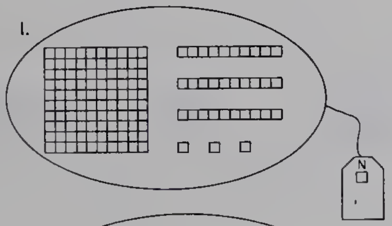
Materials for Pupils: Dittoed sheets; Cuisenaire rods.

The Lesson:

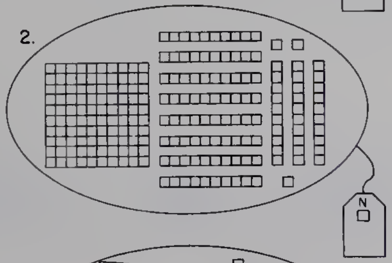
The symbols on this page are abstracted from those on pupil page 51. You can use Cuisenaire rods to show one possible source of the symbols. (100 white rods cover a floor of 10 orange rods, etc.)

Prepare the way for the special symbols, used from pupil page 54 on, by giving the following exercises on a Dittoed sheet:

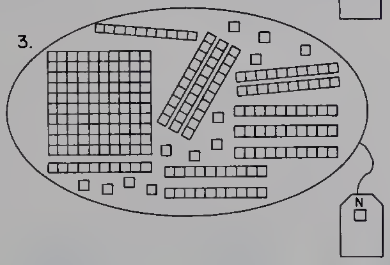
1.



2.



3.



Drill

Here is a suggestion for a drill sheet:

Add.

3 +4	2 +5	7 +3	1 +2	3 +2
4 +4	1 +5	7 +2	6 +4	5 +5
8 +2	3 +5	6 +0	5 +2	4 +3
1 +8	7 +1	6 +3	5 +3	3 +6
4 +6	3 +7	7 +0	2 +4	4 +5

Here are two stories and sample number sentences:

1) That cake has only 5 candles on it. But Bobby is 8 years old today.

$5+\square=8$ $8-5=3$ $8>5$

2) Anne has 9 cupcakes. 6 of them are chocolate and the rest are vanilla.

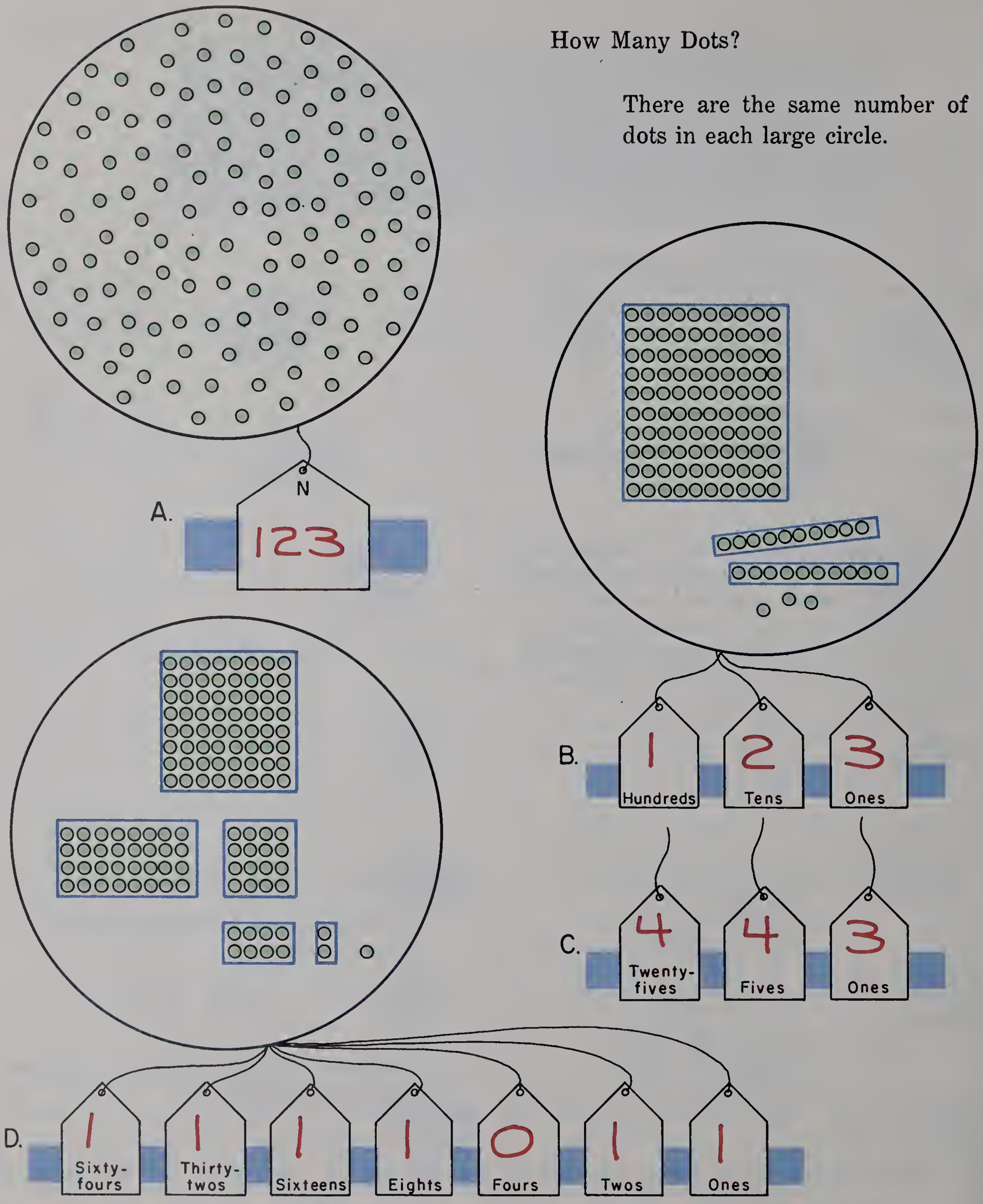
$6+\triangle=9$ $9-6=\square$ $6>3$

Headline for the Day: Extend the table:

Dollars	Dimes	Pennies
1	10	100
2	20	___
___	___	300

How Many Dots?

There are the same number of dots in each large circle.


A.
N
123
B.
1
Hundreds
2
Tens
3
Ones
C.
4
Twenty-fives
4
Fives
3
Ones
D.
1
Sixty-fours
1
Thirty-twos
1
Sixteens
1
Eights
0
Fours
1
Twos
1
Ones

E. What's My Rule?

b + b	6	2	10	20	16	24	0	40	30	50
b	3	1	5	10	8	12	0	20	15	25

Notes on Pupil Page 53

Tasks: Pupils record on the tags the number of dots displayed in each circle. They fill in the table in E.

Purpose: To extend pupils' understanding of numeration systems.

Unifying Ideas: Number and counting; numeration; functions and relations.

Materials for Teacher: Ditto masters.

Materials for Pupils: Dittoed sheets.

The Lesson:

This page is designed to show pupils the connection between decimal partitioning and expressing a number in decimal notation. The partitioning used in D is of little value in forming a decimal numeral (but of great value in forming a binary one). The partitioning used in B leads immediately to a decimal numeral and, without too much trouble, to a quinary one (C).

Ask your very able pupils to use D to solve the following problem: If a dot represents a glass of water, what is the smallest number of containers (gallons, half-gallons, quarts, half-quarts, pint, half-pints) needed to hold the water represented by all the dots?

If each dot represents a penny, what is the smallest number of coins (quarters, nickels, pennies) needed for the amount of money represented by all of the dots? The partitioning in exercise C will help.

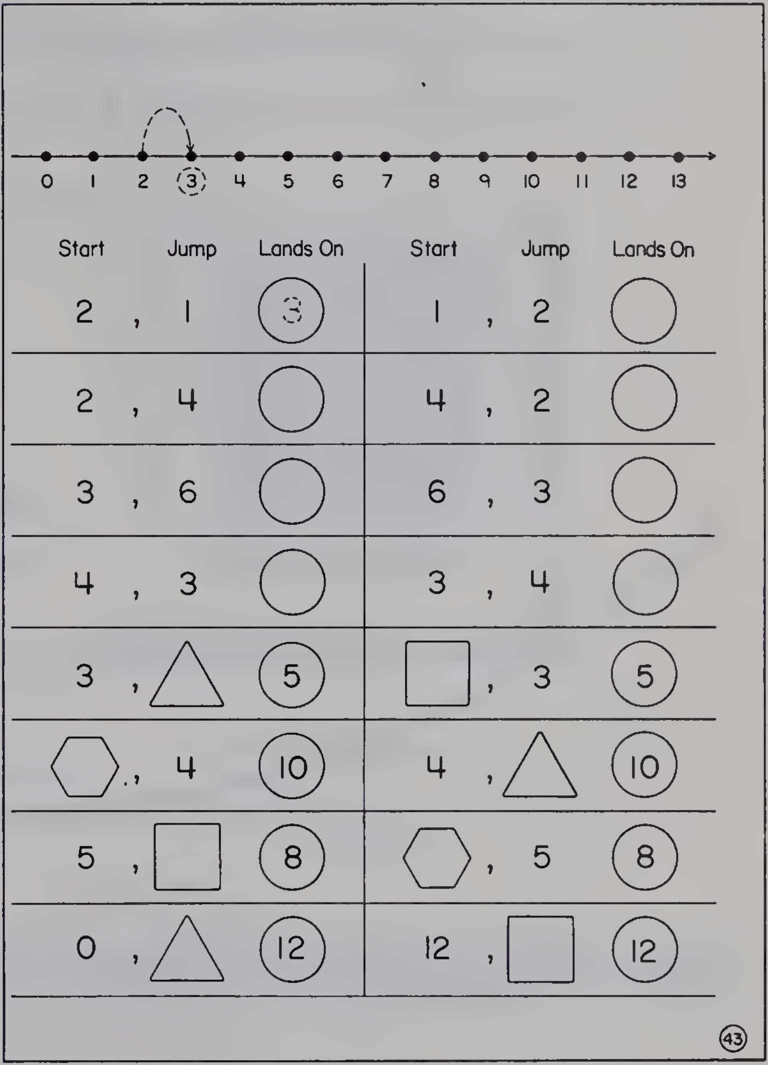
Here is a follow-up to exercise E:

In any sentence, the same number must be written in all frames of the same shape.

□ + □ = 4　　□ + □ = 12
8 = □ + □　　10 = □ + □
□ + □ = 20　　□ + □ = 24
□ = 14 − □　　18 − □ = □

Drill

Consider using as a drill sheet page 43 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Start	Jump	Lands On	Start	Jump	Lands On
2 ,	1	③	1 ,	2	○
2 ,	4	○	4 ,	2	○
3 ,	6	○	6 ,	3	○
4 ,	3	○	3 ,	4	○
3 ,	△	⑤	□ ,	3	⑤
⬡ ,	4	⑩	4 ,	△	⑩
5 ,	□	⑧	⬡ ,	5	⑧
0 ,	△	⑫	12 ,	□	⑫

㊸

Here are two stories with sample headlines:

1) Randy had 7 caps. Sue gave him 3 more.

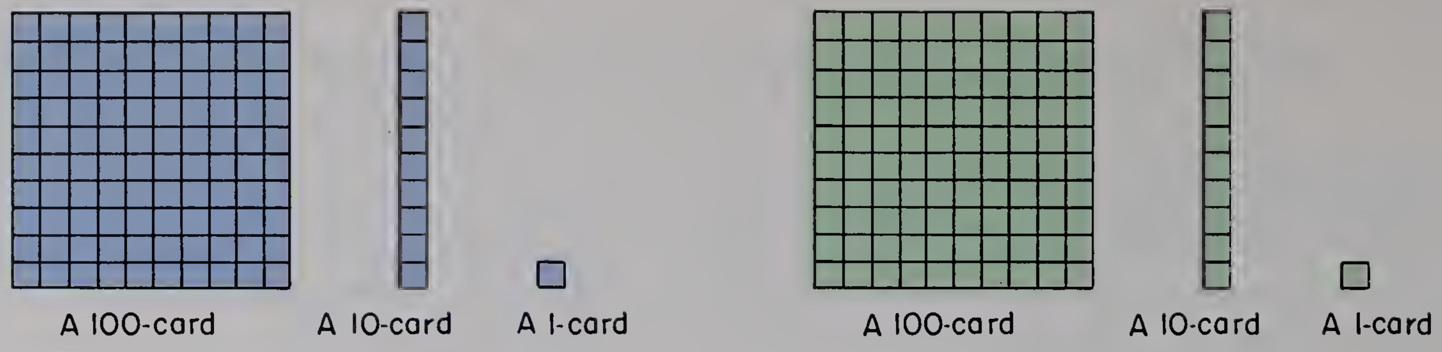
$7 + 3 = 10$　　$\square = 7 + 3$　　$7 > 3$

2) Ned's mother brought some cookies before bedtime. Ned and his 2 friends each ate 3.

$3 + 3 + 3 = 9$　　$1 + 2 = 3$

Headline for the Day: Extend the table:

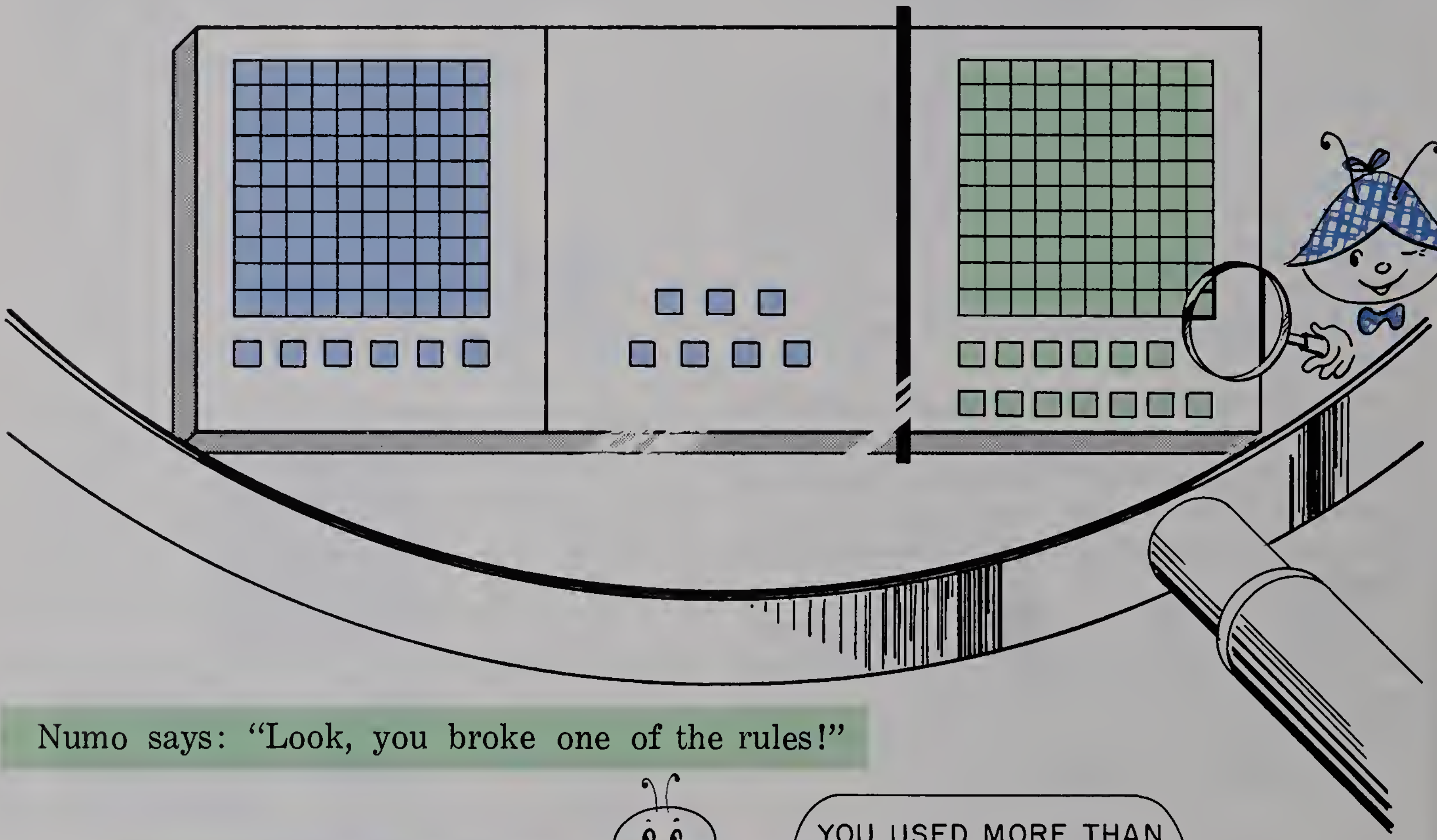
Quarts	Pints	Half-Pints
1	2	4
2	4	8


A 100-card
A 10-card
A 1-card
A 100-card
A 10-card
A 1-card

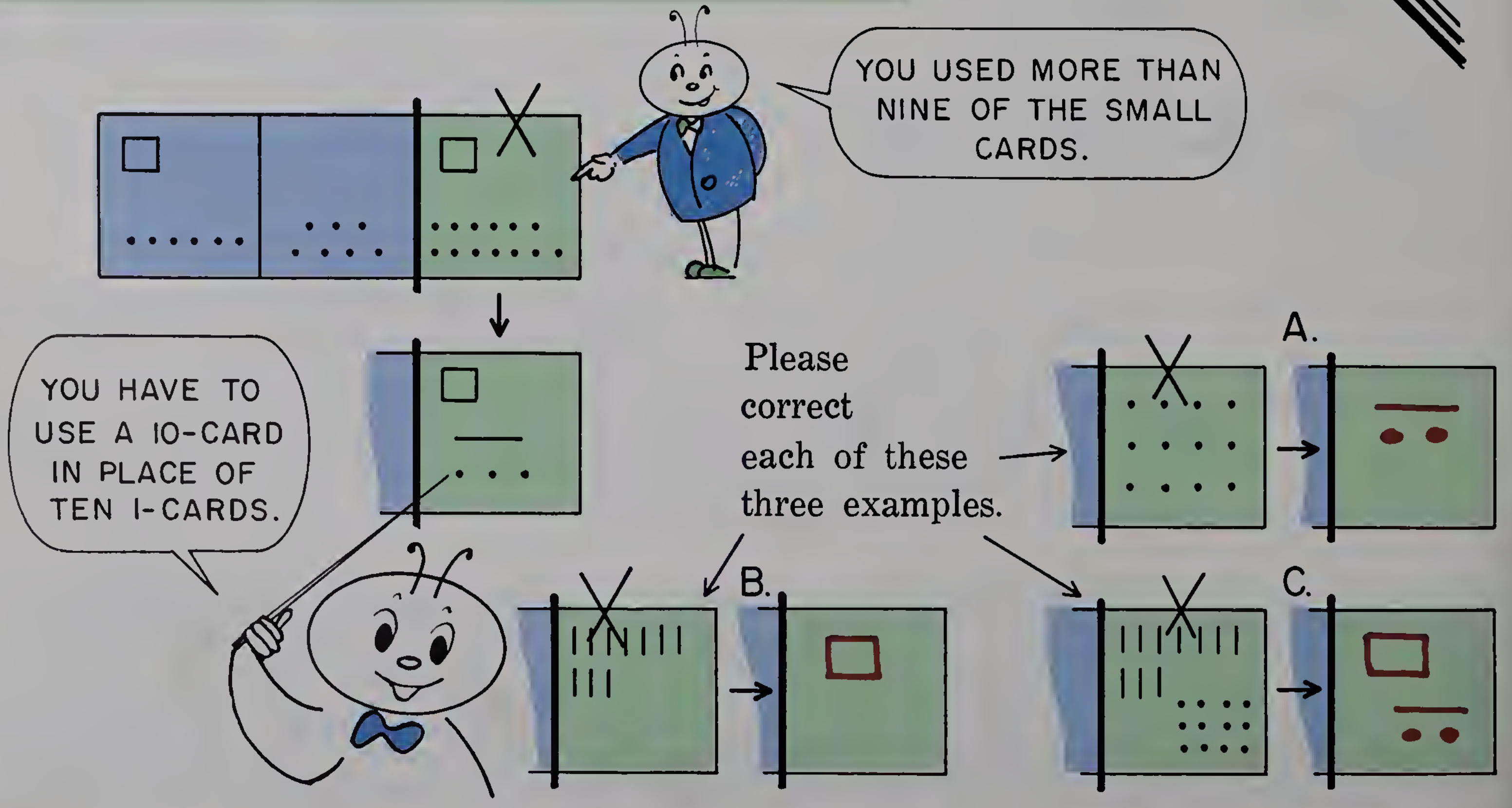
Rules:

(a) There must be the same number of little squares on each side of the heavy line.

(b) There can be no more than 9 of any card in any one of the three boxes.



Numo says: "Look, you broke one of the rules!"


YOU USED MORE THAN NINE OF THE SMALL CARDS.
YOU HAVE TO USE A 10-CARD IN PLACE OF TEN 1-CARDS.
A.
B.
C.

Please correct each of these three examples.

Notes on Pupil Page 54

Tasks: *Pupils use the rules given under the magnifying glass to correct the examples at the bottom right-hand corner of the page.*

Purpose: To extend pupils' understanding of the decimal numeration system.

Unifying Ideas: Numeration.

Materials for Teacher: 9 blue and 9 green 100-cards; 18 blue and 18 green 10-cards; 18 blue and 18 green 1-cards; flannel board.

Materials for Pupils: None.

The Lesson:

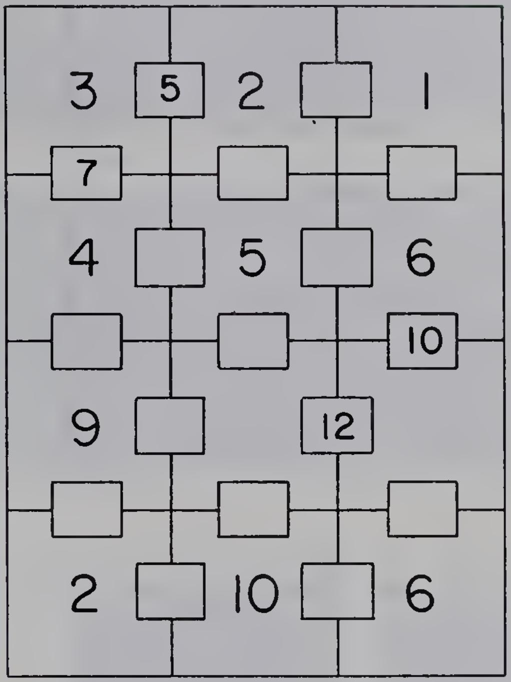
This marks the beginning of our *formal* development of the decimal numeration system, and of the algorithms for addition and subtraction. Pupils must manipulate the cards described on the page in order to fully appreciate what is happening on pupil page 54 and on pupil pages 55, 57, 58, 60, and 61.

Let John pick blue cards to represent 106 and let Mary pick blue cards to represent 7. Neither pupil may use more than 9 of any kind of card. Susan's job —rule (a)—is to represent with green cards the sum of John's number and Mary's number. If Susan tries merely to match John's and Mary's cards, she will be violating rule (b). Susan must select a 100-card, a 10-card, and three 1-cards.

Invent more problems of this type which involve exchanging cards and work on them until both rules (a) and (b) are completely understood by the class.

Drill

Here is a suggestion for a drill sheet:



Here are two stories with sample number sentences:

1) Alice had 9 cookies and ate 2 of them.

$9 - 2 = 7$ $\quad$ $2 + \square = 9$
$9 - 2 = \triangle$ $\quad$ $9 = \square + 2$

2) There are 8 pencils on the table. Frank needs 10 pencils.

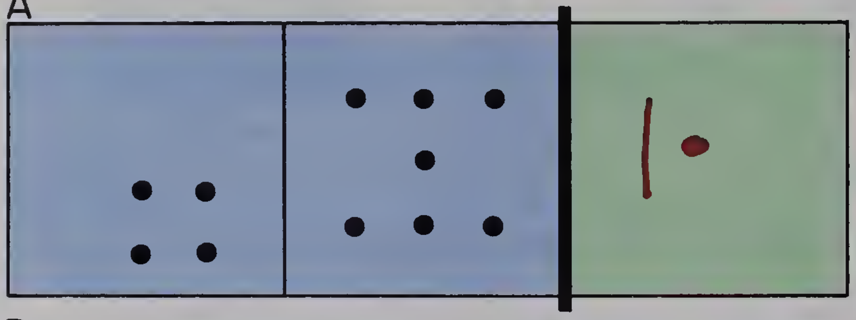
$8 + \square = 10$ $\quad$ $10 = 8 + 2$
$10 > 8$ $\quad$ $10 - 8 = \square$

Headline for the Day: In how many ways can you make this sentence true using only odd numbers? $\square + \triangle = 10$

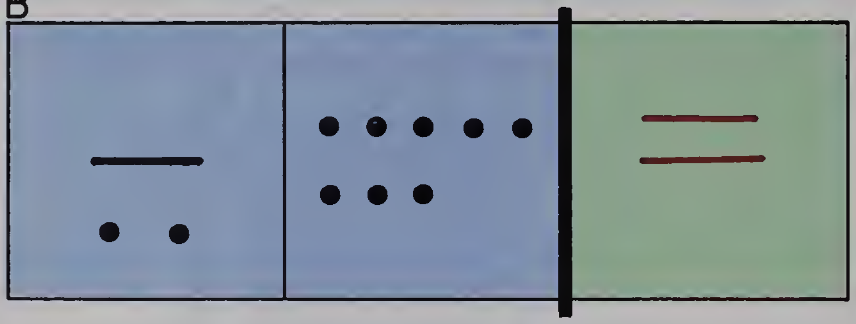
Follow Rule (a) and Rule (b) from previous page.

— or | stands for 10 objects and • stands for 1 object.

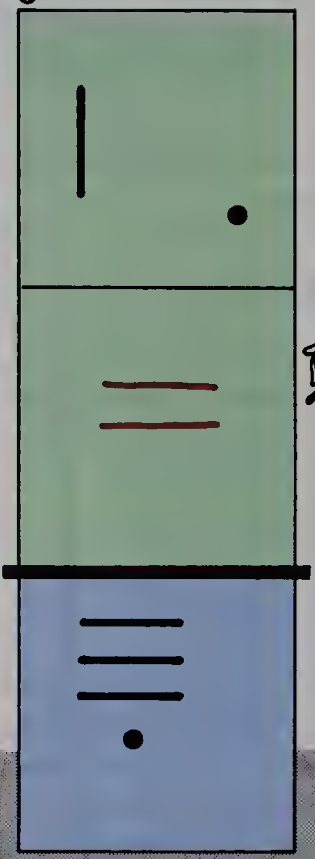
A



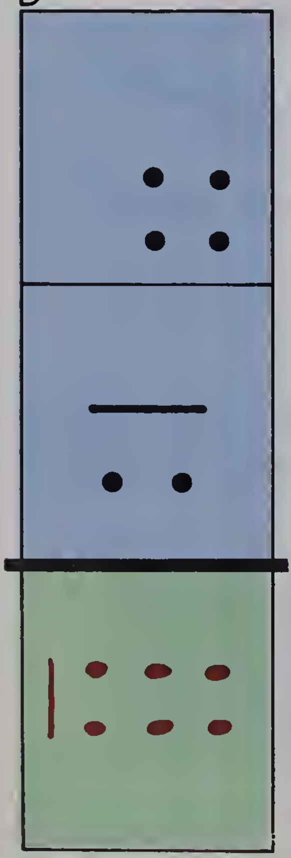
B



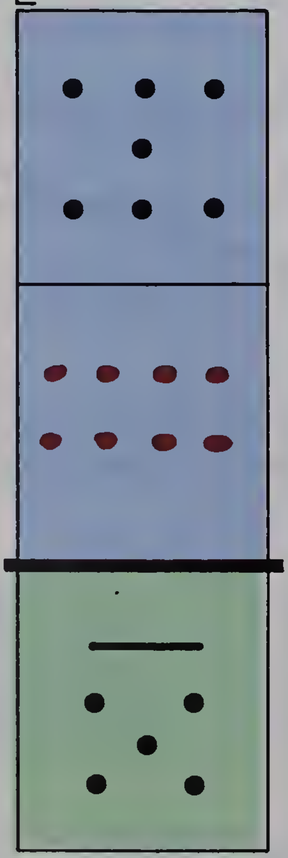
C



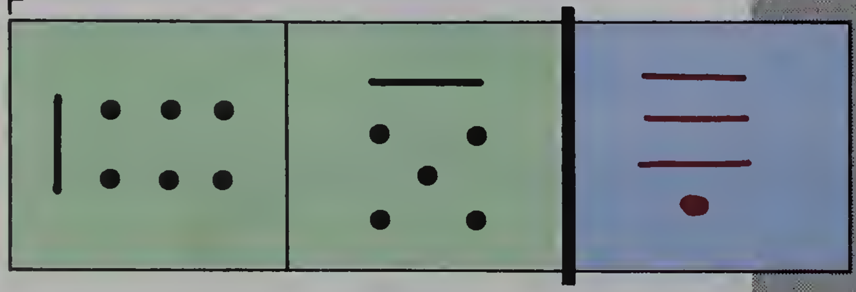
D



E



F



Record above results.

a

4	7	11

b

12	8	20

c

11
20
31

d

4
12
16

e

7
8
15

f

16	15	31

Notes on Pupil Page 55

Tasks: *Pupils use the notation suggested at the top of the page to fill in the missing information in the blue or green boxes. They then use decimal notation to record the counts in the correspondingly labeled smaller boxes in the lower right-hand corner.*

Purpose: (a) To extend pupils' understanding of numeration systems; (b) to prepare pupils for exchanging in carrying out addition or subtraction.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

Materials for Teacher: Cards described in the Notes on Pupil Page 54.

Materials for Pupils: None.

The Lesson

Use the 10-cards and 1-cards to act out the addition problems in exercises A, B, and F and the subtraction problems in exercises C and E.

Do you see that the 3-compartment rectangles in the eight exercises form a *Cross Number Puzzle?*

Drill

Here are two stories with sample number sentences:

1) Marty read 2 stories on Monday, 1 story on Thursday, and 8 stories on Friday.

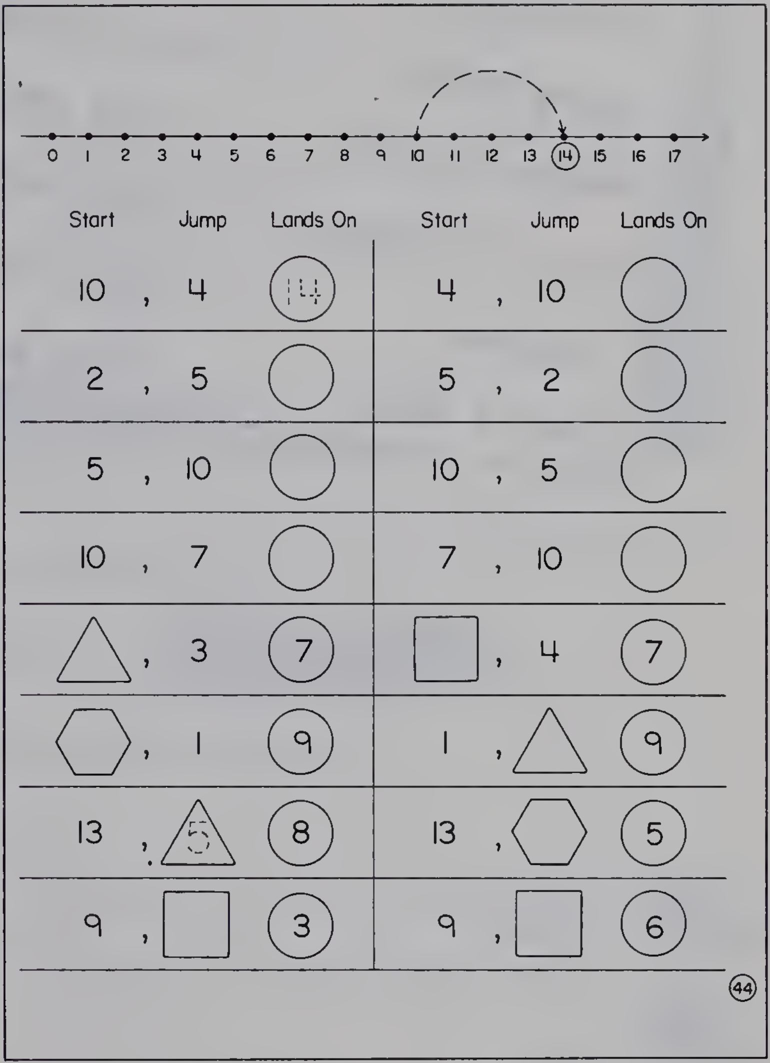
$2 + 1 + 8 = \square \qquad 8 > 2 + 1$

2) There are 8 boys and 9 girls in Alice's class. Seven of the children ride the bus to school, and the rest walk.

$7 + \square = 8 + 9 \qquad 17 - 7 = 10$

$9 > 8 \qquad 7 < 10$

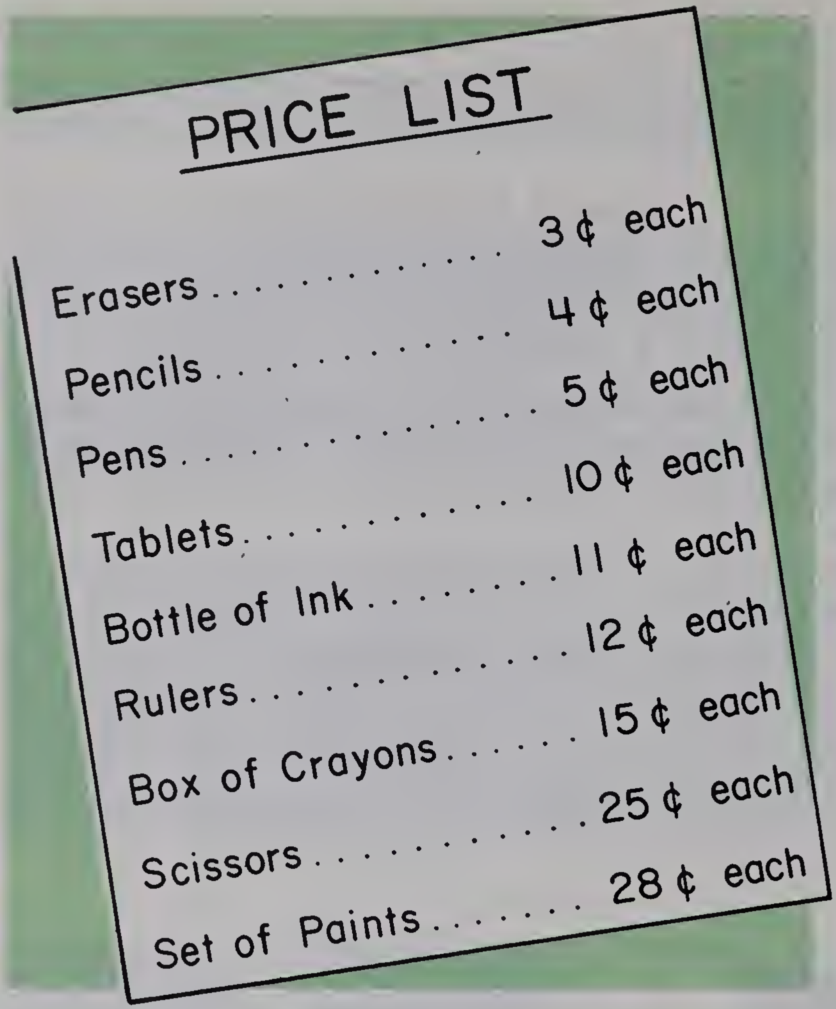
Consider using as a drill sheet page 44 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:


0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17

Start	Jump	Lands On	Start	Jump	Lands On
10 ,	4	14	4 ,	10	○
2 ,	5	○	5 ,	2	○
5 ,	10	○	10 ,	5	○
10 ,	7	○	7 ,	10	○
△ ,	3	7	□ ,	4	7
⬡ ,	1	9	1 ,	△	9
13 ,	△ 5	8	13 ,	⬡	5
9 ,	□	3	9 ,	□	6

㊹

Headline for the Day: Extend the pattern: A, d, G, j, M



PRICE LIST

Erasers	3 ¢ each
Pencils	4 ¢ each
Pens	5 ¢ each
Tablets	10 ¢ each
Bottle of Ink	11 ¢ each
Rulers	12 ¢ each
Box of Crayons	15 ¢ each
Scissors	25 ¢ each
Set of Paints	28 ¢ each

Bill spent 15¢.

What did he buy?

We don't know for sure.
But we know he could have bought:

a. 1 box of crayons
b. 1 tablet and 1 pen
c. 3 pens
d. 5 erasers
e. 1 ruler and 1 eraser
f. 1 eraser and 3 pencils

Mary spent 50¢. What did she buy?

We don't know, but she could have bought:

g. 5 tablets
h. 10 pens
i. 2 scissors
j. 1 scissors and 5 pens
k. 4 pens and 3 tablets
l. 4 bottles of ink and 2 erasers
m. 1 paint set and 2 bottles of ink

n. 10 erasers and 2 tablets
o. 2 boxes of crayons and 2 tablets
p. 4 tablets and 2 pens
q. 3 boxes of crayons and 1 pen
r. 2 tablets and 2 boxes of crayons
s. 5 pencils and 6 pens
t. 5 tablets and 0 rulers

Notes on Pupil Page 56

Tasks: *Pupils record the number of items of various kinds that could be bought at prices indicated on the list.*

Purpose: To test pupils' ability to work with one aspect of applied arithmetic—the buying and selling of goods.

Unifying Ideas: Addition and subtraction; multiplication and division; measurement.

Materials for Teacher: None.

Materials for Pupils: Graph paper; Cuisenaire rods.

The Lesson:

Suggested point distribution: 1 point for each of items *g* through *t*.

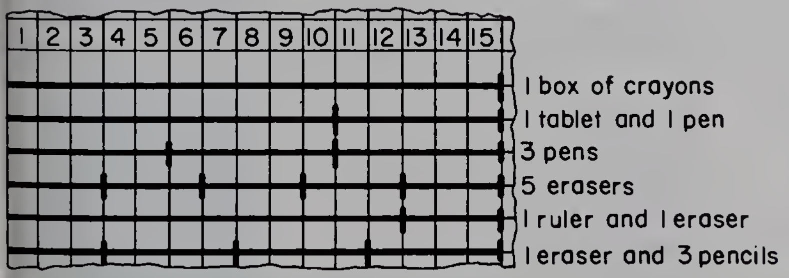
One way to prepare for this test page is to make a similar price list and consider possible purchases:

Price List

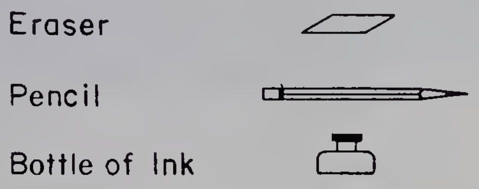
Ice Cream	10¢ each
Cake	20¢ each
Candy	5¢ each

How could you spend 10¢, 20¢, 15¢, 25¢, etc? If Mary spent 20¢ (or 30¢, 25¢, 50¢), what might she have purchased? Perhaps you can get teams to explore all the ways in which any amount (say 25¢) might be spent.

Pupils can help themselves solve the problems on pupil page 56 by using graph paper or Cuisenaire rods. Here is how they might use graph paper for exercises *a-f*:



Since a pupil's completing page 56 depends on his being able to read the items on the price list, go over the words at the board. Perhaps rough sketches next to the words would help:



Number sentences should be written for each purchase involving two or more of any item or combination of items:

1) 1 tablet and 1 pen

$10 + 5 = 15$

2) 5 erasers

$3 + 3 + 3 + 3 + 3 = 15$

Tell the combinations of items represented by the following sentences:

11 + 12 = ____
10 + 5 = ____
3 + ____ = 7

Drill

Here are two stories with sample number sentences:

1) Sally has 4 doll beds. She put 2 dolls in each bed.

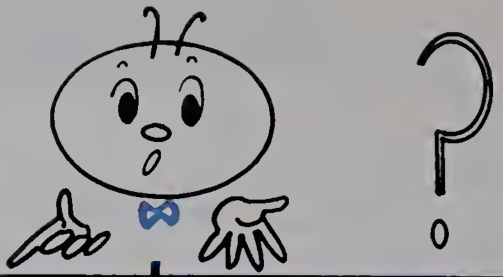
2 + 2 + 2 + 2 = □

2) There were 9 ducks in the lake. They all flew away.

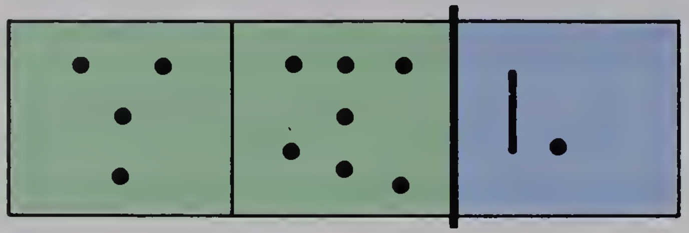
9 − 9 = □   9 − △ = 0

Headline for the Day: Write a number sentence that fits: How many hours between 7 AM.. and 7 P.M.?

On this page: ☐ stands for 100 objects
— or | stands for 10 objects
• stands for 1 object



A



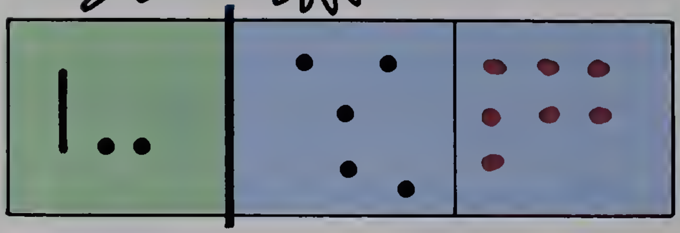
4 + 7 = 11

7 + 4 = 11

11 − 4 = 7

11 − 7 = 4

B



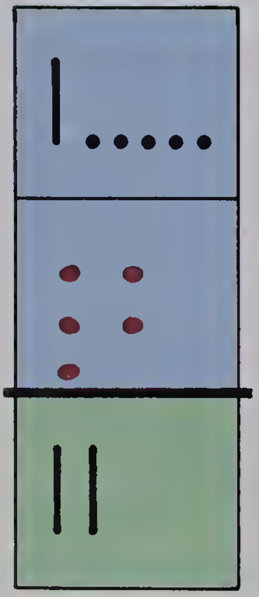
12 = 5 + 7

12 = 7 + 5

7 = 12 − 5

5 = 12 − 7

C



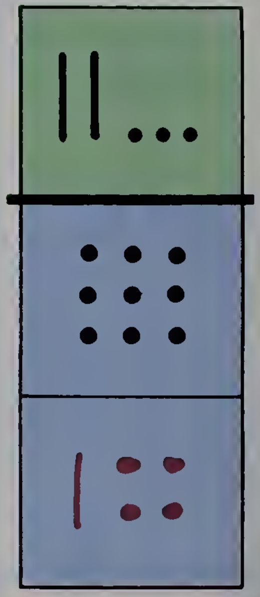
$$\begin{array}{r} 15\\ +\ 5\\ \hline 20\end{array} \qquad \begin{array}{r} 5\\ +\ 15\\ \hline 20\end{array}$$

$$\begin{array}{r} 20\\ -\ 5\\ \hline 15\end{array} \qquad \begin{array}{r} 20\\ -\ 15\\ \hline 5\end{array}$$

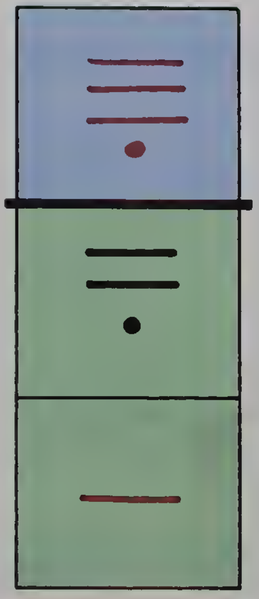
D

$$\begin{array}{r} 23\\ -\ 9\\ \hline 14\end{array} \qquad \begin{array}{r} 23\\ -\ 14\\ \hline 9\end{array}$$

$$\begin{array}{r} 14\\ +\ 9\\ \hline 23\end{array} \qquad \begin{array}{r} 9\\ +\ 14\\ \hline 23\end{array}$$



E



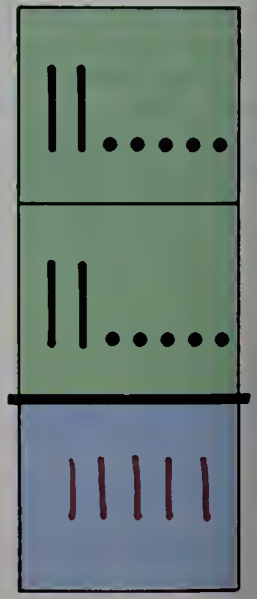
$$\begin{array}{r} 31\\ -\ 21\\ \hline 10\end{array} \qquad \begin{array}{r} 31\\ -\ 10\\ \hline 21\end{array}$$

$$\begin{array}{r} 10\\ +\ 21\\ \hline 31\end{array} \qquad \begin{array}{r} 21\\ +\ 10\\ \hline 31\end{array}$$

F

$$\begin{array}{r} 25\\ +\ 25\\ \hline 50\end{array} \qquad \cancel{\begin{array}{r} \underline{\quad}\\ +\ \underline{\quad}\\ \hline \underline{\quad}\end{array}}$$

$$\begin{array}{r} 50\\ -\ 25\\ \hline 25\end{array} \qquad \cancel{\begin{array}{r} \underline{\quad}\\ -\ \underline{\quad}\\ \hline \underline{\quad}\end{array}}$$



Notes on Pupil Page 57

Tasks: *Pupils use the notation suggested at the top of the page to fill in the missing information in either the blue or green boxes. They then use decimal notation to complete the entire fact team related to each picture.*

Purpose: To extend pupils' understanding of numeration systems and to help them view addition and subtraction via fact teams in a new setting.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

Materials for Teacher: Cards described in the Notes on Pupil Page 54; Ditto masters.

Materials for Pupils: Dittoed sheets.

The Lesson:

In this lesson we combine the manipulation of cards, of special symbols, and of number sentences in standard forms.

Each exercise should be acted out with 10-cards and 1-cards. In exercise B there might be a tendency for pupils to draw a bar and seven dots in the blue space on the right. (Of course, the first number sentence helps to forestall this misinterpretation.) This may be the time to remind pupils of the rule—rule (a) on pupil page 54—that there must be the same number of objects represented on one side of the heavy black line as there is on the other side. This misinterpretation (drawing an extra bar) may occur in exercise D. Be on the lookout for it.

Drill

Here are two stories with sample number sentences:

1) Ginger had 2 dogs, 3 cats, and 4 birds for pets.

$2 + 3 + 4 = \square \qquad 2 < 3 < 4$

2) Skip and Zeke were building a tree house. Skip had a board 14 feet long, and he sawed off a piece that was 6 feet long.

$14 - 6 = 8 \qquad 14 - \square = 6$

$6 + \square = 14 \qquad \square + 6 = 14$

Fact Teams and Stories

As indicated in the "Headline for the Day" for this lesson, pupils may be asked to tell a story for each fact team. Note that all of the sentences in a fact team are appropriate as headlines for the same story. Also, any sentence in a fact team could fit four types of stories. For example, the sentence $4 + 7 = 11$ would appropriately serve as a headline for any of the following stories:

1) Bill had 4 lemons and his friend gave him 7 grapes. How many pieces of fruit did he then have?
2) Bill's friend gave him 7 grapes. Bill already had 4 lemons. How many pieces of fruit did he then have?
3) Bill had 11 pieces of fruit. 4 of these were lemons. The rest were grapes which he got from his friend. How many grapes did he get from his friend?
4) Bill had 11 pieces of fruit. 7 of these were grapes which he got from his friend. The rest were lemons. How many lemons did he have?

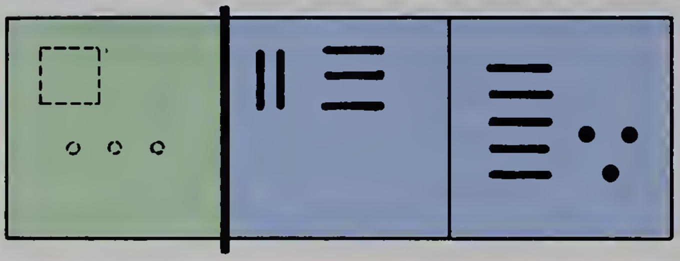
Headline for the Day: Write number stories which fit the fact teams of page 57.

Rules:

(a) In any row or column of 3 boxes, the same number of objects must be shown on each side of the heavy line.

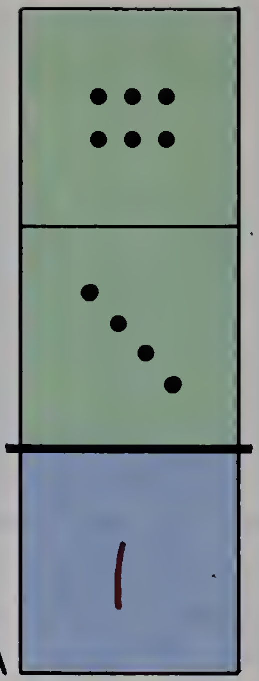
(b) No more than 9 of the same kind of shorthand mark can be in any one box.

(c) ☐ stands for 100 objects
— or | stands for 10 objects
• stands for 1 object



Practice

A



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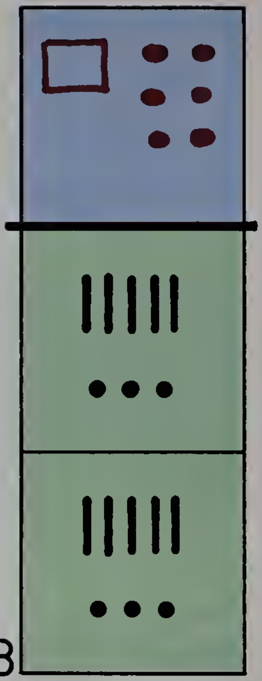

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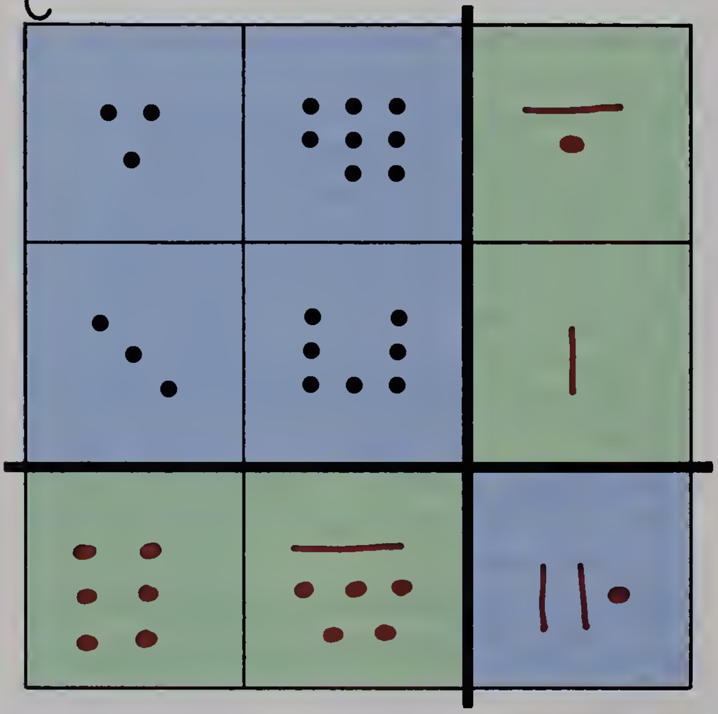
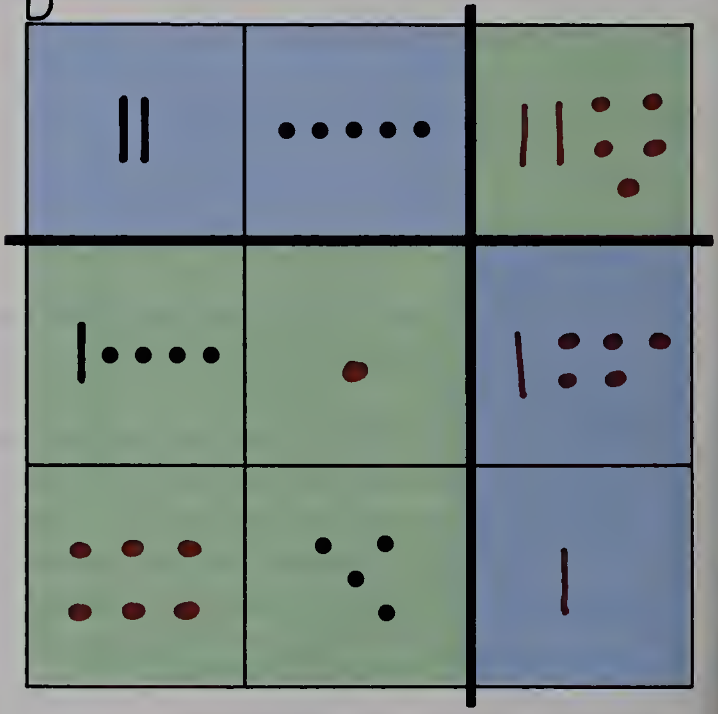


D



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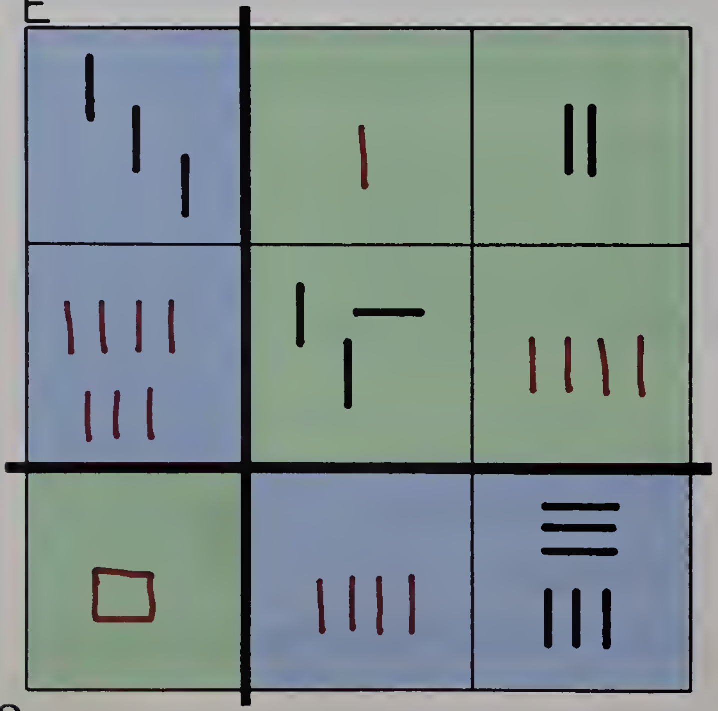
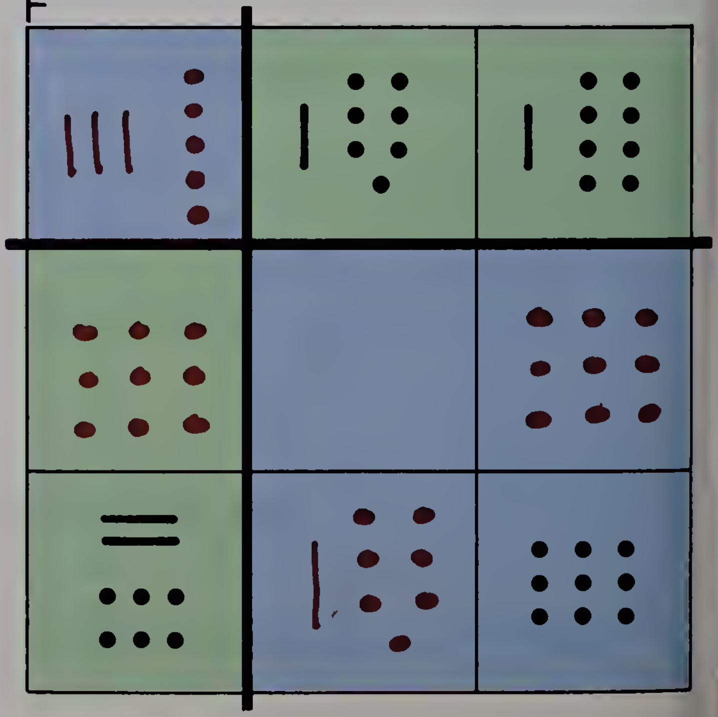


F



Notes on Pupil Page 58

Tasks: Pupils use the special symbols and rules to complete the Cross Number Puzzles.

Purpose: (a) To extend pupils' understanding of decimal numeration; (b) to prepare pupils for the exchanging step in the addition and subtraction algorithms.

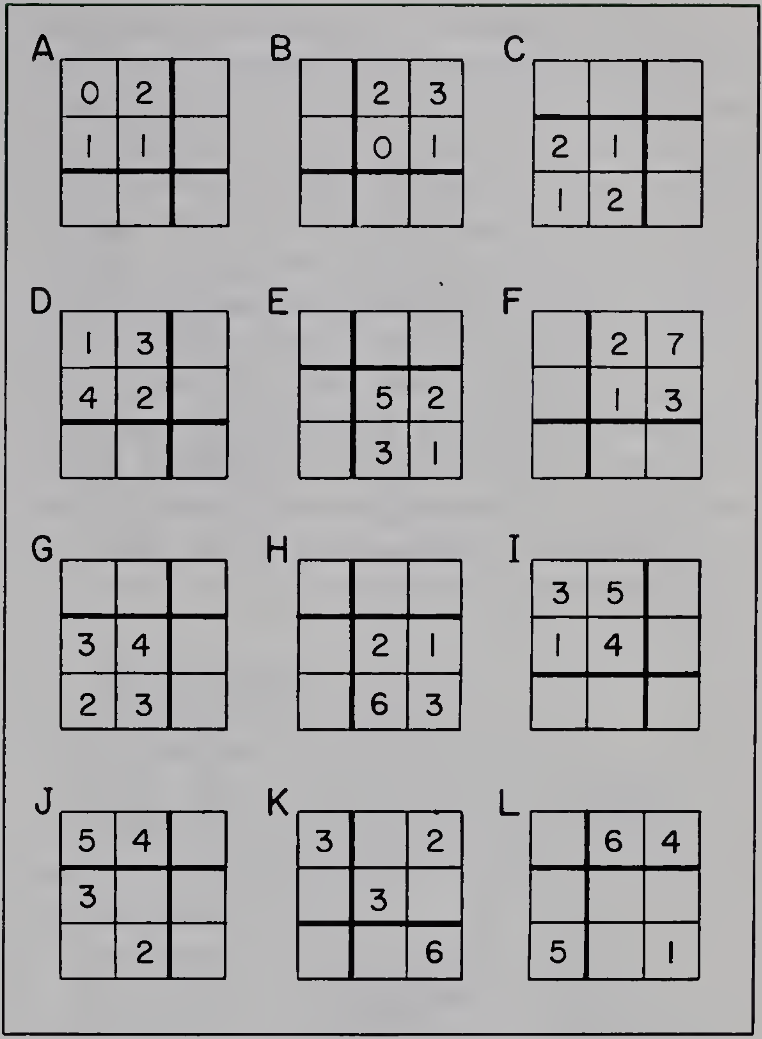
Unifying Ideas: Structure; number and counting; numeration; addition and subtraction.

Materials for Teacher: Ditto master.

Materials for Pupils: Counters (e.g., white Cuisenaire rods); short sticks each representing 10 objects (e.g., orange Cuisenaire rods); Dittoed sheets; *Cross Number Puzzle* boards (described below).

The Lesson:

Note that the *Cross Number Puzzles* on pupil page 58 have the total-box in various positions. We suggest that you prepare pupils for this by giving them a Dittoed sheet of *Cross Number Puzzles* such as the following or by working several problems on the chalkboard. Be sure to include the heavy lines as shown in the following column.



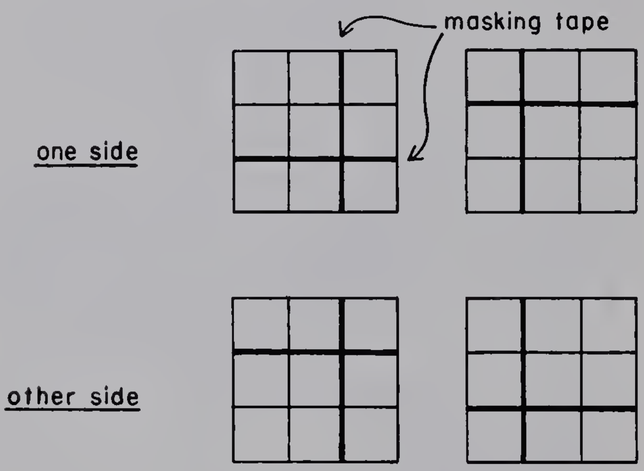
more

Headline for the Day: Write number sentences for the following:

4 of the 25 balloons he brought for the party floated away.

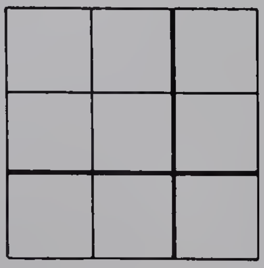
The score was 25 to 16. We were ahead by ___ points.

It is very instructive to have pupils work out some of the *Cross Number Puzzles* on pupil page 58 by manipulating objects on a *Cross Number Puzzle* board. Such a board can easily be constructed out of a piece of stiff cardboard 12″ by 12″. Rule both front and back of the board into nine 4″ by 4″ squares. Then use masking tape for the heavy lines. By putting the masking tape cross on the board to make the lower right-hand cell serve as the total-box, you automatically have a board which can be used with the upper left-hand cell serving as the total-box—just rotate the board a half turn. Similarly, the reverse side can be taped to give you a board on which either the upper right-hand cell or the lower left-hand cell can serve as the total-box:

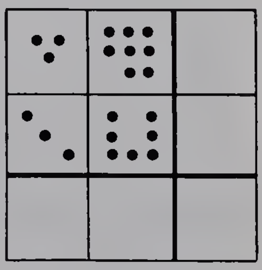


Now let's see how to use the board to solve exercise C. Position the board so that the total-box is in the lower right-hand corner (Step 1). Place counters or white Cuisenaire rods in the four cells that are above and to the left of the masking tape (Step 2):

Step 1

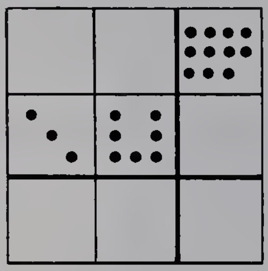


Step 2

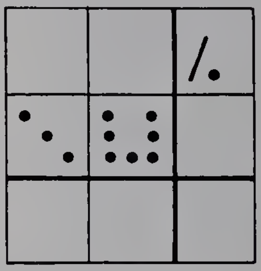


Now our job is to find what ought to go in the other cells. Let's concentrate on the upper right-hand cell. To find what goes there, one simply sweeps the contents of the other two upper cells across the masking tape into the cell in question (Step 3). But rule (b) tells us that we can't have more than 9 of the same kind of markers in a cell. So we exchange 10 of these markers for a 10-marker (Step 4). (Use either an ordinary stick or an orange Cuisenaire rod.)

Step 3



Step 4



We can now draw a bar and a dot in the upper right-hand cell in exercise C on pupil page 58. Unless we restore the board to its condition at the end of Step 2, we are now forced to determine what goes in the right-hand cell of the middle row. (If we restore the 1-markers, we have the same freedom the pupil has when he works the *Cross Number Puzzle* in his book.)

By sweeping the markers across the masking tape and exchanging, we discover that a bar should be drawn in the right-hand cell of the middle row. At this point, only two cells have markers in them. Since they are both in the right-hand column, the markers in them can be used to find out what should be drawn in the total-box—just sweep the markers down across the masking tape and exchange if necessary.

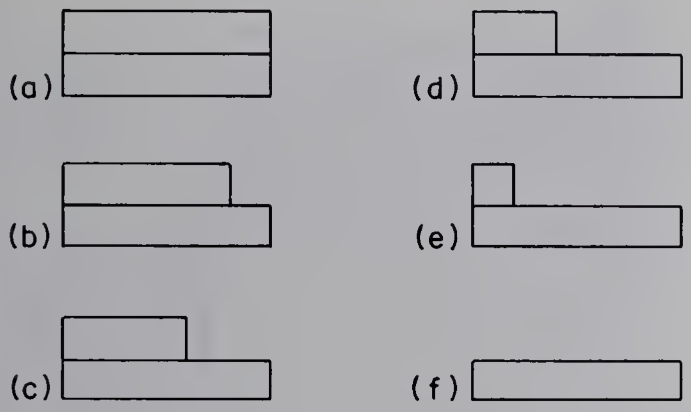
In order to find out what should be drawn in the cells in the bottom row and to the left of the masking tape, we must restore the markers in at least one of the columns. Luckily, we can use the picture on the pupil page to do this. Let's suppose that we have found out that six 1-markers belong in the lower left-hand cell. We can determine what should be drawn in the middle cell in either of two ways—restore the markers in the middle column and then add, or subtract the content of the lower left-hand cell from the content of the total-box. The subtraction procedure requires that one of the two 10-markers be exchanged for ten 1-markers.

If a pupil cannot master the intellectual tasks involved in solving a *Cross Number Puzzle* with the use of physical markers, you should regard this as evidence that he is not yet ready to understand the algorithms involving exchanging. Although he might be able to execute the algorithms on paper, he is doing so in a strictly rote fashion. There is little chance that he will be able to use his skill in an applied situation.

Cuisenaire Rods

Establish the agreement that the white rod represents 10. Then have pupils complete the following number sentences and construct models based on this agreement. The diagram below shows models for sentences (a) through (f).

(a) $50 - 50 = \triangle$
(b) $50 - 40 = \triangle$
(c) $50 - 30 = \triangle$
(d) $50 - 20 = \triangle$
(e) $50 - \triangle = 40$
(f) $50 - \triangle = 50$



Drill

Here are two *What Are My Rules?* games for a drill sheet:

$x - 1$	7	16	15			25		0			
x	8	17	16		41		6		32	49	
$x + 1$	9	18		14							51

$x - 2$	3	8	1			24					
x	5	10	3	11	17			2	6	40	51
$x + 2$	7	12					9				

Here are two stories with sample number sentences:

1) Jack had 5 marbles. Bobby had 2 marbles.

$5 + 2 = \square$ $7 - 2 = 5$
$\square - 5 = 2$ $5 > 2$

2) Andy has 12 toy boats. 7 of them are blue. The others are yellow.

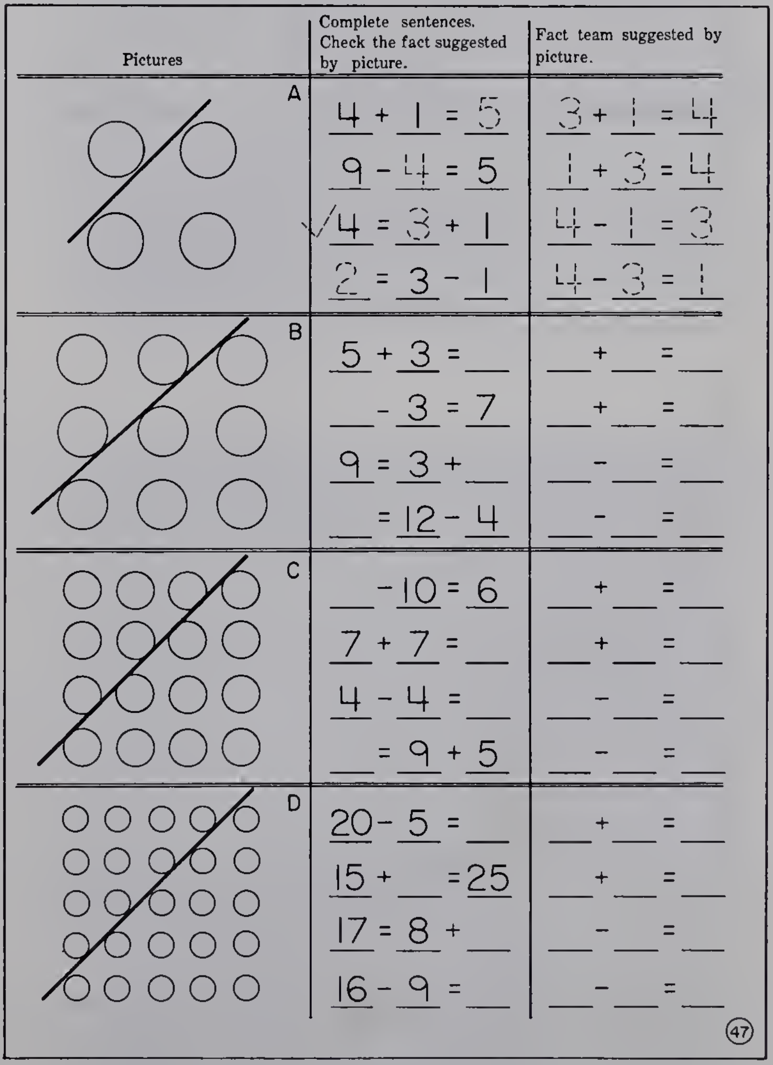
$12 - 7 = \square$ $7 + 5 = 12$

Here is another suggested drill sheet:

Subtract

8 −3	10 −7	9 −2	7 −3	7 −6
8 −5	4 −3	10 −6	7 −5	9 −7
9 −3	10 −5	7 −4	5 −3	6 −6
9 −6	8 −0	10 −8	9 −5	6 −4
6 −2	9 −1	8 −4	10 −3	10 −9

Consider using a drill sheet to follow up the work on fact teams. Try page 47 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Pictures	Complete sentences. Check the fact suggested by picture.	Fact team suggested by picture.
A	4 + 1 = 5 9 − 4 = 5 ✓ 4 = 3 + 1 2 = 3 − 1	3 + 1 = 4 1 + 3 = 4 4 − 1 = 3 4 − 3 = 1
B	5 + 3 = ___ ___ − 3 = 7 9 = 3 + ___ ___ = 12 − 4	___ + ___ = ___ ___ + ___ = ___ ___ − ___ = ___ ___ − ___ = ___
C	___ − 10 = 6 7 + 7 = ___ 4 − 4 = ___ ___ = 9 + 5	___ + ___ = ___ ___ + ___ = ___ ___ − ___ = ___ ___ − ___ = ___
D	20 − 5 = ___ 15 + ___ = 25 17 = 8 + ___ 16 − 9 = ___	___ + ___ = ___ ___ + ___ = ___ ___ − ___ = ___ ___ − ___ = ___

(47)

Rules:

(1) In any row or column of 3 boxes, the same number of objects must be shown on each side of the heavy line.

(2) No more than 4 of the same kind of shorthand mark can be in any one box.

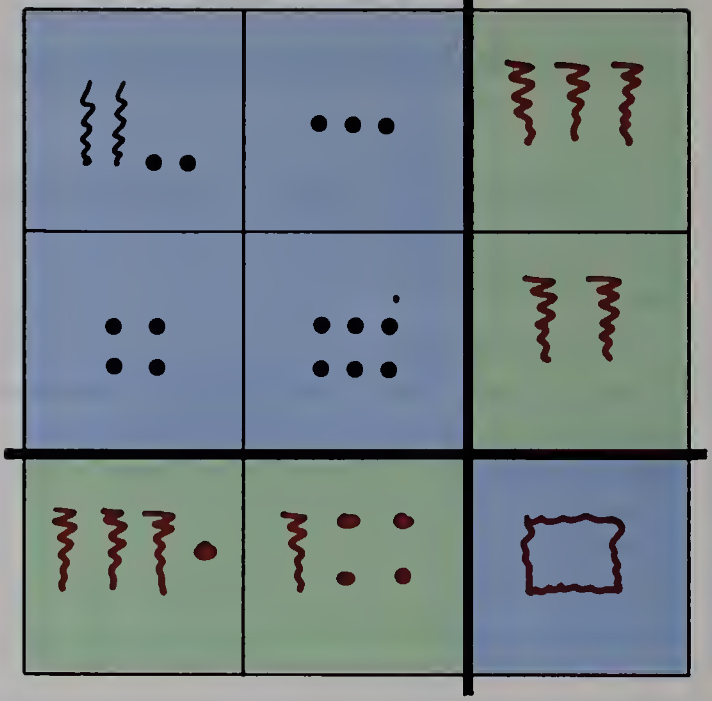
(3) For this page:

▢ stands for 25 objects

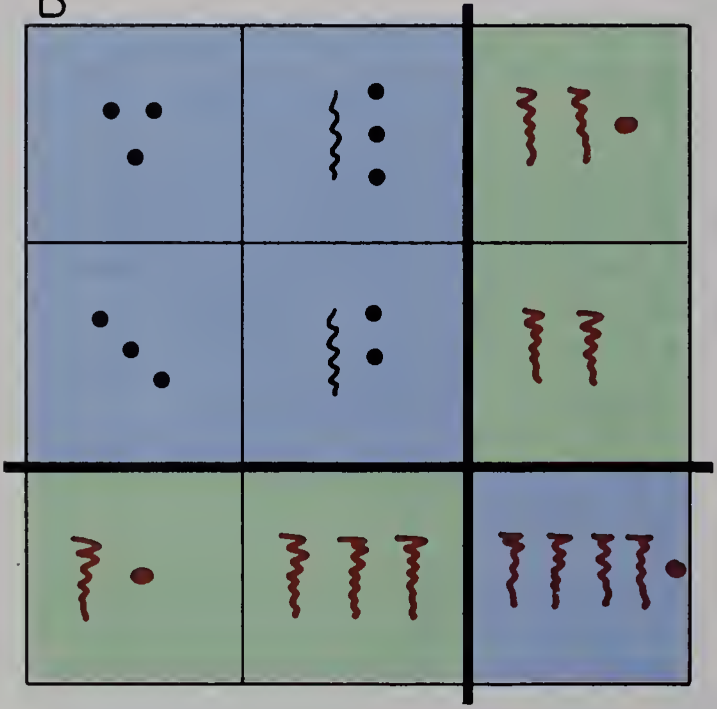
⌇ stands for 5 objects

• stands for 1 object

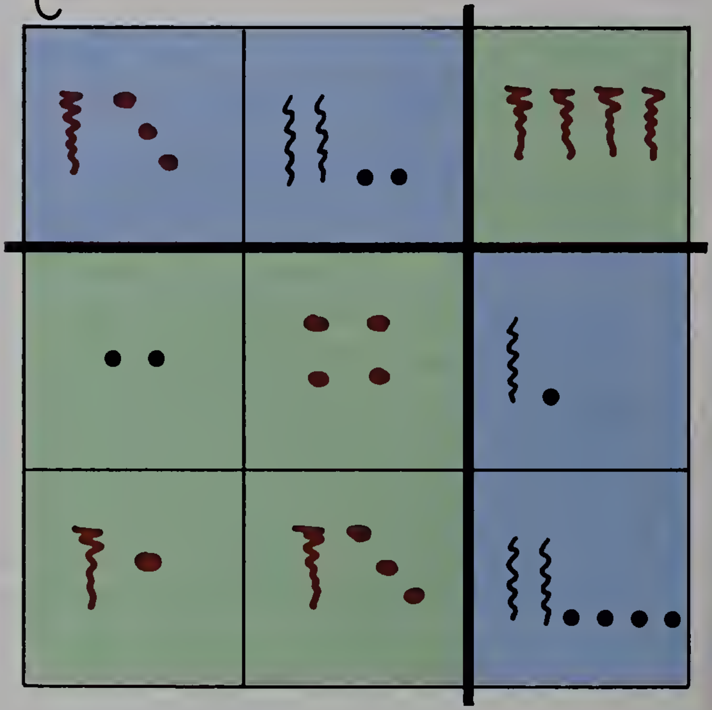
A



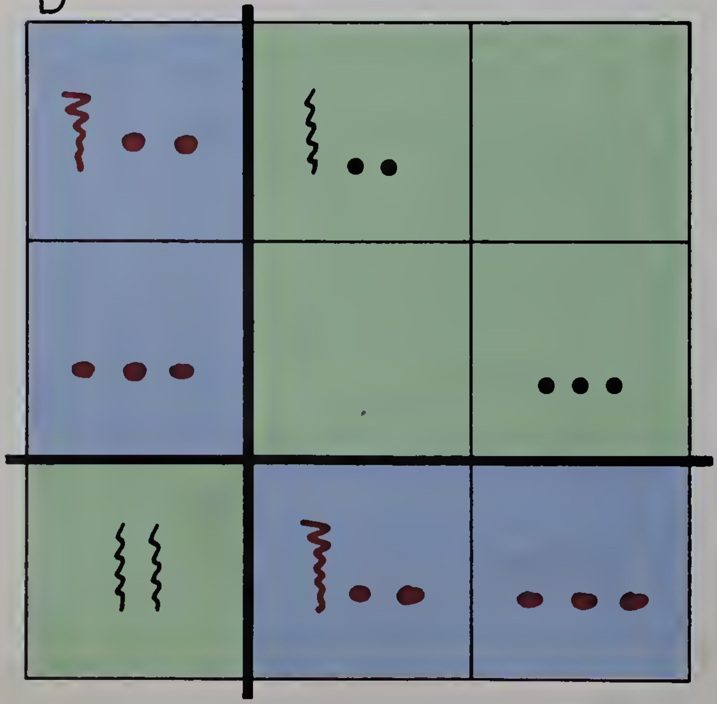
B



C



D



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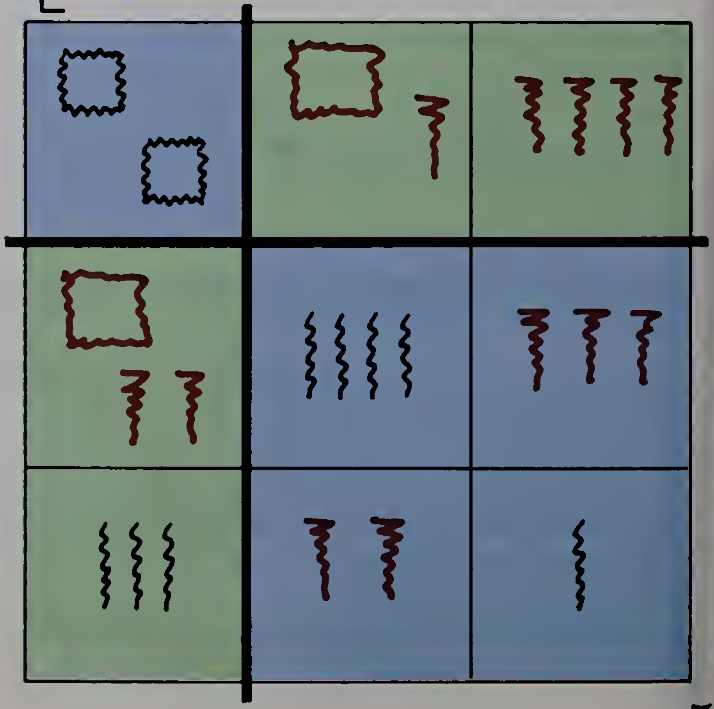


Tasks: Pupils use the special symbols and rules to complete the Cross Number Puzzles.

Purpose: (a) To extend pupils' understanding of numeration; (b) to prepare pupils for the exchanging step in the addition and subtraction algorithms.

Unifying Ideas: Structure; number and counting; numeration; addition and subtraction.

Materials for Teacher: None.

Materials for Pupils: Counters (e.g., white Cuisenaire rods); short sticks each representing 5 objects (e.g., yellow Cuisenaire rods); *Cross Number Puzzle* boards (described in the Notes on Pupil Page 58).

The Lesson:

Let pupils solve some of these exercises by manipulating markers on a *Cross Number Puzzle* board.

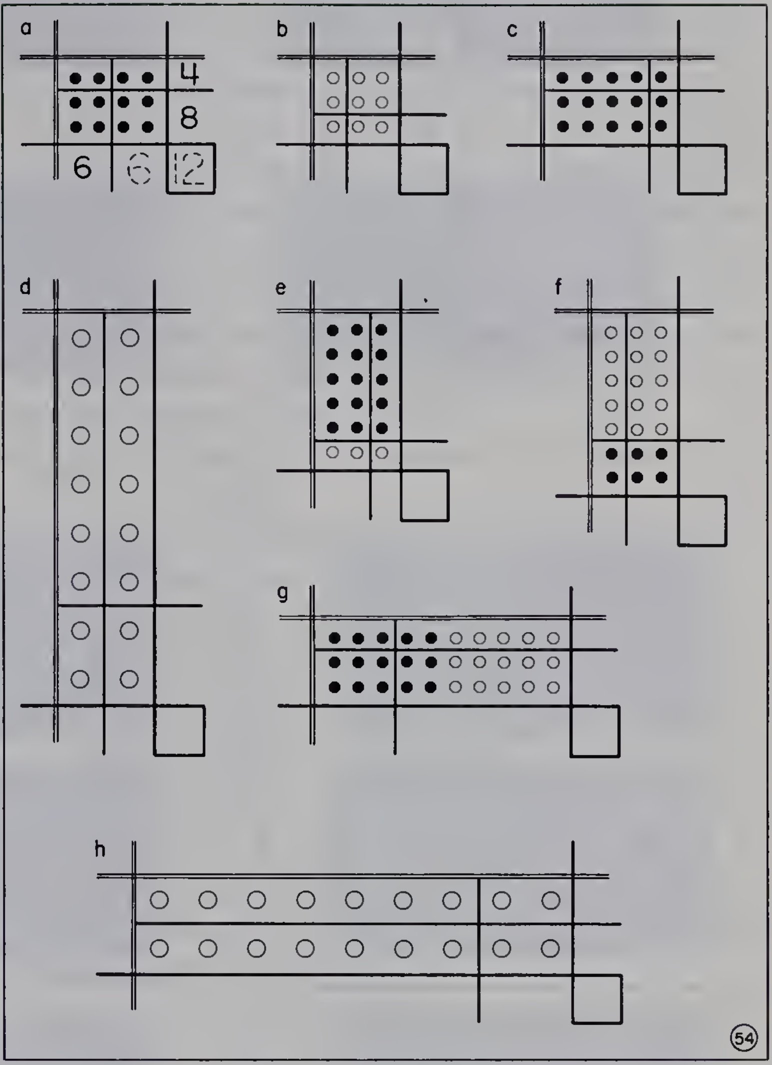
Exercise B has been designed to parallel exercise C on pupil page 58. In each case we are dealing with the same total number of objects. The grouping records in the total-boxes are quite different but the number of objects represented is the same.

The *Cross Number Puzzles* on pupil page 59 can be solved using pennies, nickels, and quarters as markers. (The ones on pupil page 58 can be solved using pennies, dimes, and dollar bills as markers.)

Exercise D on pupil page 59 is quite interesting. The markers in the total-box represent the same number of objects as the markers in the middle cell of the top row together with the markers in the right-hand cell of the middle row. Since the markers in the total-box must represent the same number of objects as are represented by the markers in all four cells above and to the right of the heavy lines, we conclude that two of those cells must be empty.

Drill

Consider using as a drill sheet page 54 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Here are two stories with sample headlines:

1) Sue saw 9 ducks in the lake. 5 of them flew away.

$9 - 5 = \square$ $5 + \triangle = 9$

$4 + 5 = 9$ $5 > 4$

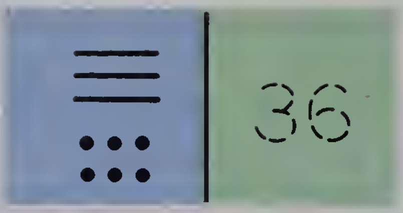
2) Clyde and Walt were measuring some sticks. Clyde found one stick that was 67 inches long. Walt found a stick that was 76 inches long.

$76 > 67$ $67 < 76$

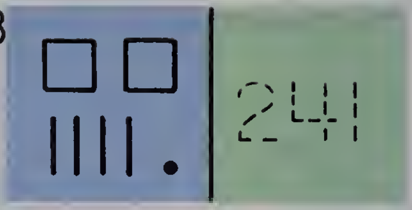
$76 - 67 = \square$ $67 + \square = 76$

Headline for the Day: I have coins worth 50¢ in my pocket. What can you say?

A



B



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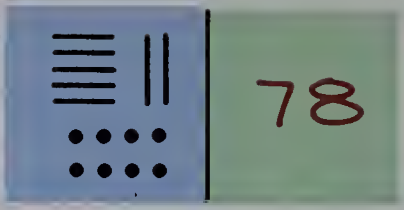
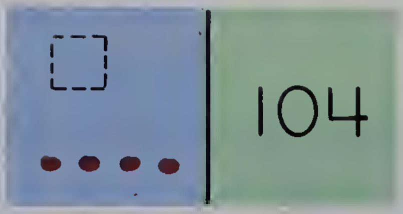


D



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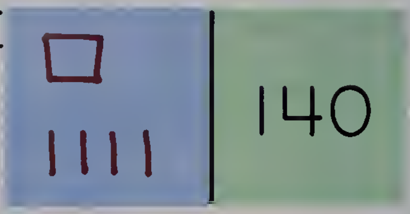

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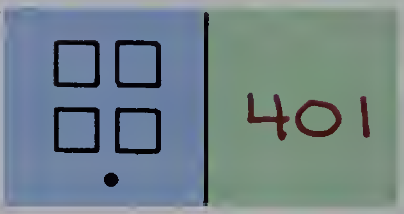

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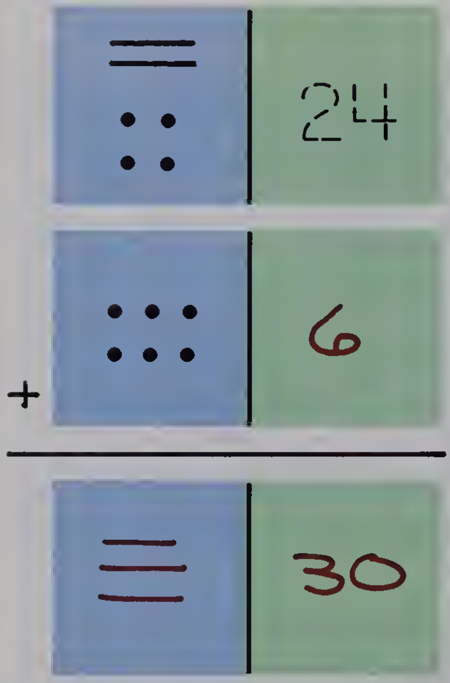

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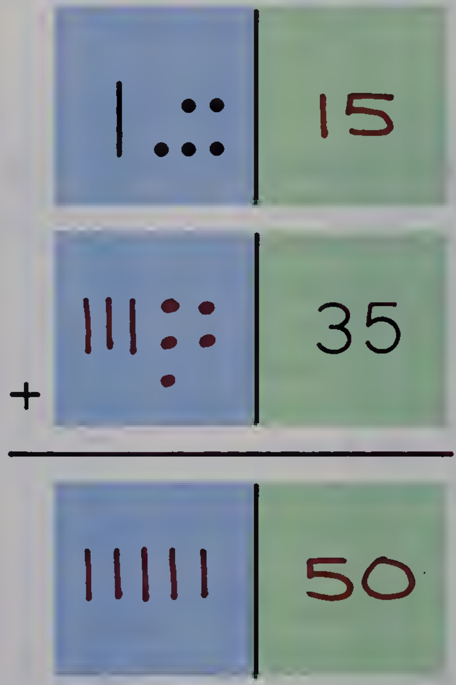
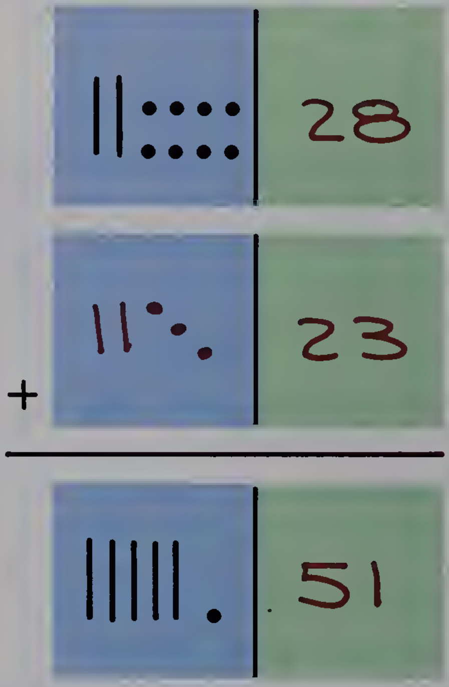


I



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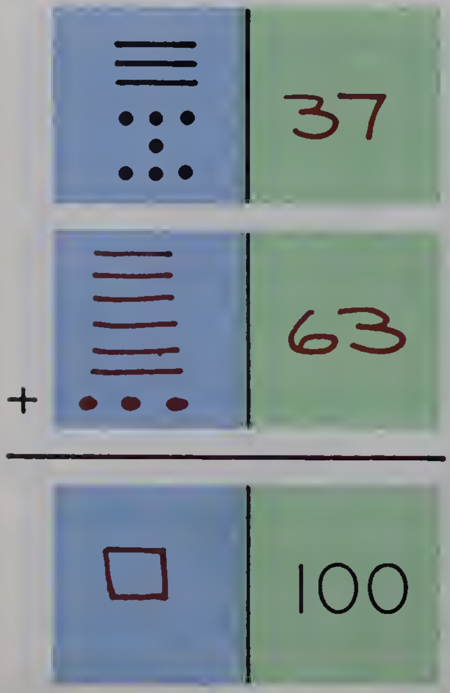

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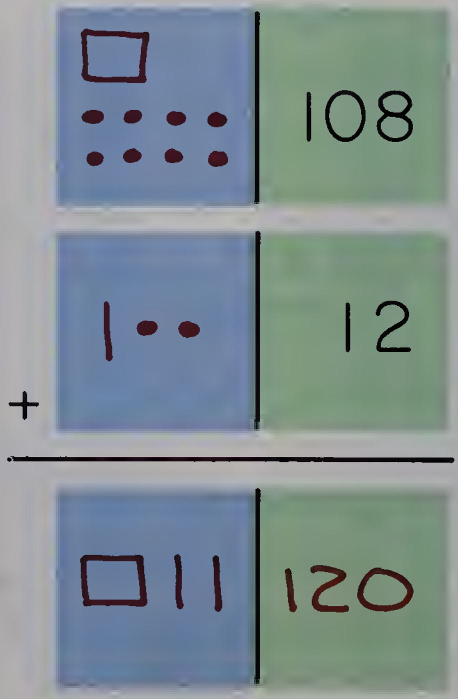
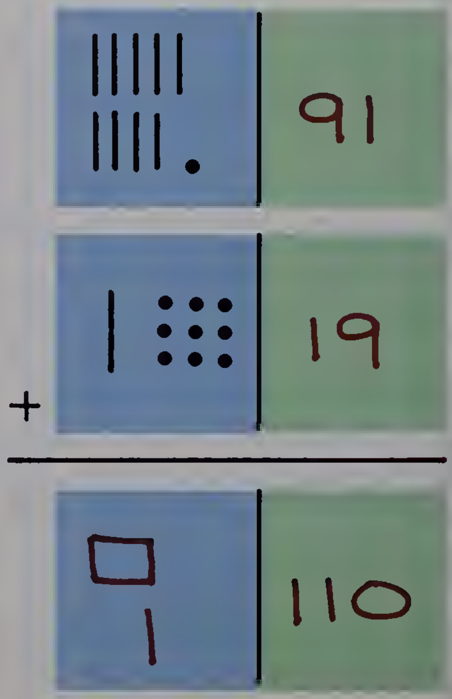


L



Notes on Pupil Page 60

Tasks: *In exercises A-F pupils translate the special shorthand notation (defined on pupil page 58) into ordinary decimal notation and vice versa. In exercises G-L pupils determine missing addends or sums and fill in all empty boxes.*

Purpose: (a) To extend pupils' understanding of decimal notation; (b) to prepare pupils for the algorithms for addition and subtraction.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

Materials for Teacher: Special flash cards (described below); cards described in the Notes on Pupil Page 54; Ditto masters.

Materials for Pupils: Dittoed sheets.

The Lesson:

Pupil pages 60 and 61 bring the pupils in sight of the algorithms for addition and subtraction.

To prepare for this page you may wish to design a set of flash cards. Use the shorthand notation on one side and ordinary decimal notation on the other side. Be sure to include cards like those in exercises D, E, and F. Flash card drill consists of holding a card up and calling on a pupil to tell you what's on the other side. If he sees the side with the special symbols, he pronounces the decimal name. If he sees the decimal notation, he draws in the air the special symbols (see the Notes on Pupil Pages 34 and 51).

Additional practice in forming decimal names can be obtained with the following activity. Display on the chalkrail two 100-cards, three 10-cards, and four 1-cards. Ask pupils to tell you the total number of little squares displayed (answer: 234). Remove a 1-card and ask for the new total (233). Remove two 10-cards and ask for the total (213). Remove one 10-card and ask for the total (203). Add one 100-card (303). Add two 1-cards (305).

Display one 100-card, three 10-cards, and four 1-cards (134). Remove a 1-card and ask for the total (133). Now take a pair of scissors, cut off one of the little squares from a 10-card, and ask for the total. We hope that your pupils will shout 132 at you.

Display four 10-cards and two 1-cards. Ask for the total (42) and write it on the board. Ask Peter to replace each 10-card with a 100-card. What is the new total? (402) Write it under the previous total. Ask Edward to replace each 1-card with a 10-card. What is the new total? (420) Write it under the previous total. Which is the largest (or greatest) number among 42, 402, and 420?

Drill

Follow up this last activity with a drill sheet:

1) Draw a ring around the greater number in each pair.

35 53	75 57	36 37
123 133	98 13	198 213
157 147	308 318	503 530

2) Write a < or a > between the numbers in each pair above.

You may wish to follow up pupil page 60 with the following Dittoed sheet:

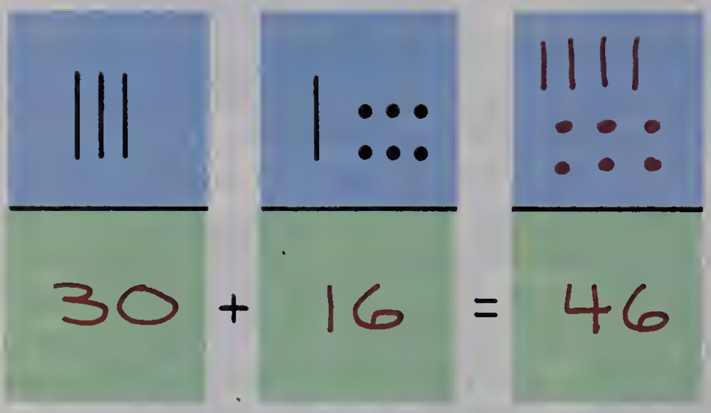
> In any sentence with frames, you must write the same number in all frames of the same size and shape.
>
> $28 = \square + \square + 8$ $\qquad$ $52 = \triangle + 2$
> $24 = \square + 4 + \square$ $\qquad$ $73 = 3 + \square$
> $59 = \square + \square + \square - 1 + \square + \square + \square$
> $28 + 23 = (\square + \square) + (8 + 3)$
> $67 = \square - 3$ $\qquad$ $67 = \square + 7$
> $24 + 6 = \square$ $\qquad$ $15 + 35 = \square$
> $28 + 23 = \square$ $\qquad$ $37 + 63 = \square$
> $108 + 12 = \square$ $\qquad$ $91 + 19 = \square$

Here are two stories:

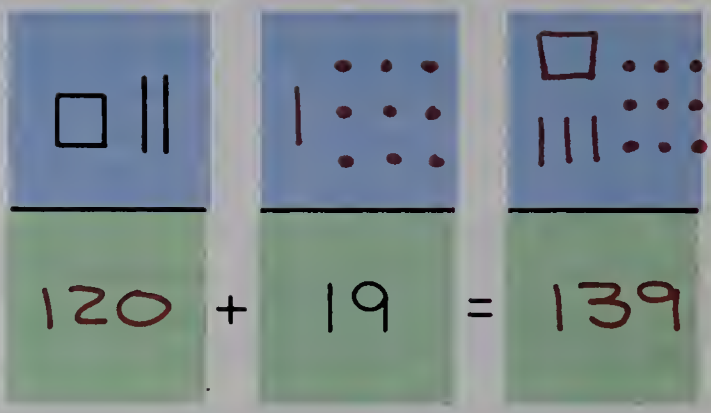
1) There were 9 roses in a vase. Jane removed 3.
2) Don's boat is 8 inches long. Sandy's is 5.

Headline for the Day: I gave the clerk a quarter. He gave me two coins as change. What can you say about this situation?

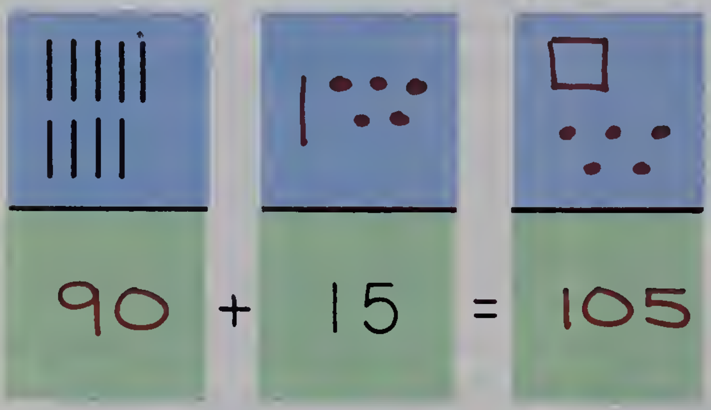
A


30 + 16 = 46

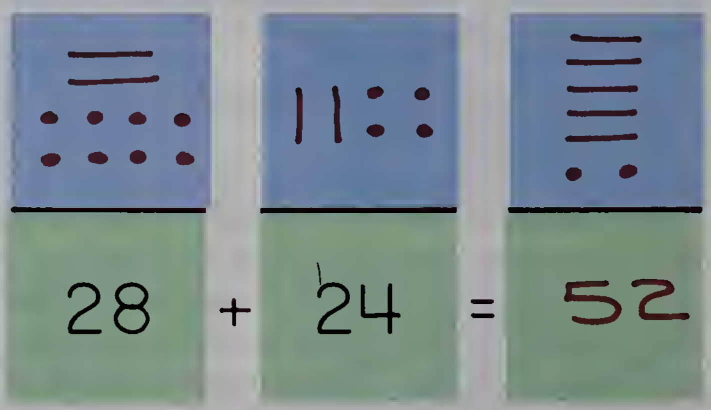
B


120 + 19 = 139

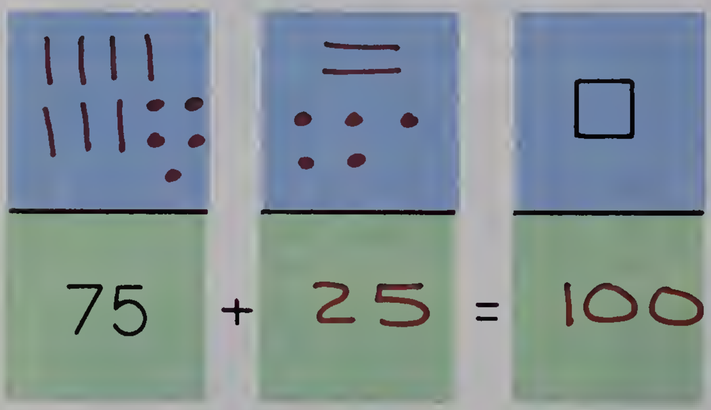
C


90 + 15 = 105

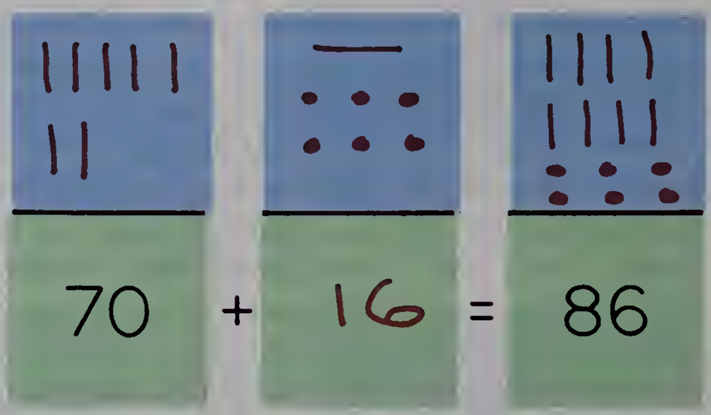
D


28 + 24 = 52

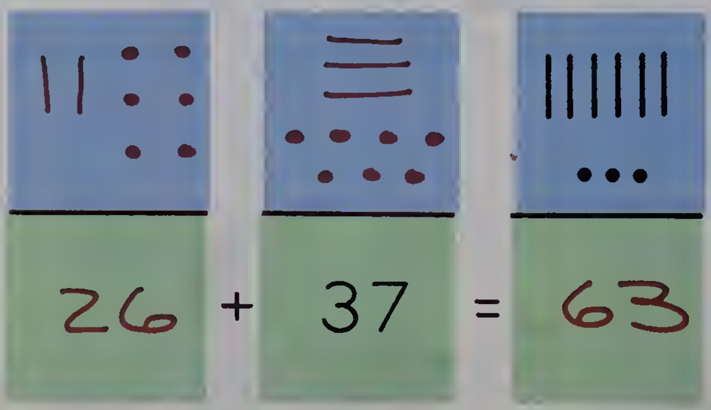
E


75 + 25 = 100

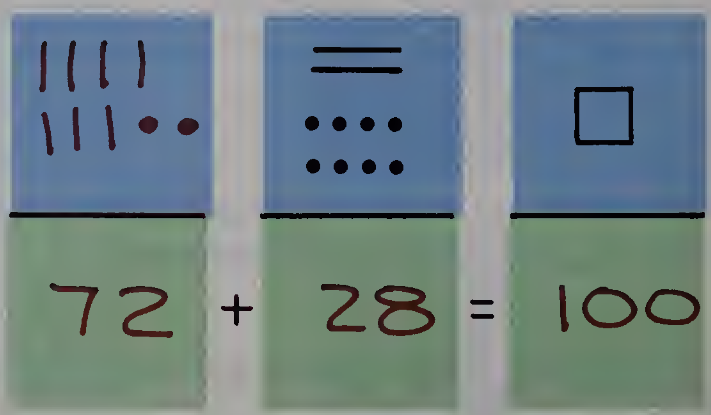
F


70 + 16 = 86

G


26 + 37 = 63

H


72 + 28 = 100

Notes on Pupil Page 61

Tasks: *Pupils determine the missing addends or sums and fill in all empty boxes.*

Purpose: (a) To extend pupils' understanding of decimal notation; (b) to prepare pupils for the algorithms for addition and subtraction.

Unifying Ideas: Number and counting; numeration; addition and subtraction.

Materials for Teacher: Flash cards described in the Notes on Pupil Page 60; cards described in the Notes on Pupil Page 54.

Materials for Pupils: None.

The Lesson:

The remarks made in "The Lesson" in the Notes on Pupil Page 60 apply here.

Oral Drill

You may wish to follow up pupil page 61 with oral drill:

30 . . . plus 6 is ______.
30 . . . plus 20 . . . plus 6 is ______.
30 . . . plus 26 is ______.
50 . . . plus 30 is ______.
50 . . . plus 38 is ______.
etc.

Try some rapid chalkboard drill:

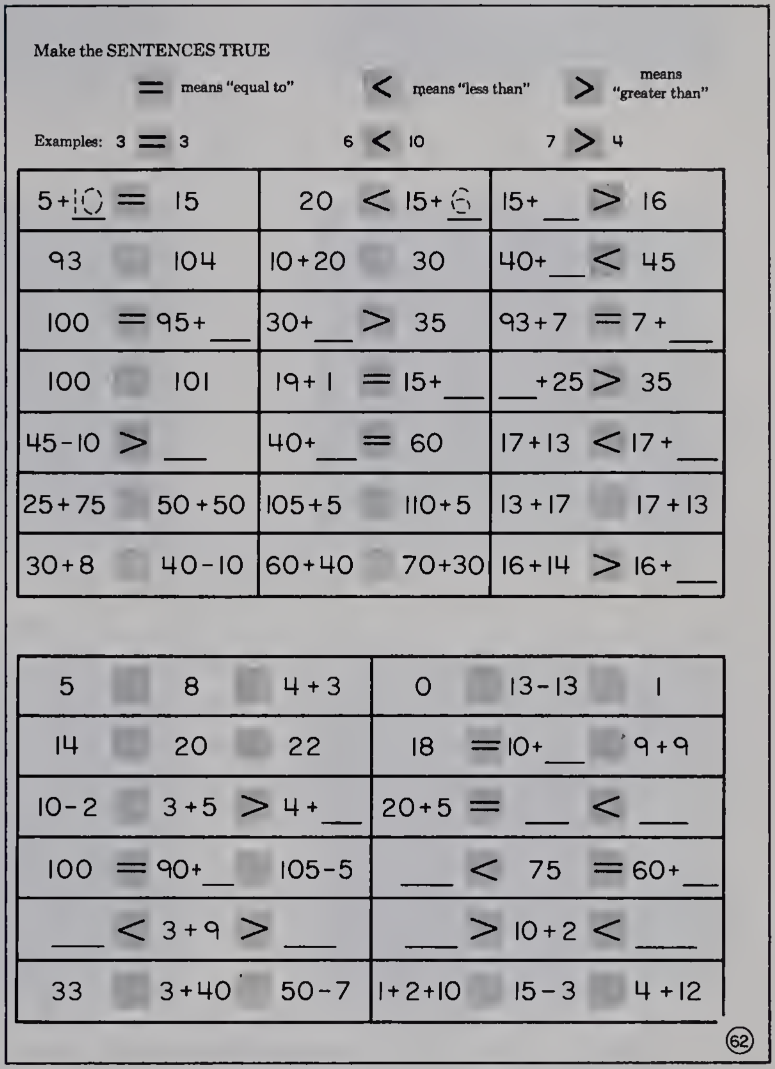
30 + 16 = ______
20 + 16 = ______
40 + 16 = ______
60 + 16 = ______

26 + 37 = ______
36 + 37 = ______
46 + 37 = ______
46 + 47 = ______

Keep the 10-cards and 1-cards handy to help pupils who hesitate in answering.

Drill

Consider using as a drill sheet page 62 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Make the SENTENCES TRUE

= means "equal to" < means "less than" > means "greater than"

Examples: 3 = 3 6 < 10 7 > 4

5 + 10 = 15	20 < 15 + 6	15 + ___ > 16
93 ☐ 104	10 + 20 ☐ 30	40 + ___ < 45
100 = 95 + ___	30 + ___ > 35	93 + 7 = 7 + ___
100 ☐ 101	19 + 1 = 15 + ___	___ + 25 > 35
45 − 10 > ___	40 + ___ = 60	17 + 13 < 17 + ___
25 + 75 ☐ 50 + 50	105 + 5 ☐ 110 + 5	13 + 17 ☐ 17 + 13
30 + 8 ☐ 40 − 10	60 + 40 ☐ 70 + 30	16 + 14 > 16 + ___

5 ☐ 8 ☐ 4 + 3	0 ☐ 13 − 13 ☐ 1
14 ☐ 20 ☐ 22	18 = 10 + ___ ☐ 9 + 9
10 − 2 ☐ 3 + 5 > 4 + ___	20 + 5 = ___ < ___
100 = 90 + ___ ☐ 105 − 5	___ < 75 = 60 + ___
___ < 3 + 9 > ___	___ > 10 + 2 < ___
33 ☐ 3 + 40 ☐ 50 − 7	1 + 2 + 10 ☐ 15 − 3 ☐ 4 + 12

(62)

Here are two stories for drill purposes:

1) Fred had 4 pennies. Pauline and Ole each gave him 3 more pennies.
2) Beulah had 1 dime, 1 nickel, and 2 pennies. She wanted to buy 3 candy bars for 6 cents each.

Headline for the Day: In how many ways can you make this sentence true using one odd and one even number? △ + □ = 10.

A.

5	4	9
6	1	7
11	5	16

B.

6	3	9
4	3	7
10	6	16

C.

4	5	9
2	5	7
6	10	16

D.

9	5	4
25	12	13
34	17	17

E.

9	9	0
25	10	15
34	19	15

F.

9	0	9
25	20	5
34	20	14

G.

30	15	15
13	4	9
17	11	6

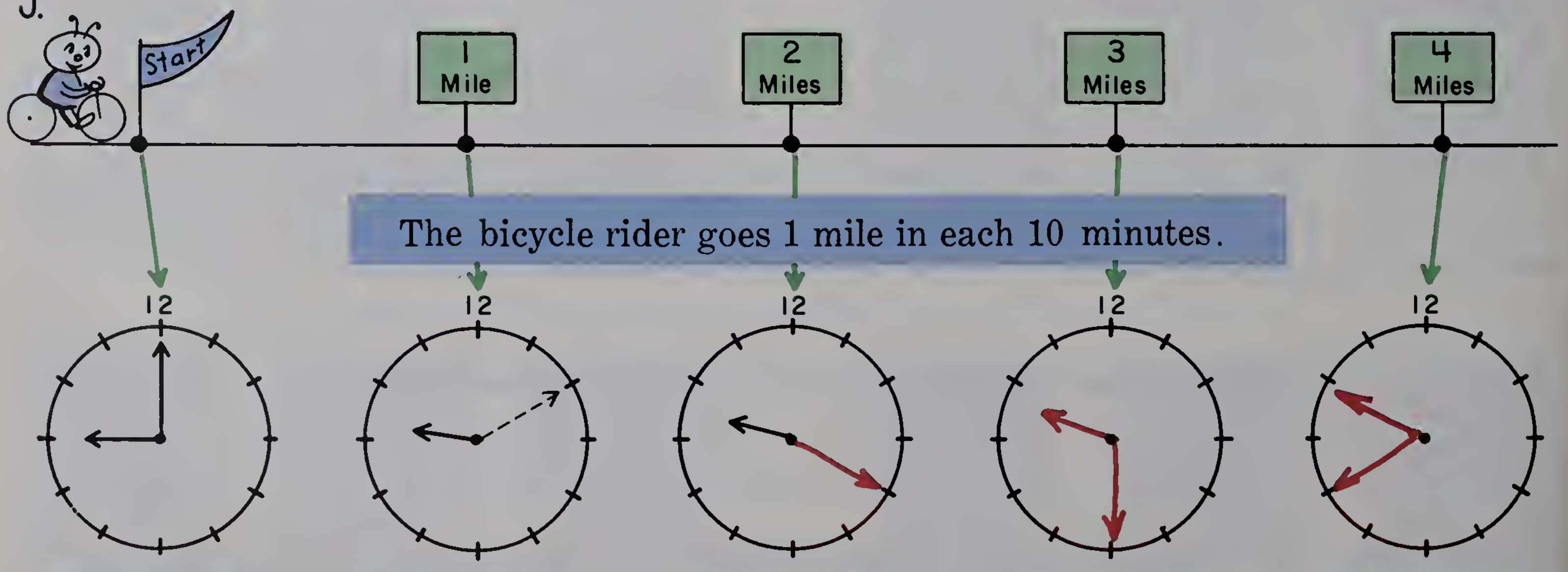
H.

30	16	14
13	6	7
17	10	7

I.

30	20	10
13	10	3
17	10	7

J.


Start
1 Mile
2 Miles
3 Miles
4 Miles
12
12
12
12
12

The bicycle rider goes 1 mile in each 10 minutes.

9:00 9:10 9:20 9:30 9:40

How far would he go in one hour? 6 mi; In $1\frac{1}{2}$ hours? 9 mi; In 2 hours? 12 mi.

Notes on Pupil Page 62

Tasks: *Pupils complete each* Cross Number Puzzle *from the given information. In exercise J pupils draw clock hands to indicate the time consumed over the distance traveled.*

Purpose: (a) To demonstrate to pupils a general partitioning procedure for finding sums or differences; (b) to introduce pupils to the idea of uniform speed.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations; measurement.

Materials for Teacher: Demonstration clockface.

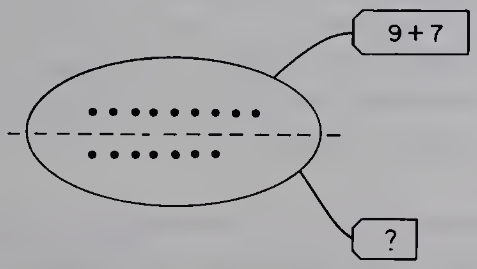
Materials for Pupils: Desk-size clockface.

The Lesson:

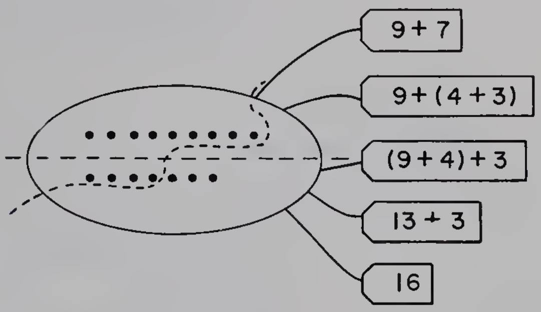
Exercises A - I are very important, for they illustrate a general procedure for finding sums and differences. The validity of the standard algorithms is based on this general procedure and on the idea of positional notation. It is important that pupils be led gradually to the standard algorithms and that they fully understand the general procedure.

Discovery Through Partitioning

Let's pretend that we have forgotten the basic fact that 9 + 7 is 16. How could we rediscover it? Our basic procedure is to take a display of 9 objects and a display of 7 objects none of which is in the display of 9 objects and combine the two displays. The sum of 9 and 7 is the number of objects in the combined display.

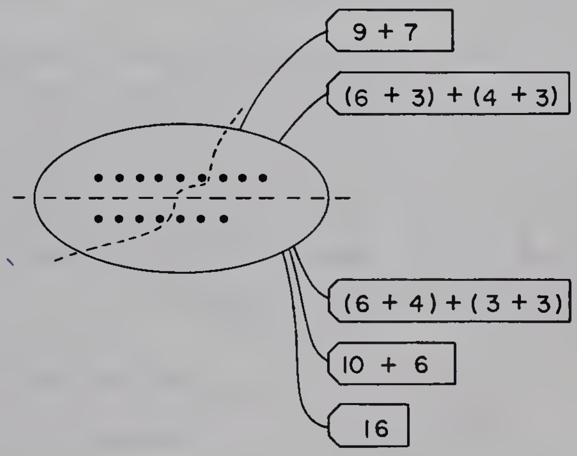


We could determine the sum by straightforward one-by-one counting. But we might shorten the counting a bit by using some basic addition facts already known to us. For example, if we had memorized the facts that 7 is 4 + 3, that 9 + 4 is 13, and that 13 + 3 is 16, our problem would be solved. For in that case, we would partition the combined display as follows:



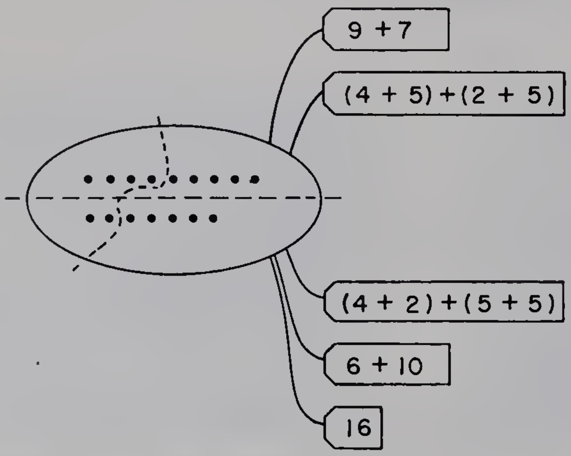
Our acquaintance with the world of physical objects assures us that this partitioning will not change the sum.

Of course there are many partitionings we might use. Our selection depends upon which addition facts we know. For example, in exercise B if we know that 9 is 6 + 3, that 7 is 4 + 3, that 6 + 4 is 10, that 3 + 3 is 6, and that 10 + 6 is 16, then we can find out that 9 + 7 is 16. With this knowledge at our disposal, we would partition the combined display as follows:



Headline for the Day: Extend the pattern: A, B, e, f, I, J

In exercise C, our discovery that 9 + 7 is 16 depended on our knowledge that 9 is 4 + 5, that 7 is 2 + 5, that 4 + 2 is 6, that 5 + 5 is 10, and that 6 + 10 is 16. For, as the completed *Cross Number Puzzle* shows, the combined display was partitioned as follows:



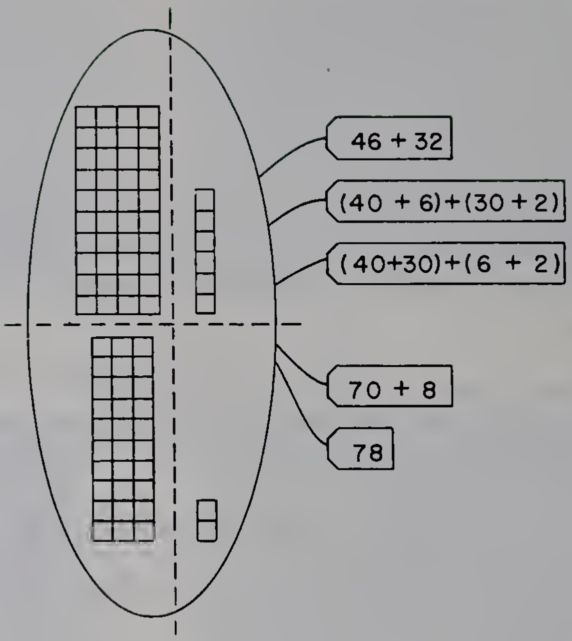
Partitioning and Positional Notation

Now, of course, it is highly unlikely that someone who had memorized all the addition facts mentioned above would have failed to memorize the fact that 9 + 7 is 16. The purpose of the foregoing discussion is to illustrate the general procedure. The general procedure becomes useful when you can *choose a partitioning which allows you to make use of memorized facts.*

Consider how we might find the sum of 46 and 32. One could not be expected to have memorized this fact and it would be absurd to try to count the combination of a display of 46 objects and a display of 32 objects by one-by-one counting. Instead one starts calling upon certain basic addition facts and his knowledge of positional notation:

1) 46 = 40 + 6 (positional notation)
2) 32 = 30 + 2 (positional notation)
3) 40 + 30 = 70 (positional notation and basic fact)
4) 6 + 2 = 8 (basic fact)
5) 70 + 8 = 78 (positional notation)

With these five facts at hand, one partitions the combined display as follows:



We can keep track of these steps in more abstract form as follows:

Problem: 46 + 32 = ?

Step 1: Get ready for a partitioning.

		46
		32
		?

Step 2: Partition in accordance with positional notation (facts 1 and 2).

40	6	46
30	2	32
		?

Step 3: Eliminate the counting by using basic memorized facts and positional notation (facts 3, 4, and 5).

40	6	46
30	2	32
70	8	78

Our experience that no rearrangement of a physical display will change the count is the basis for asserting that the sum of 70 and 8 is the sum of 46 and 32.

Finally, positional notation is such a powerful device that one need not be conscious of choosing a partitioning. The choice is made as soon as you decide to line up the places and add in columns:

$$\begin{array}{c|c} 4 & 6 \\ 3 & 2 \\ \hline 7 & 8 \end{array}$$

Understanding Comes First

Positional notation is so powerful that one can get correct answers by following some very simple rules. One is tempted to move immediately to these rules and let pupils become proficient in their use. If you yield to this temptation, your pupils may get right answers but they will probably not understand why their answers are correct. Indeed, after a summer vacation, you might find some pupils who find that the sum of 19 and 3 is 112:

$$\begin{array}{r}19\\3\\\hline 112\end{array}$$

Other Shortcuts to Counting are Valid

Certainly, we do expect pupils to become proficient in the use of the standard algorithms. This proficiency will come in time. But we should not try to achieve it by rejecting all other procedures. A search for alternatives often leads to greater understanding. For example, a pupil who can tell you that 48 + 46 is 94 because 45 + 45 is 90, 3 + 1 is 4, and 90 + 4 is 94, should be praised. On the other hand, you would be doing him a disservice if you didn't make sure that he knew how to find this sum by using the standard procedure. Common sense should guide your actions.

Note that exercises D, E, and F all deal with the problem of finding the sum of 9 and 25. Exercises G, H, and I deal with the problem of finding the difference between 30 and 13. Note that the partitioning exhibited in exercise I helps to explain the exchanging (borrowing) maneuver.

Time-Rate-Distance

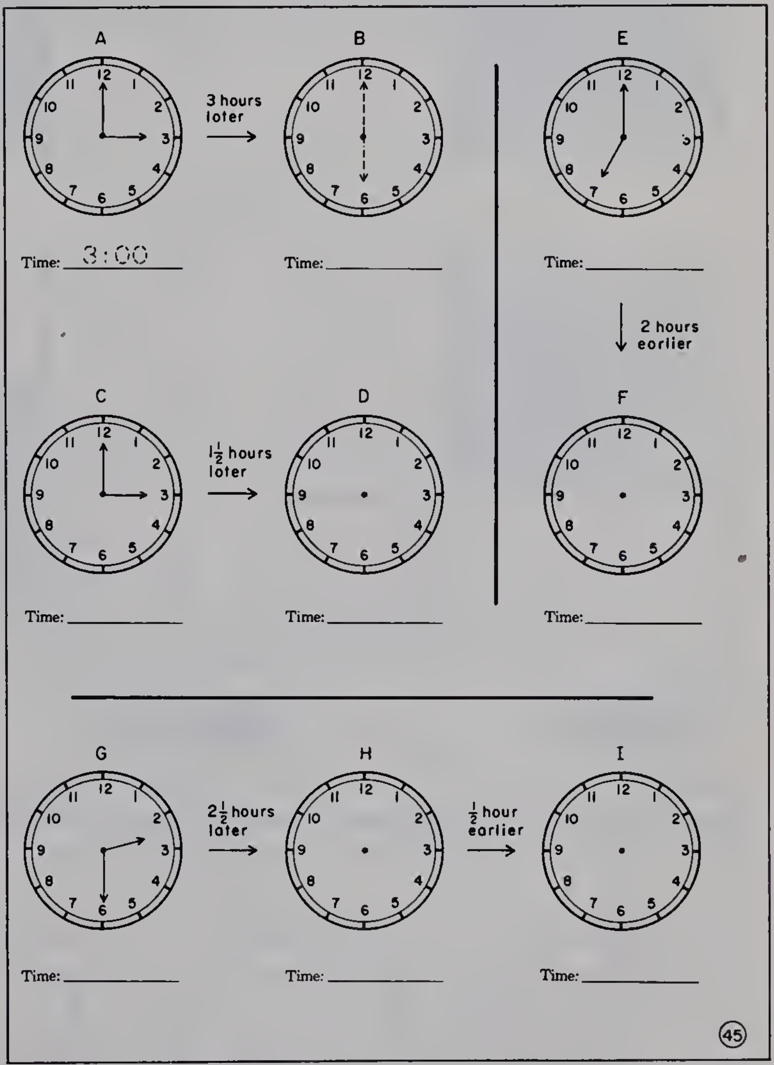
The problem of the bicycle rider and similar problems involving rates are best handled by keeping a record (first two columns below) as each step is worked out. Pupils may answer the questions about how far he would go, by counting or by extending the record and including the third column:

Miles	Time	Total Time in Minutes
0	9:00	0
1	9:10	10
2	9:20	20
3	9:30	30
4	9:40	40

Do similar problems on the board. Someone walks 1 block in 2 minutes, etc.

Drill

Consider as a drill sheet page 45 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Here are two stories for drill purposes:

1) Mary had 10 cents. Billy had 6 cents.
2) There are 14 families in Joe's apartment house. There are 8 families in Audrey's apartment house.

Green boxes are used only for 0, 10, 20, 30, 40, 50, 60, 70, 80, 90, or 100.
Blue boxes are used only for 0, 1, 2, 3, 4, 5, 6, 7, 8, or 9.

A

10	7	17
10	2	12
20	9	29

B

14	10	4
26	20	6
40	30	10

C

20	3	23
10	8	18
30	11	41

D

10	15	25
10	7	17
0	8	8

E

30	20	10
12	10	2
18	10	8

F

20	17	37
20	9	29
0	8	8

G

30	7	37
20	5	25
50	12	62

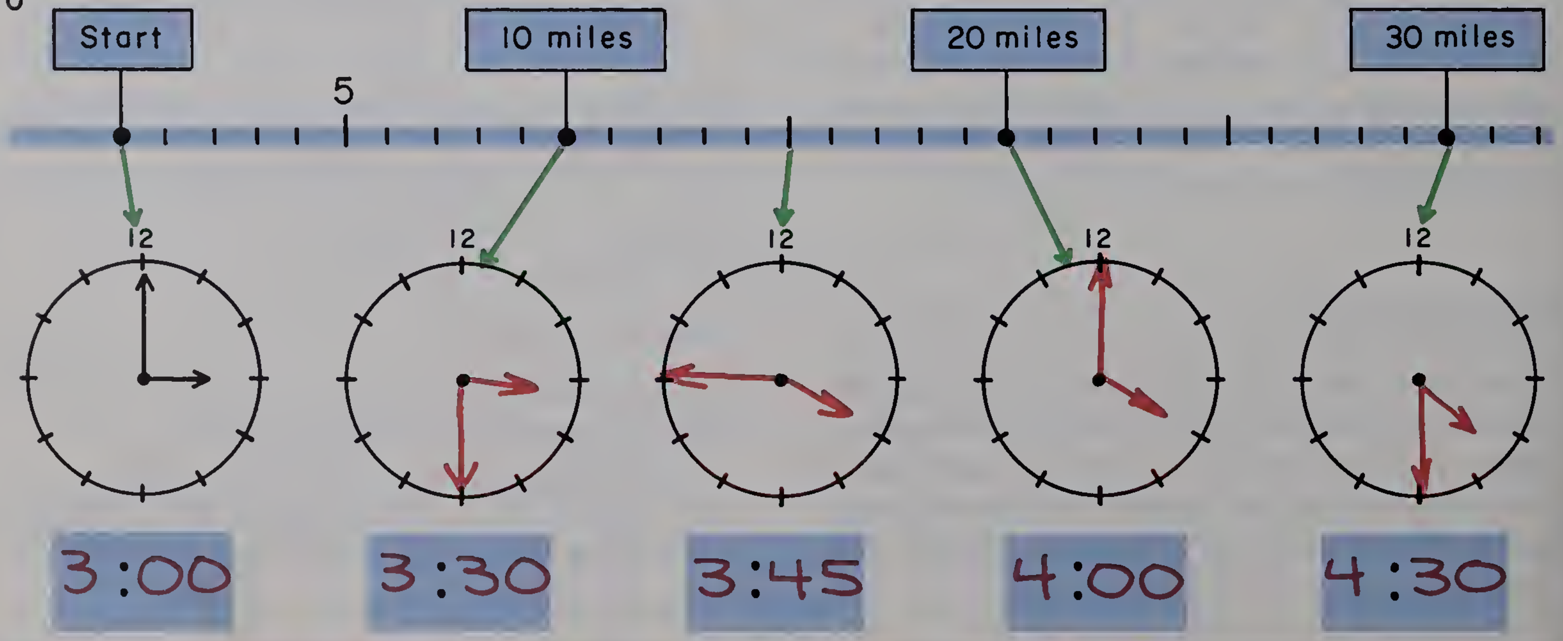
H

50	40	10
26	20	6
24	20	4

I

45	40	5
27	20	7
72	60	12

J


Start
10 miles
20 miles
30 miles
5
12
12
12
12
12
3:00
3:30
3:45
4:00
4:30

A truck driver started at 3 o'clock and drove at a steady rate. An hour later he had gone 20 miles. What time would he reach each of the points marked above?

Notes on Pupil Page 63

Tasks: *Pupils reconstruct each* Cross Number Puzzle *from the given information and the rules given at the top of the page. For the time-rate-distance problem at the bottom of the page pupils draw the clock hands to indicate the time of arrival at each sign.*

Purpose: (a) To demonstrate to pupils a special partitioning procedure for finding sums or differences; (b) to give pupils practice in problems involving uniform speed.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations; measurement.

Materials for Teacher: Ditto masters.

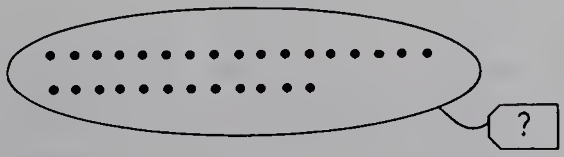
Materials for Pupils: Cuisenaire rods; Dittoed sheets.

The Lesson:

The rules given at the top of the page require the pupils to use a decimal partitioning in finding sums or differences. This special type of partitioning turns out to be most useful because it leads to a mechanical procedure for expressing sums and differences.

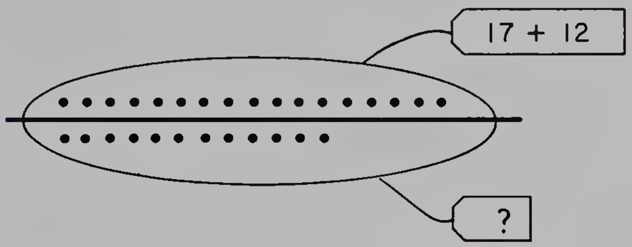
An Analysis of Addition Without Exchanging

Let's consider exercise A. Our job is to find the sum of 17 and 12. In physical terms this amounts to finding the number of objects in a combined display of two nonoverlapping displays—a display of 17 objects and a display of 12 objects. Here is a picture of such a combined display:



The job of finding the number really amounts to reporting the number in such a way that a person receiving the report will get an accurate picture of the numerosity of the combined display. We ordinarily assume that such a person has an excellent acquaintance with the numbers from 0 through 10. So we ordinarily give our report in terms of a partition each of whose parts consists of no more than 10 entities. In other words, we ordinarily give our report in terms of a decimal partitioning. (If, however, our respondent were limited in his acquaintance to just the numbers from 0 through 5, we would give our report in terms of a quinary partitioning.)

One of the things we know immediately about the combined display is that it can be partitioned into a 17-group and a 12-group, as follows:

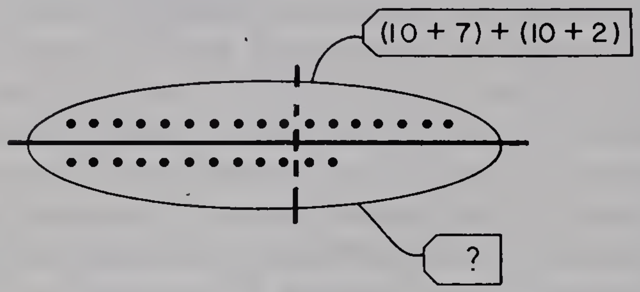


If our respondent were well acquainted with the numbers 17 and 12, all we would need to report to him is that the combined display could be partitioned into one 17-group and one 12-group. This report would give him a clear picture of the numerosity of the combined display. He would feel perfectly comfortable with the symbol 17 + 12 as a name for the sum. But we are assuming that our respondent feels comfortable only with reports which refer to displays that consist of at most 10 objects.

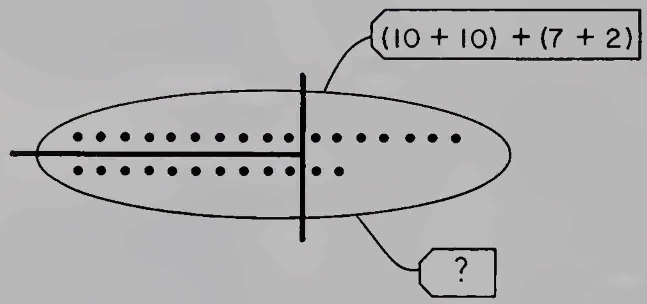
Now how do we go about obtaining a decimal partitioning of the combined display? We start with the knowledge that one partitioning is 17 + 12. Because of our understanding of positional notation and because we know that a decimal partitioning was used in reporting the numbers for the given dis-

Headline for the Day:

plays, we know that one of the given displays can be partitioned into one 10-group and seven 1-groups, and the other into a one 10-group and two 1-groups:



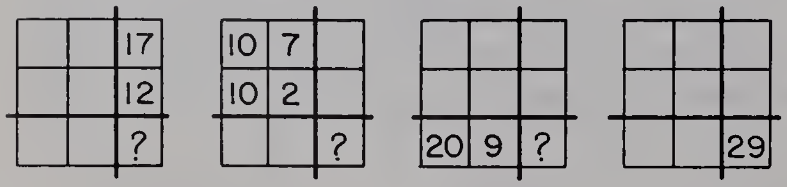
We have discovered that the combined display can be partitioned into two 10-groups and one display consisting of 7 + 2 members:



Since 7 + 2 is less than 10, the partitioning job is finished.

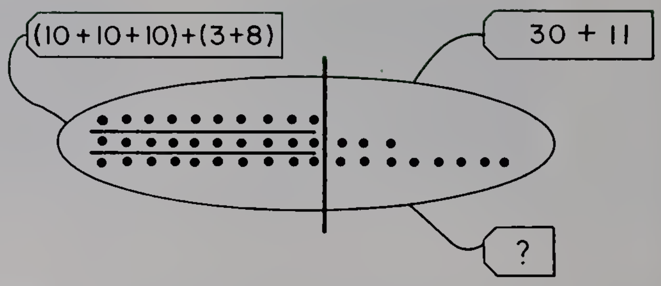
We are now ready to give the report. Because we have memorized certain basic facts we know that 7 + 2 = 9. So we can tell our decimal-minded respondent that the combined display can be partitioned into two 10-groups and nine 1-groups. For short, we tell him that the combined display consists of 29 members. (The symbol 29 means nothing more than two 10-groups and nine 1-groups.)

The *Cross Number Puzzle* form shows the steps very neatly:

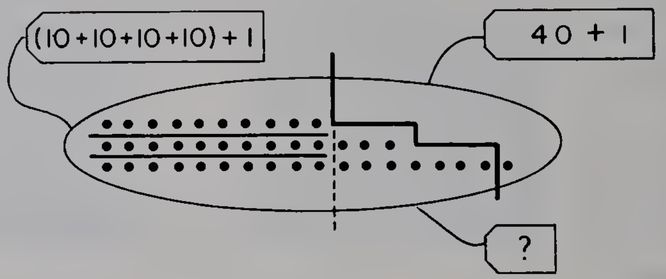


Addition With Exchanging

In exercises B and C we have to do more work to achieve the partition we are seeking. For example, here might be one of the steps for exercise C:



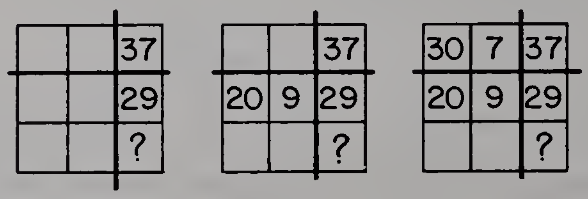
Since 3 + 8 is greater than 10, our job is not finished. Here is the finished job:



The maneuver from the partition 30 + 11 to the partition 40 + 1 is commonly called *carrying*. But this term is certainly not suggested by any of the work we have done. It is best to introduce this term when you introduce the formal algorithm. Even then it might be better to use the word *exchanging* because it is more suggestive of what is done.

An Analysis of Subtraction

In exercises D, E, and F we have three subtraction problems. A pupil can solve these problems by reconstructing addition *Cross Number Puzzles*. Let's consider exercise F. A pupil might first partially reconstruct the *Cross Number Puzzle* as follows:



At this stage he realizes that there is no whole number he can add 9 to in order to get the sum 7. Given complete freedom, he might decide to change his partition of 29 from 20 + 9 to 22 + 7. But the rules at the top of the page forbid this. So he must change the partition of 37. Any of the following partitions could help:

28	9	37
20	9	29
		?

27	10	37
20	9	29
		?

26	11	37
20	9	29
		?

20	17	37
20	9	29
		?

But only the last partition capitalizes fully on positional notation. (Otherwise, one would be hard-pressed to defend it over the second.)

The maneuver from the partition 30 + 7 to the partition 20 + 17 is commonly called *borrowing.* Clearly, this term is not suggested by any of the work we have done. As with *carrying,* avoid this term at least until the formal algorithm is introduced. Even then you may wish to use the word *exchanging* instead.

Follow-up

Follow the *Cross Number Puzzle* work with a Dittoed sheet such as the following:

Find the answers to the addition and subtraction problems. Use the boxes to help you if you need help.

$$\begin{array}{r} 24 \\ +37 \\ \hline \end{array} \qquad \begin{array}{r} 38 \\ +15 \\ \hline \end{array} \qquad \begin{array}{r} 72 \\ +18 \\ \hline \end{array}$$

$$\begin{array}{r} 58 \\ -32 \\ \hline \end{array} \qquad \begin{array}{r} 74 \\ -39 \\ \hline \end{array} \qquad \begin{array}{r} 65 \\ -26 \\ \hline \end{array}$$

Also, consider having the pupils use Cuisenaire rods as physical representations of these problems.

Time-Rate-Distance

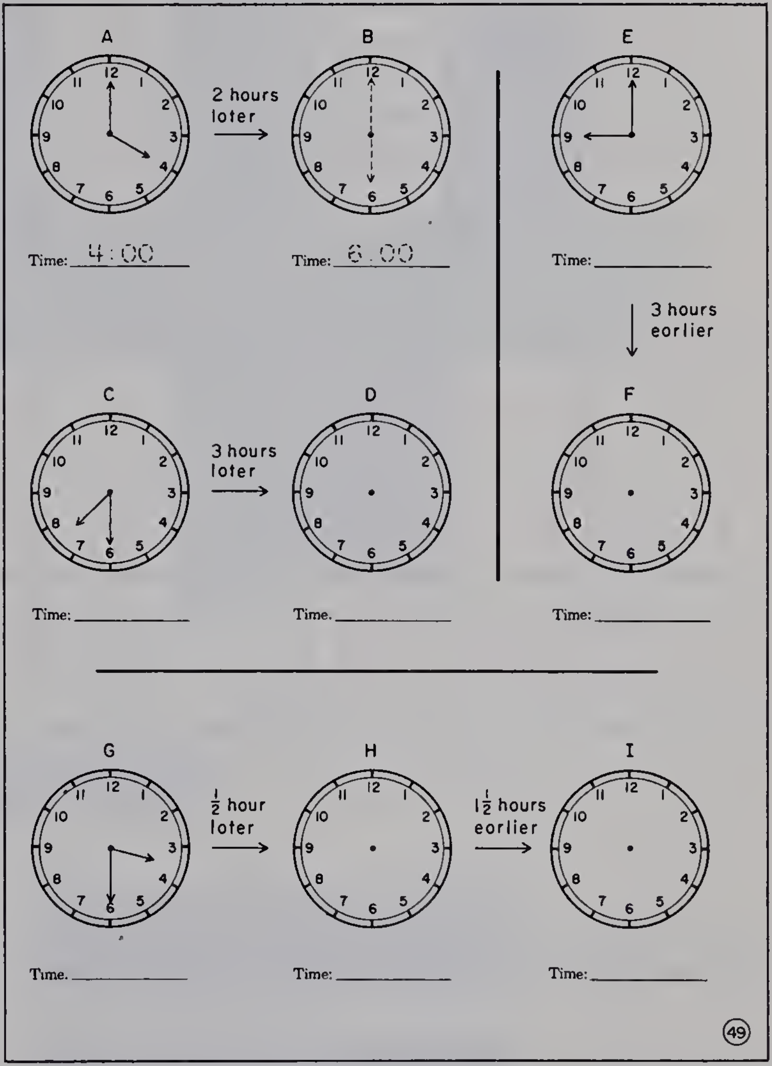
A problem similar to the truck problem (exercise J) should be discussed at the board. Consider the bicycle rider in the lesson on pupil page 62. The bicycle rider went 6 miles in one hour. If he left at 5 o'clock, how far would he have gone by 5:30? by 6:00? by 8:00? Or what time would it be when he had gone 3 miles? 6 miles? 18 miles?

Sketch a chart similar to the one in the text showing time and distance on the same number line. Build a table showing the relationship in a *What's My Rule?* format.

distance	0	10	20	30	
time	3		4		

Drill

Consider using as a drill sheet page 49 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Here are two stories for drill purposes:

1) Mona made 10 peanut butter sandwiches. She gave 3 to her brother and 2 to her sister.
2) Carla and Charlene went to the seashore. They found 16 pretty sea shells. Charlene found 7 of them.

Blue boxes are used only for numbers less than 10.

(a)

8	8	16
23	2	25
31	10	41

(b)

7	9	16
20	5	25
27	14	41

(c)

10	6	16
20	5	25
30	11	41

(d)

17	17	34
11	8	19
6	9	15

(e)

20	14	34
10	9	19
10	5	15

(f)

40	5	45
30	5	35
70	10	80

* MANY ANSWERS POSSIBLE.

(g) *

17	10	7
26	20	6
43	30	13

(h) *

15	10	5
17	10	7
32	20	12

(i) *

65	5	60
25	5	20
90	10	80

Twice as much . . .

(j) 7 in. → 1 ft. and 2 in.

(k) 5 da. → 1 wk. and 3 da.

(l) 10 oz. → 1 lb. and 4 oz.

(m) 10 in. → 1 ft. and 8 in.

(n) 6 da. → 1 wk. and 5 da.

(o) 9 oz. → 1 lb. and 2 oz.

(p) 4 da. → 1 wk. and 1 da.

(q) 11 in. → 1 ft. and 10 in.

(r) 7 oz. → 0 lb. and 14 oz.

(s) 6 in. → 1 ft. and 0 in.

Tasks: Pupils reconstruct each Cross Number Puzzle *based on the given information and the rule at the top of the page. In exercises* j-s *pupils record measurements in the grouping or partitioning pattern specified.*

Purpose: (a) To remind pupils of the general partitioning principle which underlies the addition and subtraction algorithms; (b) to extend pupils' understanding of various conventional notations for expressing measurements.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations; measurement.

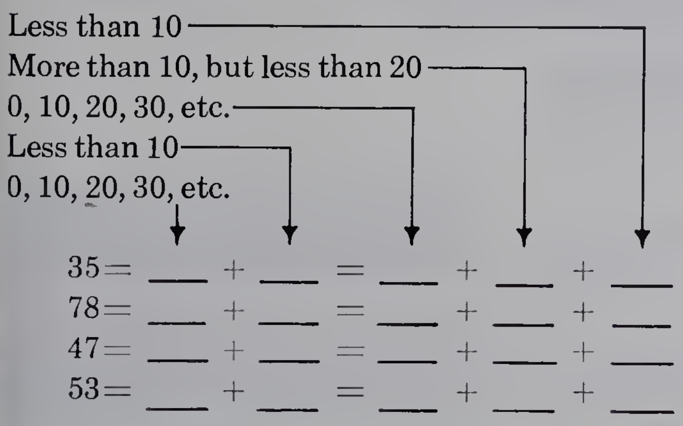
Materials for Teacher: Various measuring instruments.

Materials for Pupils: None.

The Lesson:

Here we remind pupils that the partitioning used for the addends is arbitrary. Each of exercises *a-c* asks for the sum of 16 and 25, and each exercise enables pupils to arrive at 41 in a different way. Exercises *d* and *e* show two ways of finding the difference 34 − 19. Exercises *g-i* allow pupils to choose their own partitions.

The idea of choosing a partitioning according to specified rules can be introduced to pupils by the following kind of board work:



This and many similar activities will do much to help pupils gain an understanding of the great freedom they have in partitioning. Though we are focusing attention on arbitrary rules which provide the flavor of a game the pupils are developing a keen sense of the many ways in which they can partition a number. This understanding will help them find many shortcuts which are often more effective than those traditionally used.

Conversion of Measurements

In the conversion of measurements on pupil pages 64 and 65, pupils have another experience with partitioning. By partitioning measurements for which the base is other than 10 or for which there may be several bases, they will gain a deep understanding of the general notion of partitioning.

Begin work by developing and extending these patterns on the board:

12 in. = 1 ft.
14 in. = 1 ft. 2 in.
15 in. = 1 ft. 3 in.
. . .
. . .
. . .
24 in. = 2 ft. 0 in.
etc.

16 oz. = 1 lb.
16 oz. = 1 lb. 0 oz.
32 oz. = 2 lb. 0 oz.
48 oz. =
etc., for other measurement units

Drill

Here are two stories for drill purposes:

1) Jack has 2 pockets. He put 5 pennies in each pocket.
2) Pat had 17 peanuts. She gave 7 of them to Carla.

Headline for the Day: I have an odd number of coins, but fewer than 6 coins. What possible amounts of money do I have?

Find the number for the green box the easiest way you can.

(a)

10	3	13
20	9	29
30	12	42

(b)

26		
26		
52		

(c)

31	20	11
18	10	8
13	10	3

(d)

		50
		14
		36

(e)

		37
		53
		90

(f)

34		
55		
89		

(g)

63		
36		
27		

(h)

		87
		56
		31

(i)

		29
		58
		87

The same as . . .

(j) 15 in. → 1 ft. and 3 in.

(k) 70 min. → 1 hr. and 10 min.

(l) 30 hrs. → 1 da. and 6 hrs.

(m) 23 oz. → 1 lb. and 7 oz.

(n) 30 in. → 2 ft. and 6 in.

(o) 42 min. → $\frac{1}{2}$ hr. and 12 min.

(p) 35 oz. → 2 lb. and 3 oz.

(q) 35 in. → 2 ft. and 11 in.

(r) 48 hrs. → 2 da. and 0 hrs.

(s) 90 min. → $1\frac{1}{2}$ hrs. and 0 min.

Notes on Pupil Page 65

Tasks: *Pupils determine the required sums and differences in the easiest way they can. In exercises j-s pupils record measurements in the grouping or partitioning pattern specified.*

Purpose: (a) To prepare pupils for the standard addition and subtraction algorithms; (b) to extend pupils' understanding of various conventional notations for expressing measurements.

Unifying Ideas: Structure; numeration; addition and subtraction; functions and relations; measurement.

Materials for Teacher: Various measuring instruments.

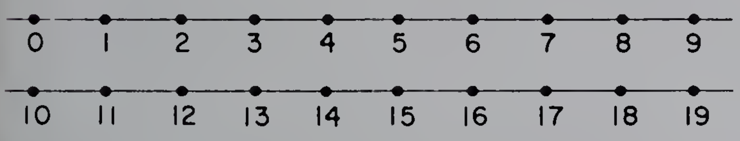
Materials for Pupils: Cuisenaire rods.

The Lesson:

Here we allow pupils complete freedom in finding sums and differences. If they wish to express the partitioning by reconstructing the *Cross Number Puzzles*, let them do so. If they would rather do the partitioning "in their heads" encourage them to do so. If you wish to find out how they do it, don't ask them to verbalize their procedure—this could be very tedious. Instead ask them to use the *Cross Number Puzzle* form in explaining their work. Or have them use Cuisenaire rods to explain it.

Number Line Chunks

Chop a number line from 0-149 into fifteen horizontal segments or chunks of 10 numbers each.



A large copy of such a number line chart can be put to good use in the classroom. It can be used to demonstrate that it is possible to jump from 2 to 12, or from 5 to 15, or from 7 to 37 with great ease. Pupils won't need to count in order to do this, either. Every jump down (↓) is equivalent to adding 10. A jump ↓ followed by a jump $\overset{\curvearrowright}{2}$ is equivalent to adding 12. A jump ↓ followed by $\overset{\curvearrowleft}{2}$ is equivalent to adding 8. To find the sum of 10 and 17, you start at 10, jump ↓ and $\overset{\curvearrowright}{7}$, and you're on 27. Or start at 10, jump $\overset{\curvearrowright}{7}$, and then jump ↓.

Before going on, allow your pupils to spend a few lessons in learning how to move about the number line grid. They will enjoy exercises such as these: Where will you land if you start at 0 and take three jumps down? Four jumps down? Two jumps down? Six jumps down? Ten jumps down?

Describe ways to make the trip from 11 to 22. From 5 to 12. From 3 to 24. From 7 to 18. From 25 to 37. From 111 to 123.

Where are we if we start on 15, jump down two rows and four jumps to the right? ($15+20+4=39$) If we start at 15, take four jumps to the right, and jump down two rows? ($15 + 4 + 20 = 39$) If we start at 15, take two jumps to the right, jump down two rows, and jump two to the right? ($15 + 2 + 20 + 2 = 39$)

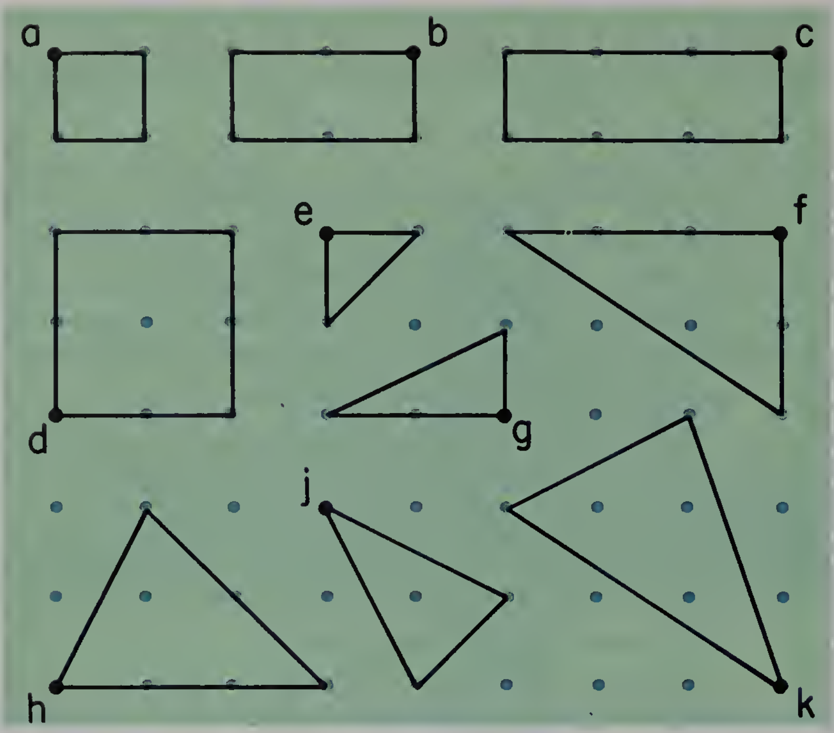
Have pupils jump forward or back, and then up, to subtract on the number line. Just as there are many ways to get from 15 to 39, there are also many ways to get from 39 to 15.

Games

When pupils understand how to use the number line chart, it is possible to play games with it. For instance, one game is started by saying: I'm thinking of a number. Start at 15, jump three rows down and four to the left. What is the number? The first pupil to tell you what the number is becomes the leader and has to think of a number for which he must give appropriate clues. Some pupils may become so proficient that they won't need to use the number line chart. Encourage such pupils to "do it in their heads."

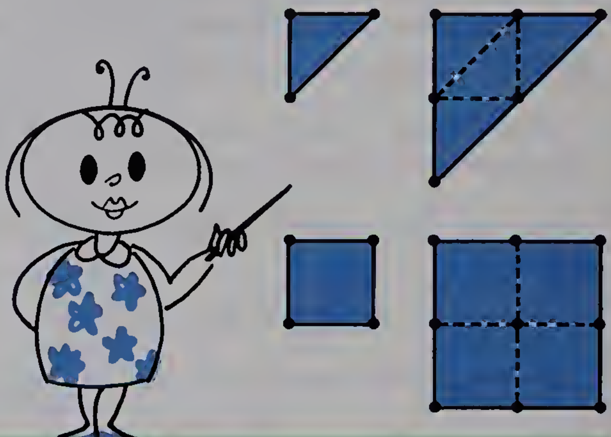
Other games could proceed like this: Start at 0, add 11. Add 11 more. Add 11 more. Where are you?

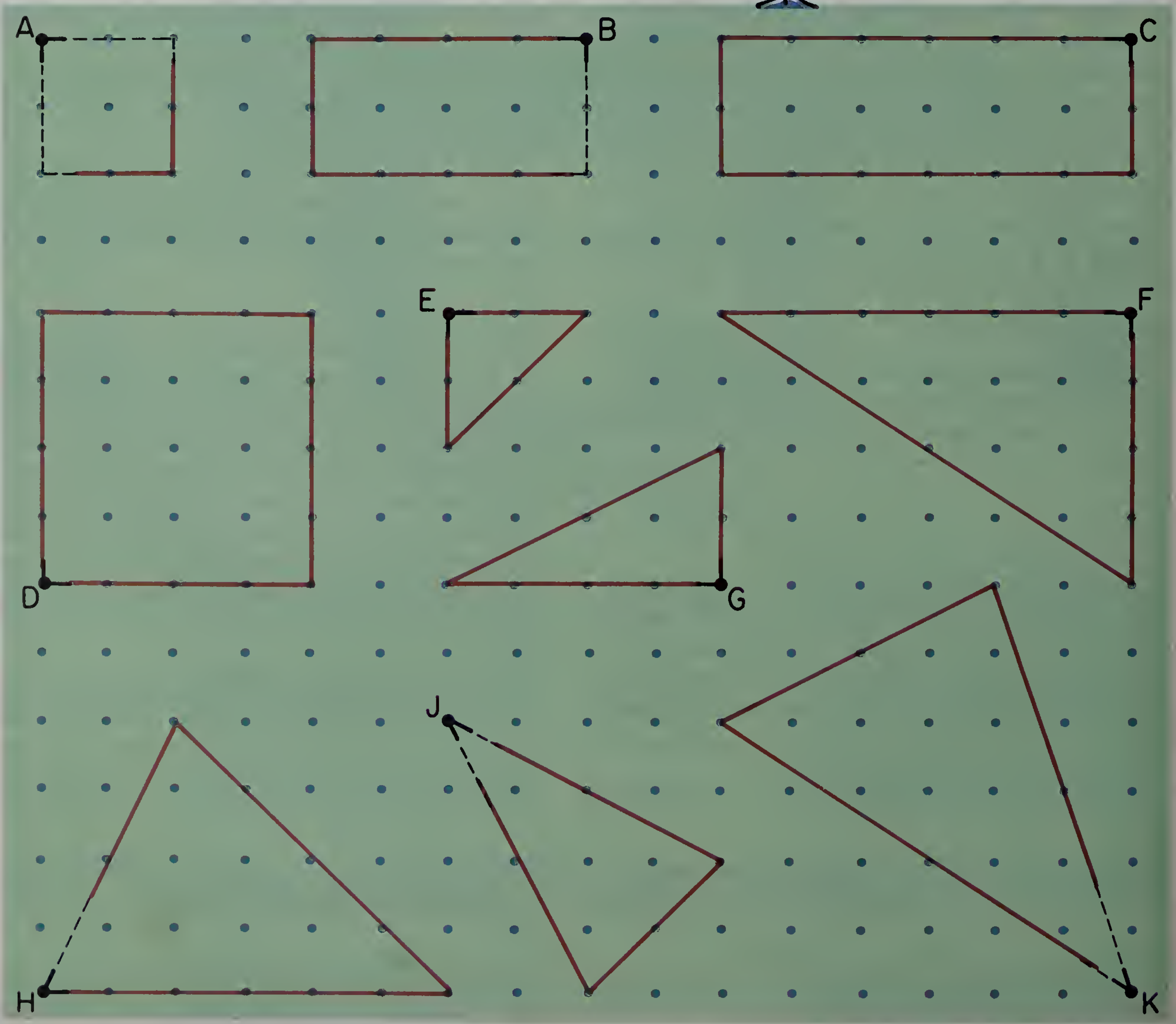
Headline for the Day: Write a number sentence that fits: Mrs. Brown had 9 eggs. She bought a dozen eggs and used a half-dozen eggs. How many eggs has she left?


a
b
c
d
e
f
g
h
j
k

In the large box below, copy the shapes on the left so the lines in your drawings are twice as long as those in the smaller box.

How many of the smaller shapes on the left will fit into the similar shapes you drew below?




A
B
C
D
E
F
G
H
J
K

Notes on Pupil Page 66

Tasks: *Pupils draw in the lower box a figure similar to each figure that appears in the upper box but with a perimeter twice that of the similarly labeled original figure. They then compare the areas in class discussion.*

Purpose: To review and extend pupils' understanding of linear and surface measure.

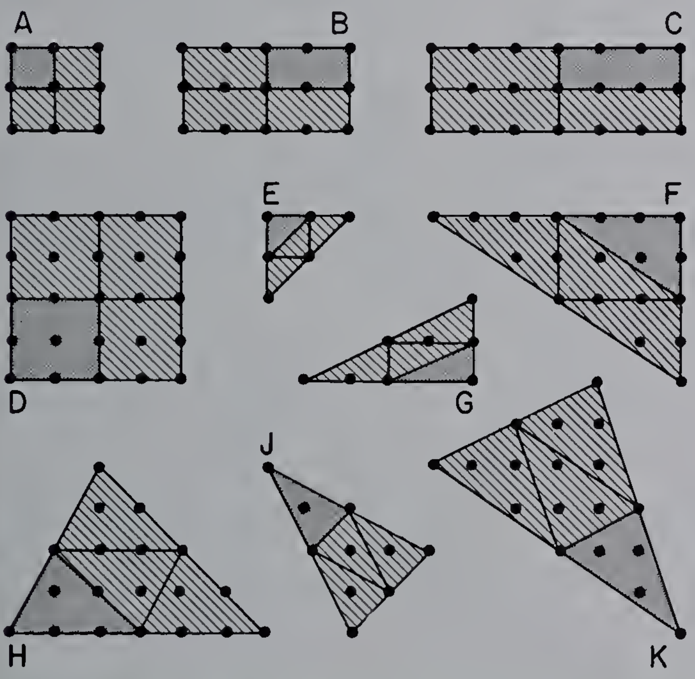
Unifying Ideas: Geometry; measurement; functions and relations.

Materials for Teacher: Overhead projector; transparencies of lattice points; demonstration-size peg board; pegs; rubber bands; Counting Board described in the Notes on Pupil Page 33.

Materials for Pupils: Straightedges; Geo-boards.

The Lesson:

It will be easy for pupils to see that each of the larger figures has twice the perimeter of the corresponding smaller figure. But the area relationship is a bit harder to see.



One approach to the area problem is to show that four of each of the smaller figures can be fit exactly into the corresponding larger figure. Do this kind of work on the overhead projector, on the Counting Board, or on a peg board. Let the pupils use a Geo-board (described in pages 80-86 of the booklet *Games and Enrichment Activities*) for their experimentation.

The diagram above shows the relationship between the area of the smaller and larger figures.

A second approach is to compute areas. This is more difficult. (See the Notes on Pupil Page 125.)

Drill

Here is a suggestion for a drill worksheet:

Write the correct signs.

13=19 ↙−6

10 ↙+4=14

18=12 ↙6

15=19 ↙4

9 ↙2=11

7=5 ↙12

16 ↙3=13

16 ↙3=19

10 ↙5=5

18 ↙7=11

16=(10 ↙4)+2

10=10 ↙0

25=23 ↙2

19=20 ↙1

15 ↙3=16 ↙2

18 ↙5=5 ↙18

58+97=97 ↙58

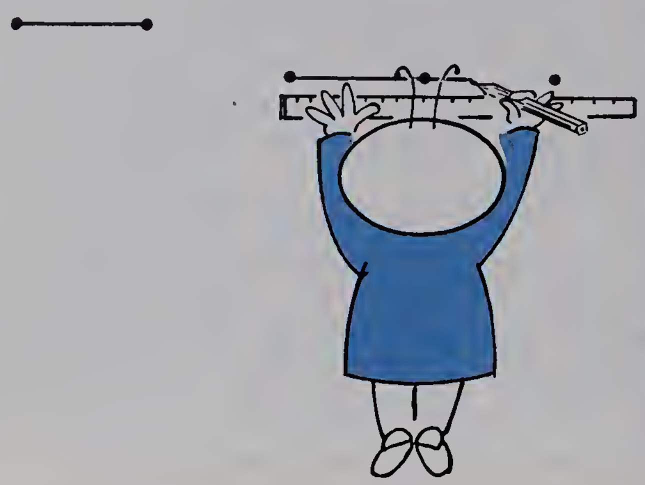
6 ↙2=10 ↙2

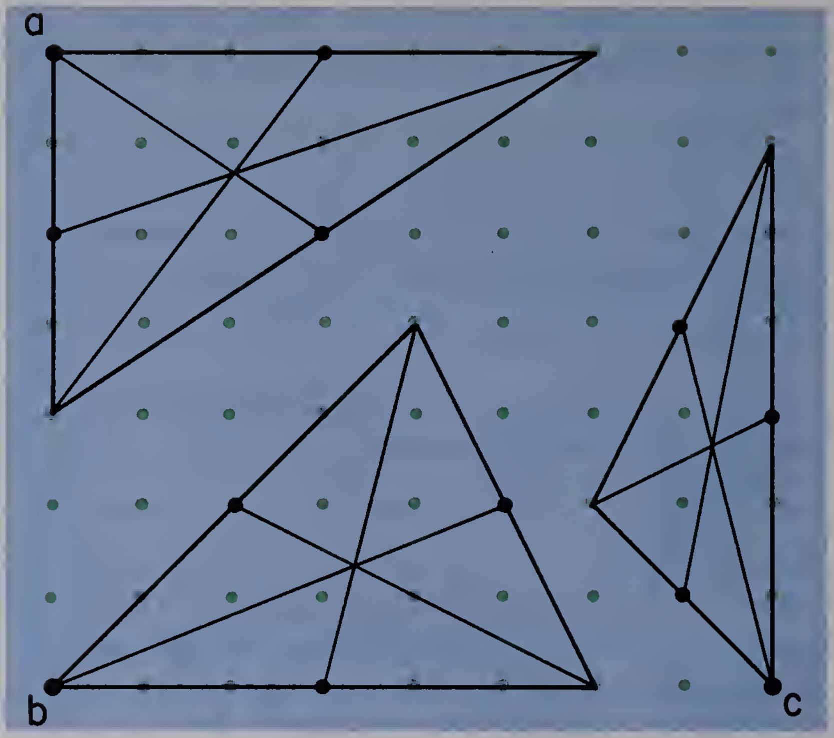
Here are two stories for drill purposes:

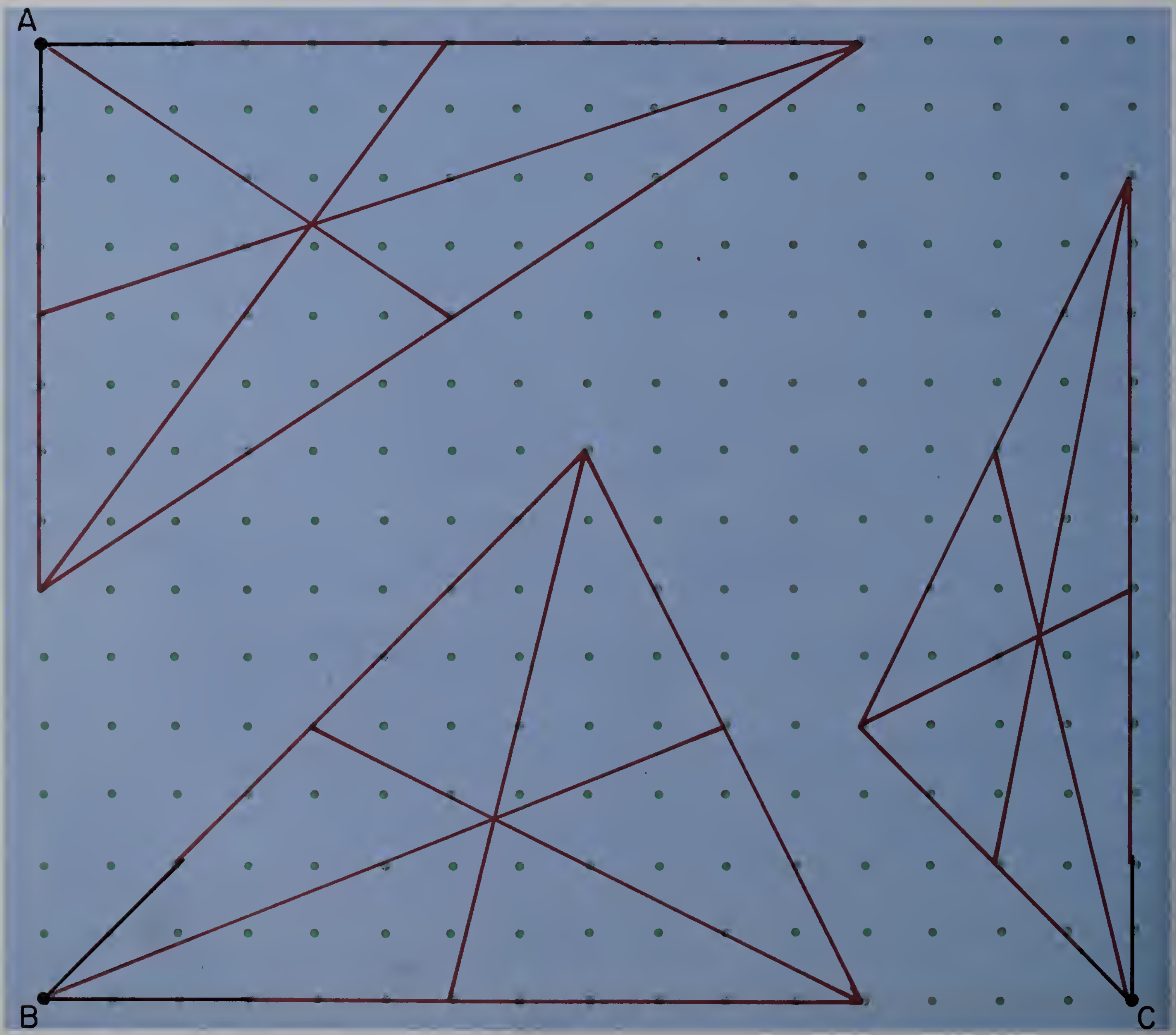
1) Inez had 8 dolls. Her father gave her 4 more.
2) Pete gave 15 toy rockets to Lou. 9 of them were 5 inches long. The others were 7 inches.

Headline for the Day: Bill earned 1¢ on Monday, 2¢ on Tuesday, 4¢ on Wednesday, 8¢ on Thursday, ... Make a table for eight days of work.

In the large box below, copy the drawings on the right so that the lines in your drawings are twice as long as those in the smaller box.







Notes on Pupil Page 67

Tasks: *Pupils draw in the lower box a figure similar to each figure that appears in the upper box so that each line segment in the new figure is twice as long as the corresponding line segment in the original figure.*

Purpose: To review and extend pupils' understanding of linear measure and similarity.

Unifying Ideas: Geometry; measurement; functions and relations.

Materials for Teacher: None.

Materials for Pupils: Straightedges.

The Lesson:

Pupils should be pleased to see that the three medians of each of the larger triangles cross in one point.

Drill

Here are two stories for drill purposes:

1) Sid paid 6 cents for a pencil and 4 cents for a pad.
2) Helen and Ginnie were jumping rope. During one turn, Helen jumped the rope 37 times before missing. Ginnie jumped 2 less times than Helen.

Here are suggestions for drill sheets:

Add.

$$\begin{array}{r}4\\2\\\underline{7}\end{array}\qquad\begin{array}{r}3\\5\\\underline{3}\end{array}\qquad\begin{array}{r}4\\3\\\underline{5}\end{array}\qquad\begin{array}{r}5\\3\\\underline{4}\end{array}\qquad\begin{array}{r}7\\1\\\underline{7}\end{array}\qquad\begin{array}{r}5\\3\\\underline{3}\end{array}$$

$$\begin{array}{r}5\\2\\\underline{4}\end{array}\qquad\begin{array}{r}3\\4\\\underline{5}\end{array}\qquad\begin{array}{r}6\\3\\\underline{6}\end{array}\qquad\begin{array}{r}5\\3\\\underline{6}\end{array}\qquad\begin{array}{r}2\\6\\\underline{3}\end{array}\qquad\begin{array}{r}4\\5\\\underline{4}\end{array}$$

54 + 5 = 50 + □ = △
25 + 3 = 20 + □ = △
72 + 6 = 70 + □ = △
56 + 3 = 50 + □ = △
63 + 4 = 60 + □ = △
15 + 2 = □ + 7 = △
28 + 1 = □ + 9 = △
46 + 3 = □ + 9 = △
74 + 2 = □ + ○ = △

Headline for the Day: Make a list of things which are (1) triangular, (2) rectangular.

Correct answer → C
Answer that is furthest from correct answer → D
Difference between these answers → ____

A B C D
○ ○ ● ⊗ ____

A

$3 + 7 = 10$

(A) 4
(B) 9
(C) 10
(D) 21

A B C D
○ ○ ● ⊗ 11

B

$9 + 7 = 16$

(A) 26 (C) 2
(B) 16 (D) 18

A B C D
○ ● ⊗ ○ 14

C

$12 - 4 = 8$

(A) 13
(B) 16
(C) 2
(D) 8

A B C D
○ ⊗ ○ ● 8

D

$30 - 10 = 20$

(A) 20 (C) 31
(B) 40 (D) 45

A B C D
● ○ ○ ⊗ 25

E

$14 + 6 = 20$

(A) 24
(B) 20
(C) 8
(D) 30

A B C D
○ ● ⊗ ○ 12

F

$25 + 7 = 32$

(A) 22 (C) 42
(B) 18 (D) 32

A B C D
○ ⊗ ○ ● 14

TOY SALE: Drums 12¢ each Horns 35¢ each Boats 25¢ each

G

How much did Al pay for a boat and a drum?

(A) 35¢ (C) 27¢
(B) 37¢ (D) 40¢

A B C D
○ ● ⊗ ○ 10 ¢

H

Helen bought a horn and gave the clerk 50¢. How much change did Helen get?

(A) 15¢ (C) 38¢
(B) 25¢ (D) 40¢

A B C D
● ○ ○ ⊗ 25 ¢

I

How much did Jack pay for two drums?

(A) 25¢ (C) 24¢
(B) 35¢ (D) 30¢

A B C D
○ ⊗ ● ○ 11 ¢

J

Mary bought a drum. Tom bought a boat. How much more did Tom spend than Mary?

(A) 13¢ (C) 9¢
(B) 23¢ (D) 29¢

A B C D
● ○ ○ ⊗ 16 ¢

K

Harry bought two boats and gave the clerk a dollar. How much change did he get?

(A) 35¢ (C) 50¢
(B) 30¢ (D) 60¢

A B C D
○ ⊗ ● ○ 20 ¢

L

Sarah bought a drum, a horn, and a boat. How much did she spend?

(A) 62¢ (C) $1.00
(B) 47¢ (D) 72¢

A B C D
○ ○ ⊗ ● 28 ¢

Notes on Pupil Page 68

Tasks: *In each problem pupils determine the correct answer, the answer which is furthest from the correct one, and the difference between these answers.*

Purpose: (a) To test pupils' mastery of ideas and information presented in the text; (b) to teach pupils a new format for recording answers.

Unifying Ideas: Addition and subtraction; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Suggested point distribution: 3 points per item, 11 items—total, 33 points.

The purposes of this test page are several. In the first place, it is a test of mastery of ideas taught in the text. Secondly, it asks pupils to discriminate among the alternative responses in order to note both the correct answer *and* the answer furthest from the correct one. Then pupils must determine the difference between these two responses and record it. Finally, pupils are given an opportunity to learn another common way of recording answers.

Do several examples like those on pupil page 68 with all pupils until they understand how to proceed.

If pupils have difficulty reading the words in the story problems, help them without comment.

Drill

Here is a suggested drill sheet:

Add.

75	42	83	54	62	75
+ 4	+ 7	+ 6	+ 5	+ 4	+ 3
67	22	53	72	25	36
+ 12	+ 25	+ 36	+ 16	+ 24	+ 42
95	36	58	85	23	45
+ 8	+ 18	+ 24	+ 17	+ 28	+ 49

Here are two stories for drill purposes:

1) 8 children were playing. 3 of them went away.
2) There were 13 animal story books on a library shelf. There were 8 pictures of animals on the bulletin board.

Headline for the Day: Make up stories about school or vacation for all six number sentences in the top section of page 68.

Write number sentences which could be headlines for the six stories in the bottom section of page 68.

In each problem, write the given numbers in figures. Then label the largest by writing an A in the colored box next to it; the next largest by a B; the next by a C; etc.

1
fifty-seven . . . . 57 B
ninety-one . . . . 91 A

2
one hundred . . . . 100 A
eighty-seven . . . . 87 B

3
one hundred eighty . . . . 180 B
one hundred twenty-five 125 C
three hundred . . . . 300 A

4
three hundred twenty . . 320 B
two hundred eighteen 218 C
one thousand . . . . 1,000 A

5
ten more than nine . . . . 19 D
thirty-six . . . . 36 A
half of fifty . . . . 25 B
two dozen . . . . 24 C

6
twice fifty . . . . 100 D
seven hundred . . . . 700 B
eighty plus thirty . . . . . 110 C
eight hundred seven . . . 807 A

7
three hundred forty-one . . 341 A
three hundred forty . . . . . 340 B
two hundred fifty-nine . . 259 D
two hundred sixty . . . . . . 260 C

8
ten more than four hundred eleven . . . . 421 A
ten less than four hundred twenty . . . . 410 B

9
half of sixty . . . . 30 C
three dozen . . . . 36 A
twice eleven . . . . 22 F
four sevens . . . . 28 D
thirty-one . . . . 31 B
ten twos . . . . 20 G
two dozen . . . . 24 E

10
four hundred eighty . . . . 480 E
three thousand four hundred . . . . 3,400 B
seven hundred more than one thousand . . . 1,700 C
ten thousand . . . . 10,000 A
nine hundred . . . . 900 D

Tasks: In each exercise pupils record in decimal notation the numbers referred to by word names. Then they rank the numbers in descending order.

Purpose: (a) To review and extend pupils' understanding of decimal notation; (b) to give pupils practice in ranking numbers in order.

Unifying Ideas: Numeration; functions and relations.

Materials for Teacher: Cards described in the Notes on Pupil Page 54.

Materials for Pupils: Cuisenaire rods.

The Lesson:

Before working on this page, examine and extend patterns of numbers and their word names:

11 eleven 12 twelve 13 thirteen
. . .
20 twenty 21 twenty-one
. . .
90 ninety 91 ninety-one
. . .
100 one hundred 200 two hundred
. . .
143 one hundred forty-three
. . .
1143 one thousand, one hundred forty-three
2245 two thousand, two hundred forty-five

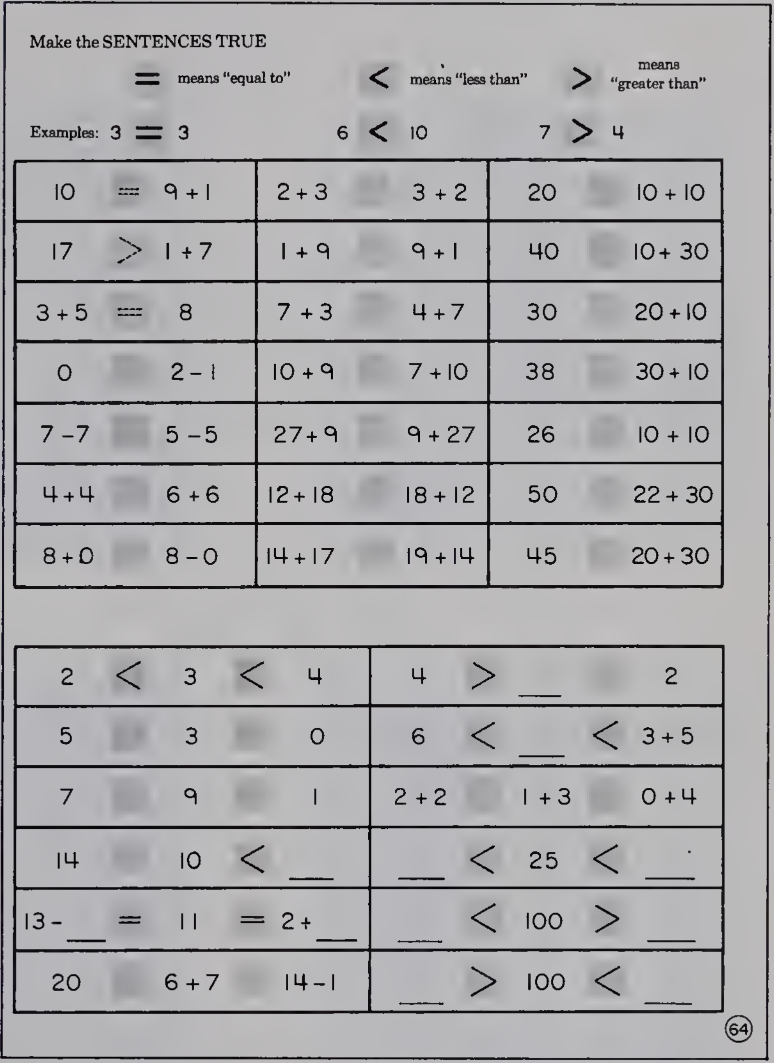
twice ten	20	half of ten	5
twice twenty	40	half of twenty	__
twice fifty	__	half of fifty	__

five tens	50	ten more than ten	20
six tens	__	ten more than eleven	__
seven tens	__	ten more than twenty	__

Use the 100-cards, 10-cards, and 1-cards to help in translating the word names. Let pupils use their Cuisenaire rods for these translations.

Drill

Consider using as a drill sheet page 64 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Make the SENTENCES TRUE

= means "equal to" < means "less than" > means "greater than"

Examples: 3 = 3 6 < 10 7 > 4

10	=	9 + 1	2 + 3		3 + 2	20		10 + 10
17	>	1 + 7	1 + 9		9 + 1	40		10 + 30
3 + 5	=	8	7 + 3		4 + 7	30		20 + 10
0		2 − 1	10 + 9		7 + 10	38		30 + 10
7 − 7		5 − 5	27 + 9		9 + 27	26		10 + 10
4 + 4		6 + 6	12 + 18		18 + 12	50		22 + 30
8 + 0		8 − 0	14 + 17		19 + 14	45		20 + 30

2	<	3	<	4	4	>	__		2
5		3		0	6	<	__	<	3 + 5
7		9		1	2 + 2		1 + 3		0 + 4
14		10	<	__	__	<	25	<	__
13 − __	=	11	=	2 + __	__	<	100	>	__
20		6 + 7		14 − 1	__	>	100	<	__

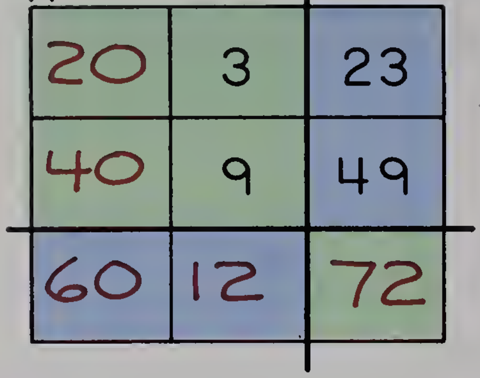
⑥④

Here are four stories for drill purposes:

1) Pedro got a lollipop for 2¢, a candy bar for 6¢, and a pencil for 3¢.
2) Jan and Fran were playing ringtoss. In three tries, Jan scored 100, 200, and 300. Fran scored 200, 300, and 200.
3) Dino got a ball for 8¢, and a candy bar for 6¢.
4) Flora had 59¢, and bought a jump rope for 7¢.

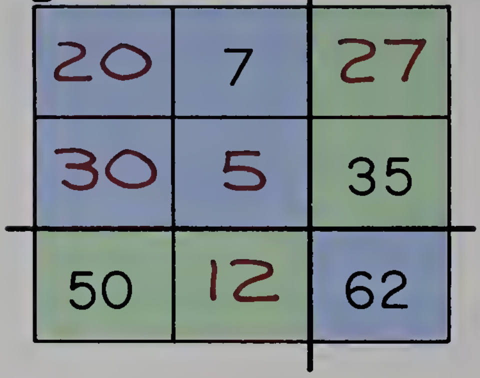
Headline for the Day: Extend the pattern: 2, 7, 12, 17, 22, 27

A



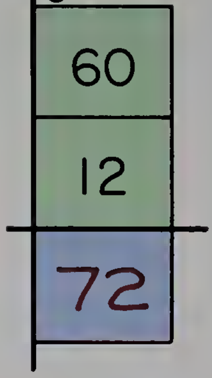
20	3	23
40	9	49
60	12	72

B



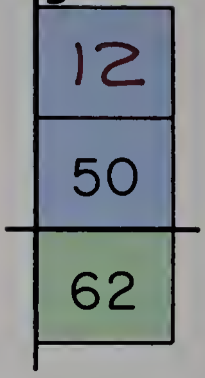
20	7	27
30	5	35
50	12	62

C



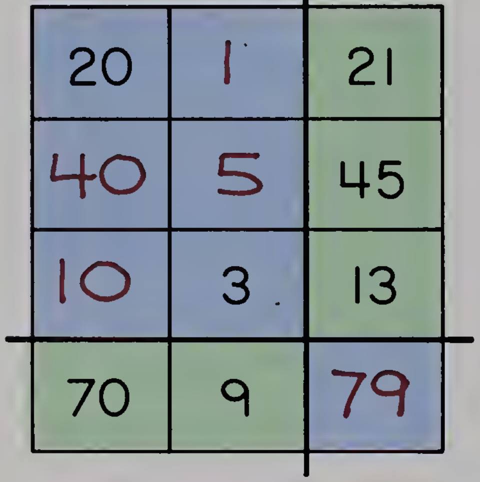
60
12
72

D



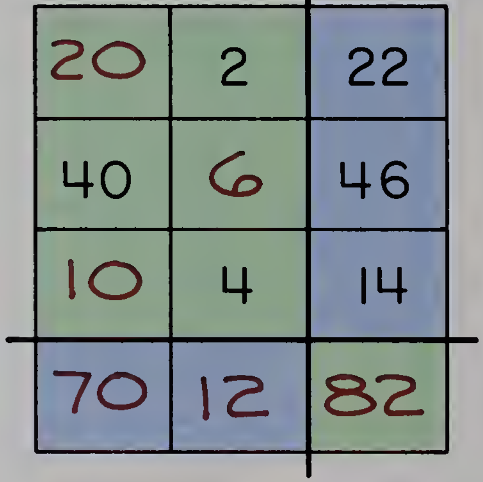
12
50
62

E



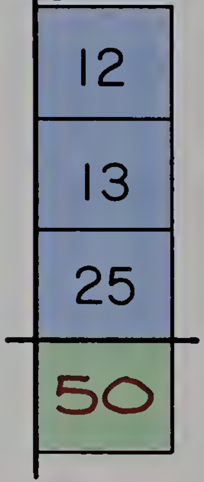
20	1	21
40	5	45
10	3	13
70	9	79

F



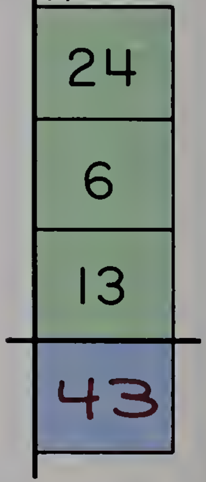
20	2	22
40	6	46
10	4	14
70	12	82

G



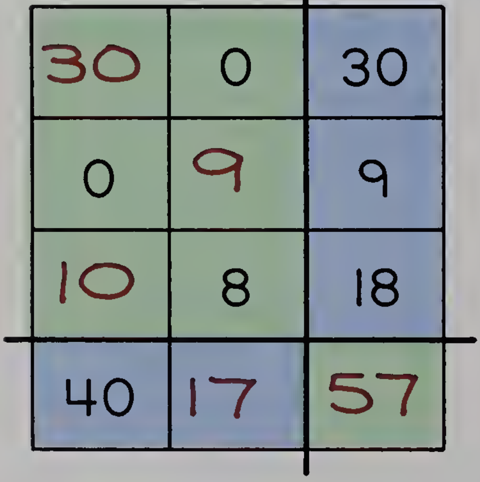
12
13
25
50

H



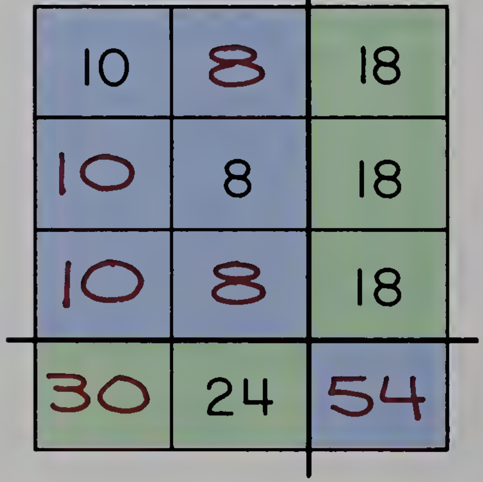
24
6
13
43

I



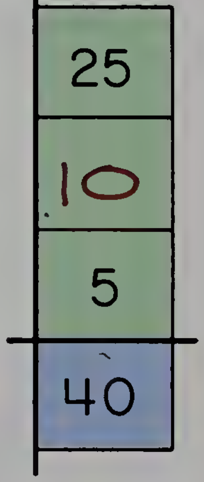
30	0	30
0	9	9
10	8	18
40	17	57

J



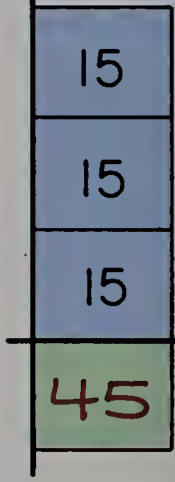
10	8	18
10	8	18
10	8	18
30	24	54

K



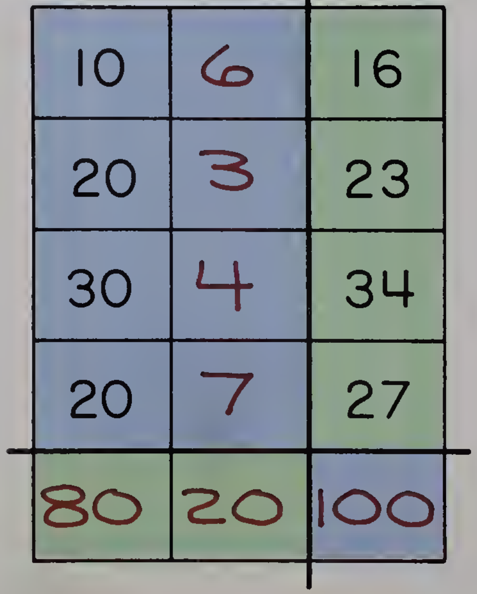
25
10
5
40

L



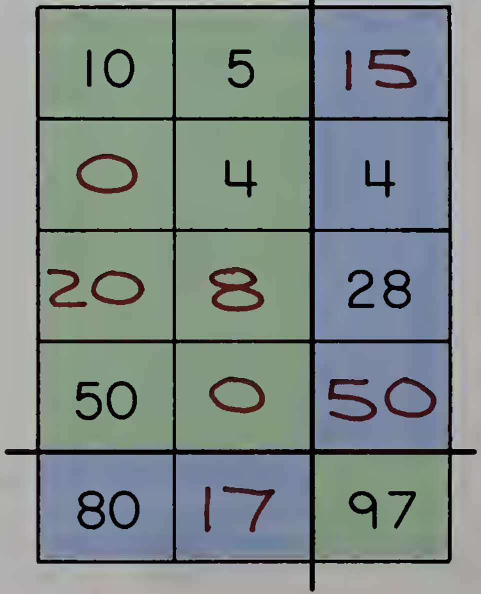
15
15
15
45

M



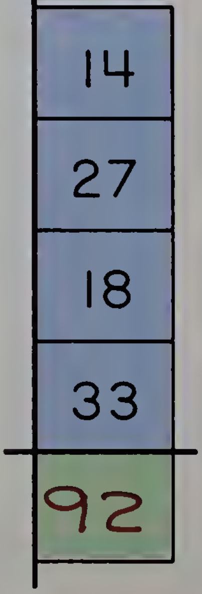
10	6	16
20	3	23
30	4	34
20	7	27
80	20	100

N



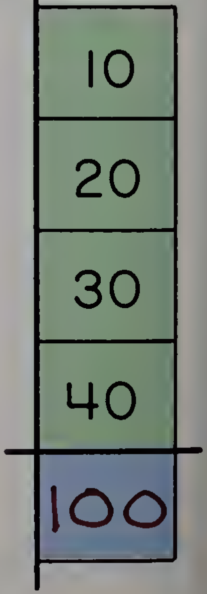
10	5	15
0	4	4
20	8	28
50	0	50
80	17	97

O



14
27
18
33
92

P



10
20
30
40
100

Notes on Pupil Page 70

Tasks: *In each exercise pupils reconstruct the* Cross Number Puzzle *or (in the exercises in the third and fourth columns) find the missing sums or addends.*

Purpose: (a) To further extend pupils' knowledge of decimal numeration and skill in addition and subtraction; (b) to help pupils develop the standard algorithms for addition and subtraction.

Unifying Ideas: Structure; numeration; addition and subtraction.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Be sure to check work carefully and correct any misconceptions you may find. Pupils who have difficulty with the exercises in the third and fourth columns should be encouraged to use the *Cross Number Puzzle* format for such problems.

Shortcuts

List addition facts in groups of three and add the numbers written in each column as shown below:

$$\begin{array}{ccccc} & & \text{(a)} & & \\ 2 & + & 1 & = & 3 \\ 1 & + & 4 & = & 5 \\ 0 & + & 2 & = & 2 \\ \overline{3} & & \overline{7} & & \overline{10} \end{array} \qquad \begin{array}{ccccc} & & \text{(b)} & & \\ 3 & + & 1 & = & 4 \\ 2 & + & 5 & = & 7 \\ 5 & + & 4 & = & 9 \\ \overline{10} & & \overline{10} & & \overline{?} \end{array}$$

Notice that in example (a) the results we found by adding the columns (3, 7, 10) suggest that we write 3 + 7 = 10. If this holds in all such cases then we have a shortcut in example (b)—instead of finding the sum of 4, 7, and 9, we might find the sum of 10 and 10:

$$\begin{array}{r} 4 \\ 7 \\ 9 \\ \hline 10 + 10 = 20 \end{array}$$

If we count out 4 objects, then 7 more, and then 9 more—and count the whole collection—we find that we do have 20.

more

Headline for the Day: Write a number sentence that fits: It's 8:15. What time will it be in 20 minutes?

Try several groups of three statements. Does this method work in each case?

Consider a group of three subtraction facts:

$$\begin{array}{ccc}
3 - 2 = 1 & \qquad & 5 - 4 = 1 \\
5 - 1 = 4 & & 8 - 5 = 3 \\
4 - 2 = 2 & & 7 - 2 = 5 \\
\overline{12} \;\; \overline{5} \;\; \overline{7} & & \overline{20} \;\; \overline{11} \;\; \overline{9}
\end{array}$$

Is it the case that 12 — 5 = 7 and 20 — 11 = 9?

Here is a puzzle: With each of the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 used only once, complete the following:

$$\begin{array}{lcl}
\underline{\quad + \quad + \quad} = 15 & \qquad & \underline{8 + 1 + 6} = 15 \\
\underline{\quad + \quad + \quad} = 15 & & \underline{3 + 5 + 7} = 15 \\
\underline{\quad + \quad + \quad} = 15 & & \underline{4 + 9 + 2} = 15 \\
15 + 15 + 15 = \overline{45} & & 15 + 15 + 15 = \overline{45}
\end{array}$$

(answer)

(Rows and columns in the answer can be interchanged, but the same three numbers will always appear in the same column or the same row.)

Combining Numbers Game

Think of a number greater than 4 and less than 16. Think of another, and another, until five numbers have been given and written on the board, such as:

5		13
	10	
8		15

Combine these to make true sentences in the following forms (you may use only the numbers given, but any number may be repeated):

$\underline{\quad + \quad} = 20$

$\underline{20 - \quad} = \underline{\quad}$

For example:

$$\left.\begin{array}{l}
5 + 15 = 20 \\
15 + 5 = 20 \\
20 - 15 = 5 \\
20 - 5 = 15
\end{array}\right\}\text{ fact team}$$

$$\left.\begin{array}{l}
10 + 10 = 20 \\
20 - 10 = 10
\end{array}\right\}\text{ fact team}$$

Divide the class into two teams, A and B, and let each team select its own five numbers greater than 4 and less than 16. The team wins that makes the most sentences from these numbers in the forms:

$\underline{\quad + \quad} = 20$ and: $\underline{20 - \quad} = \underline{\quad}$

Suppose that teams A and B select numbers as follows:

A

5		13
	10	
8		15

B

7	10	13
9		11

Team A can make the 6 sentences listed above—one 4-member fact team and one 2-member fact team. The selection by team B will afford 10 true statements in the form specified – two 4-member fact teams and one 2-member fact team:

7 + 13 = 20 (4 members)
11 + 9 = 20 (4 members)
10 + 10 = 20 (2 members)

The game can be made more exciting and challenging by changing the forms permitted, such as:

$\underline{\quad + \quad + \quad} = 20 \qquad \underline{\quad + \quad - \quad} = 20$

Now team A, with the above selections, could write such statements as:

5 + 5 + 10 = 20 15 + 13 — 8 = 20 etc.

Drill

Here are three suggested drill sheets:

Subtract:

59 − 8	45 − 2	38 − 6	42 − 3	34 − 7	75 − 6
88 −16	77 −23	85 −25	55 −17	64 −15	32 −11
75 − 3	28 −17	79 −72	65 −27	72 −38	38 −19

$\begin{array}{r} 5 \\ +\square \\ \hline 9 \end{array}$	$\begin{array}{r} \square \\ +2 \\ \hline 9 \end{array}$	$\begin{array}{r} 3 \\ +\square \\ \hline 6 \end{array}$	$\begin{array}{r} 7 \\ +3 \\ \hline \square \end{array}$	$\begin{array}{r} 4 \\ +\square \\ \hline 4 \end{array}$
$\begin{array}{r} 6 \\ +4 \\ \hline \square \end{array}$	$\begin{array}{r} 2 \\ +3 \\ \hline \square \end{array}$	$\begin{array}{r} \square \\ +5 \\ \hline 5 \end{array}$	$\begin{array}{r} 1 \\ +\square \\ \hline 9 \end{array}$	$\begin{array}{r} 2 \\ +\square \\ \hline 3 \end{array}$
$\begin{array}{r} 9 \\ +\square \\ \hline 9 \end{array}$	$\begin{array}{r} 3 \\ +\square \\ \hline 5 \end{array}$	$\begin{array}{r} \square \\ +6 \\ \hline 11 \end{array}$	$\begin{array}{r} 3 \\ +5 \\ \hline \square \end{array}$	$\begin{array}{r} 4 \\ +4 \\ \hline \square \end{array}$

$18 = 13 + \square$

$10 = \square + 6$

$19 = 2 + \square$

$21 = \square + 8$

$20 = 3 + \square$

$13 = \square + 13$

$19 - \square = 2$

$21 - 5 = \square$

$18 - 13 = \square$

$\square - 12 = 9$

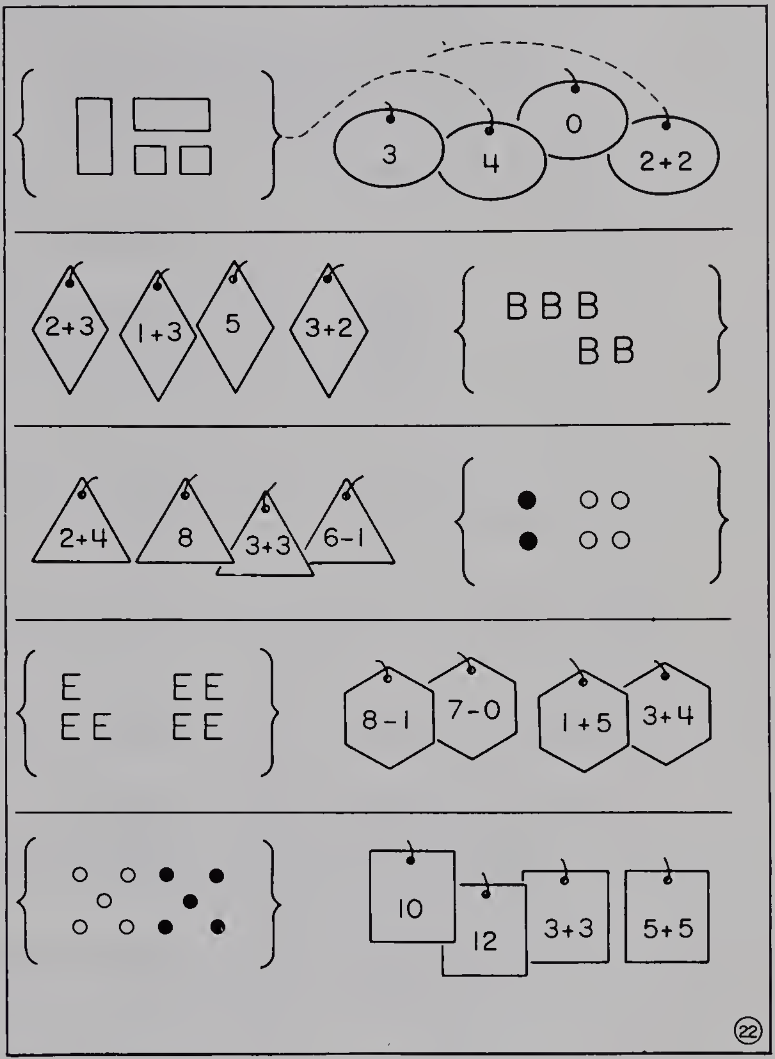
$16 - \square = 16$

$13 - \square = 2$

Here are two stories for drill purposes:

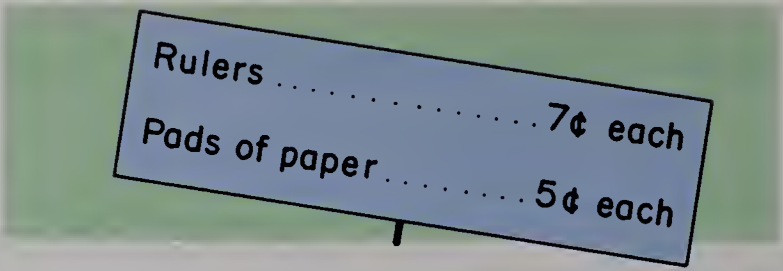
1) Jim put 18 plants into boxes. He put 3 plants in each box.
2) There are 17 boys in our class. There are 11 girls in our class.

Also consider using as a drill sheet page 22 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



A Lot About 25¢

Bill had a quarter. He saw a sign which said that rulers cost 7¢ each and pads of paper cost 5¢ each.


Rulers . . . . 7¢ each
Pads of paper . . . . 5¢ each

1. If he bought as many pads as he could, he would spend 25 ¢ and get 0 ¢ change.
2. If he bought as many rulers as he could, he would spend 21 ¢ and get 4 ¢ change.
3. If he bought one ruler and one pad of paper, he would spend 12 ¢ and get 13 ¢ change.
4. If he bought two rulers and two pads, he would spend 24 ¢ and get 1 ¢ change.

Each sentence below tells about a purchase Bill might have made. Fill in the blanks in each sentence.

5. 1 ruler(s) and 2 pad(s) would cost 17 ¢ leaving 8 ¢ change.
6. 2 ruler(s) and 1 pad(s) would cost 19 ¢ leaving 6 ¢ change.
7. 0 ruler(s) and 3 pad(s) would cost 15 ¢ leaving 10 ¢ change.
8. 2 ruler(s) and 0 pad(s) would cost 14 ¢ leaving 11 ¢ change.
9. 1 ruler(s) and 3 pad(s) would cost 22 ¢ leaving 3 ¢ change.
10. 0 ruler(s) and 1 pad(s) would cost 5 ¢ leaving 20 ¢ change.
11. 1 ruler(s) and 0 pad(s) would cost 7 ¢ leaving 18 ¢ change.
12. 0 ruler(s) and 4 pad(s) would cost 20 ¢ leaving 5 ¢ change.
13. 0 ruler(s) and 2 pad(s) would cost 10 ¢ leaving 15 ¢ change.

What's My Rule?

△	4	1	10	13	6	15	23	19	28	37	93
△+7	11	8	17	20	13	22	30	26	35	44	100

Notes on Pupil Page 71

Tasks: *Pupils record the number and costs of the various items which could be bought under the terms of the price list for not more than 25 cents. They also compute the change they would receive from 25 cents. Finally, they complete the* What's My Rule? *table.*

Purpose: (a) To give pupils practice in an aspect of applied arithmetic—the buying and selling of goods; (b) to provide pupils with practice in adding and subtracting 7.

Unifying Ideas: Addition and subtraction; measurement; functions and relations.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Begin by considering similar items and conditions:

Pencils	2¢ each
Erasers	1¢ each

If Tom had 10 cents, in how many ways could he spend it on pencils and erasers? Write the number sentences suggested: $(2 + 2 + 2 + 2 + 1 + 1 = 10,$ $2 + 2 + 2 + 1 + 1 + 1 + 1 = 10,$ etc.)

Another problem:

Pencils	4¢ each
Erasers	3¢ each

What purchase would cost exactly 10 cents? (2 erasers and 1 pencil.) Write the number sentence. $(3 + 3 + 4 = 10)$ What purchase would cost exactly 18 cents? Is there more than one way to spend 24 cents?

Drill

Here is a suggested drill sheet:

8+6=____	8+7=____	4+6=____
3+9=____	8+5=____	5+6=____
6+6=____	4+8=____	8+2=____
5+9=____	9+9=____	8+9=____
3+7=____	9+2=____	7+8=____
9+4=____	6+7=____	4+9=____
9+6=____	7+7=____	8+4=____
7+3=____	7+5=____	4+7=____
3+8=____	6+8=____	2+8=____
9+3=____	8+8=____	6+9=____

Here are two stories for drill purposes:

1) Dina has 7 dimes and 4 pennies.
2) Rachel and Barbara helped Miss Warden count the books in a library bookcase. Rachel counted 56 books on the first three shelves, and Barbara counted 39 books on the other shelves.

Headline for the Day: Write a number sentence that fits: In 1975 I will be____years old.

A.

15		10 + 5		8 + 7		5 + 10		9 + 6		20 − 5
+ 18		10 + 8		12 + 6		5 + 13		11 + 7		20 − 2
33	=	20 + 13	=	20 + 13	=	10 + 23	=	20 + 13	=	40 − 7

B.

34		9 + 25		30 + 4		20 + 14		31 + 3		40 − 6
+ 9		4 + 5		0 + 9		3 + 6		9 + 0		10 − 1
43	=	13 + 30	=	30 + 13	=	23 + 20	=	40 + 3	=	50 − 7

C.

26		25 + 1		20 + 6		20 + 6		30 − 4		35 − 9
+ 26		25 + 1		22 + 4		20 + 6		30 − 4		35 − 9
52	=	50 + 2	=	42 + 10	=	40 + 12	=	60 − 8	=	70 − 18

D.

37		20 + 17		30 + 7
+ 19		16 + 3		10 + 9
56	=	36 + 20	=	40 + 16

E.

37		7 + 30		15 + 22
+ 13		3 + 10		10 + 3
50	=	10 + 40	=	25 + 25

F.

14		7 + 7		4 + 10
+ 28		14 + 14		8 + 20
42	=	21 + 21	=	12 + 30

G.

29		25 + 4		30 − 1
+ 29		25 + 4		30 − 1
58	=	50 + 8	=	60 − 2

H. What Are My Rules?

a + 2	10	8	19	17	20	11	28	12	34	27	21
a	8	6	17	15	18	9	26	10	32	25	19
b	2	3	5	10	6	8	9	5	18	25	17
a + b	10	9	22	25	24	17	35	15	50	50	36

Tasks: Pupils determine and record the missing addends and sums in exercises A-G. They complete the table in exercise H.

Purpose: (a) To remind pupils of the general procedure for finding a sum by partitioning; (b) to provide pupils with practice in addition, adding 2, and subtracting 2.

Unifying Ideas: Structure; addition and subtraction; functions and relations.

Materials for Teacher: The chart described in the Notes on Pupil Page 27.

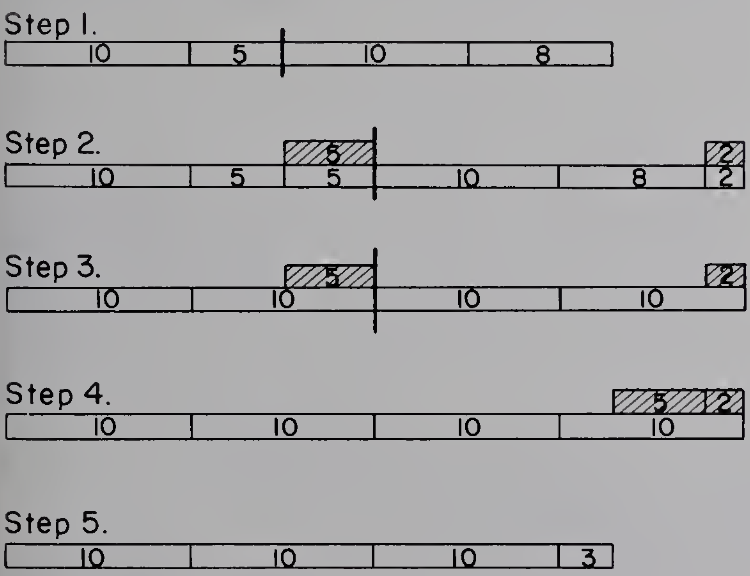
Materials for Pupils: Cuisenaire rods.

The Lesson:

These exercises are designed to remind pupils of the freedom they have in partitioning addends when finding sums.

Cuisenaire Rods

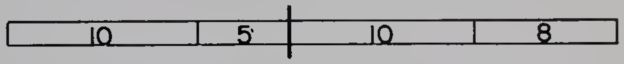
Pupils can carry out some of these exercises with Cuisenaire rods or with ordinary counters. The last problem in exercise A is nicely illustrated with rods. One starts with two pairs of rods to show 15 + 18:



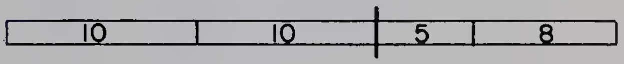
If you use new rods in Step 2 and leave the rods of Step 1 untouched, you can compare the trains in Step 1 and Step 5, and thus verify that 15 + 18 = 33.

Here is how one might use rods to illustrate the first problem in exercise A:

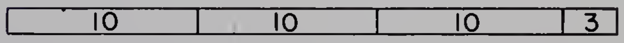
Step 1.



Step 2. Interchange the pair of rods in the middle of the train.



Step 3. Exchange the last two rods for an orange rod and a light green one.



Cross Number Puzzles

Each problem in A-G can be put into the *Cross Number Puzzle* format. For example, here are the five problems of exercise A formulated as *Cross Number Puzzles:*

15	10	5
18	10	
?	20	13

15	8	7
18	12	6
?		

15	5	10
18	5	13
?		

15		6
18		
?		13

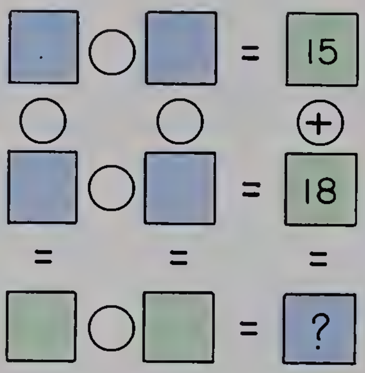
15	5	
18	2	20
?	7	40

Headline for the Day: Cola costs 10¢ a bottle and ice cream 15¢ a cone. What can you buy if you spend 15¢, 20¢, 25¢, 30¢, etc.?

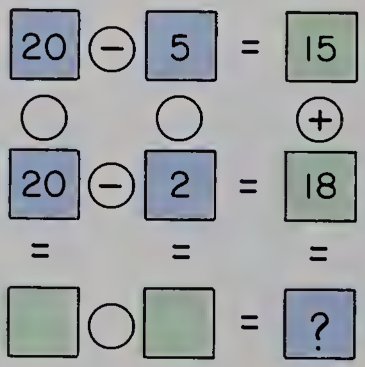
Watch the Signs!

The problems in exercises A-G can be carried out in a *Watch the Signs!* format. Consider the last problem in exercise A. Use the chart described in the Notes on Pupil Page 27.

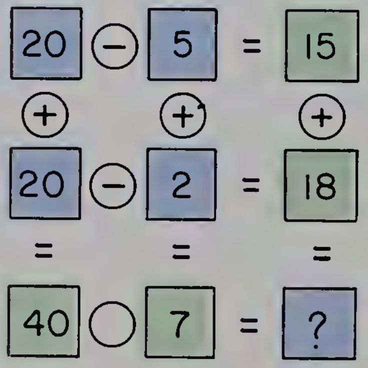
Step 1



Step 2



Step 3



Step 4 consists of noticing that the missing sign is a subtraction sign, writing it in the blank, and computing the number for the empty box by subtracting 7 from 40.

A Deeper Look for the Teacher

Let's take a look at the last problem in exercise A. According to the information given on the pupil page, the claim is made that since

$$\text{(I)}\quad 15=20-5$$

and $$\text{(II)}\quad 18=20-2$$

it follows that

$$\text{(III)}\quad 15+18=(20+20)-(5+2)$$

Why does sentence (III) follow from (I) and (II)? Let's see. Since adding a number is the inverse of subtracting it, we know from (I) and (II) that

$$\text{(IV)}\quad 15+5=20$$

and $$\text{(V)}\quad 18+2=20$$

By the property of addition which underlies the *Cross Number Puzzle* (see below), it follows from (IV) and (V) that

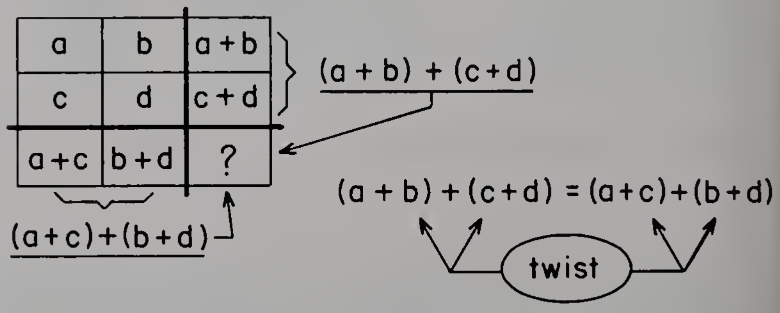
$$\text{(VI)}\quad (15+18)+(5+2)=20+20$$

And, since subtracting a number is the inverse of adding it, it follows from (VI) that

$$15+18=(20+20)-(5+2)$$

which is what we wished to show.

The property of addition which underlies the *Cross Number Puzzle* is sometimes called *the twist property*, a name coined by Professor H. E. Vaughan of the University of Illinois.



The fact that addition has this property is based on the connection between addition and displays and on the fact that partitioning a display does not affect the number of objects in the display.

Another way of looking at the twist property of addition is to notice that it follows from the fact that addition has the *associative* and *commutative* properties. Here is the connection:

$$
\begin{aligned}
&(a+b)+(c+d) \\
&\qquad \Big\} \text{(associativity)} \\
&=[(a+b)+c]+d \\
&\qquad \Big\} \text{(associativity)} \\
&=[a+(b+c)]+d \\
&\qquad \Big\} \text{(commutativity)} \\
&=[a+(c+b)]+d \\
&\qquad \Big\} \text{(associativity)} \\
&=[(a+c)+b]+d \\
&\qquad \Big\} \text{(associativity)} \\
&=(a+c)+(b+d)
\end{aligned}
$$

So $(a+b)+(c+d)=(a+c)+(b+d)$.

The foregoing discussion of the twist, associative, and commutative properties of addition is strictly for your own information. We do not recommend that you deal with these matters with your pupils. It is quite enough for them to understand that the general procedure for finding sums by partitioning addends depends upon the fact that changing the arrangement of the objects in a display does not change the number of objects.

Drill

Here is a suggested drill sheet:

$79 + 10 = \square + 9 = \triangle$

$43 + 10 = \square + 3 = \triangle$

$47 + 40 = \square + 7 = \triangle$

$76 + 10 = \square + 6 = \triangle$

$25 + 60 = \square + 5 = \triangle$

$73 + 20 = 90 + \square = \triangle$

$42 + 50 = 90 + \square = \triangle$

$13 + 70 = 80 + \square = \triangle$

$49 + 30 = 60 + \square = \triangle$

The last item is a pattern breaker. Warn pupils to be careful when they work it.

Here are two stories for drill purposes:

1) Sylvia has 9 cents. She needs 16 cents to buy a boat for Jose.
2) Our class had 75 tickets to sell. The girls sold 40 of them.

I

+	one ten	four ones	14
10	20	14	24
7	17	11	21
15	25	19	29
20	30	24	34
28	38	32	42
0	10	4	14

II

−	one ten	three ones	13
15	5	12	2
20	10	17	7
27	17	24	14
30	20	27	17
13	3	10	0
31	21	28	18

III

+	three ones	two tens	one hundred	123	three hundreds	305
100	103	120	200	223	400	405
10	13	30	110	133	310	315
110	113	130	210	233	410	415
70	73	90	170	193	370	375
80	83	100	180	203	380	385
605	608	625	705	728	905	910

IV What's My Rule?

□	7	10	8	13	19	24	5	33	40	42	100
□−5	2	5	3	8	14	19	0	28	35	37	95

Notes on Pupil Page 73

Tasks: *Pupils determine the missing numbers (sums or differences) as required. They also complete the* What's My Rule? *table at the bottom of the page.*

Purpose: (a) To provide pupils with practice in addition and subtraction; (b) to reinforce pupils' understanding of the connection between the operations and the work with concrete objects; (c) to provide pupils with practice in adding 5 and subtracting 5.

Unifying Ideas: Numeration; addition and subtraction; functions and relations.

Materials for Teacher: None.

Materials for Pupils: Play money or Cuisenaire rods.

The Lesson:

The addition and subtraction tables are designed to help pupils find sums and differences by taking advantage of the decimal nature of the common numeration system. For example, in exercise I the main problem is that of adding 14. This is carried out in two steps—adding 10 and adding 4. By using word descriptions of one of the addends—one ten, four ones, etc.—we push the pupil into thinking about manipulating objects. A word description such as two tens refers to two 10-groups. So, as in exercise III, when we ask a pupil to add 123 to 100, we are asking him to find the number of objects obtained by starting with the objects in a 100-group, adding to these the objects in three 1-groups, adding to this the objects in two 10-groups, and adding to this the objects in a 100-group.

You can prepare pupils for this page by allowing them to examine and extend patterns involving ordinary decimal notation and word names:

1—one one—one penny
1+1→ 2—two ones—two pennies
1+1+1→ 3—three ones—three pennies

• • •

10—one ten—one dime
10+10→20—two tens—two dimes

• • •

100—one hundred—one dollar
100+100→200—two hundreds—two dollars

• • •

	Tens	Ones	Cents
16	1	6	sixteen
22	2	2	twenty-two
	3	4	

• • •

	Hundreds	Tens	Ones	Money
123	1	2	3	one dollar and twenty-three cents
___	2	1	3	___________

• • •

One ten and four ones are 14 ones.
Two tens and four ones are 24 ones.

• • •

Before starting the pupil page you may wish to practice on two-dimensional tables such as:

+	one ten	two ones	12
12	22		
14		16	
16			28

−	one ten	two ones	12
12	2		
14		12	
16			8

Drill

Here are some exercises for a drill sheet:

46 + ☐ = 48
14 + ☐ = 19
35 + ☐ = 38
23 + ☐ = 29
66 + ☐ = 69

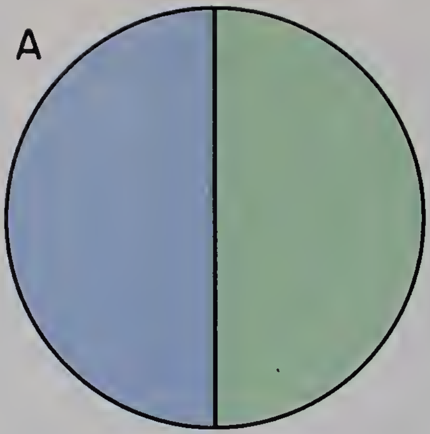
65 − 2 = ☐
74 − 3 = ☐
88 − 5 = ☐
59 − 4 = ☐
26 − 2 = ☐

Here are two stories for drill purposes:

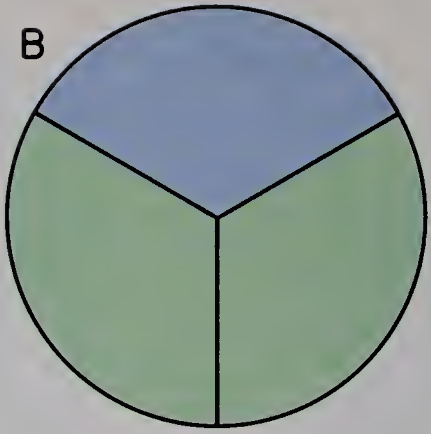
1) Sarah gave her brother 1 dime and 4 pennies.
2) Jules weighed 83 pounds and Rex weighed 74.

Headline for the Day: Write a story suggested by the headline: 25>17

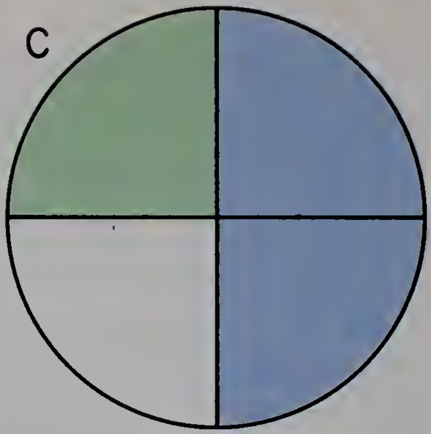
A



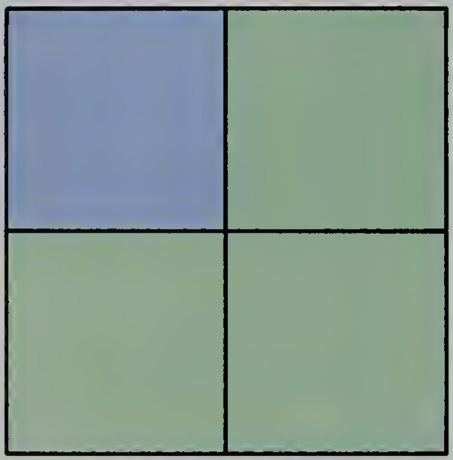
B



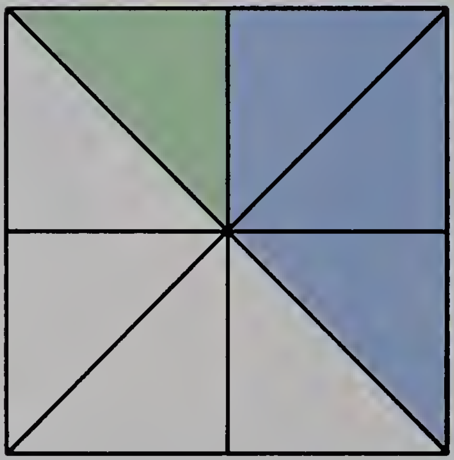
C



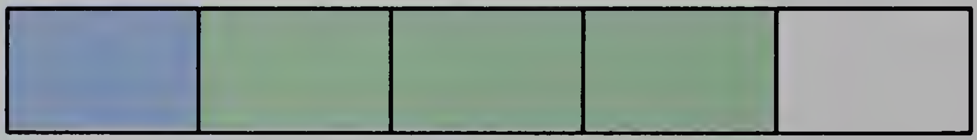
D



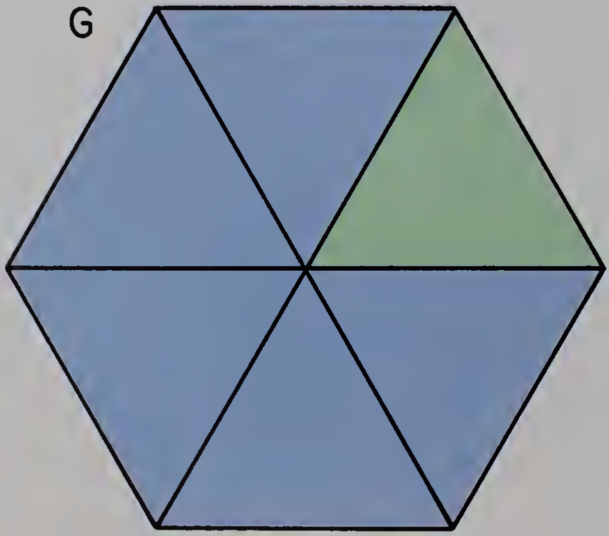
E



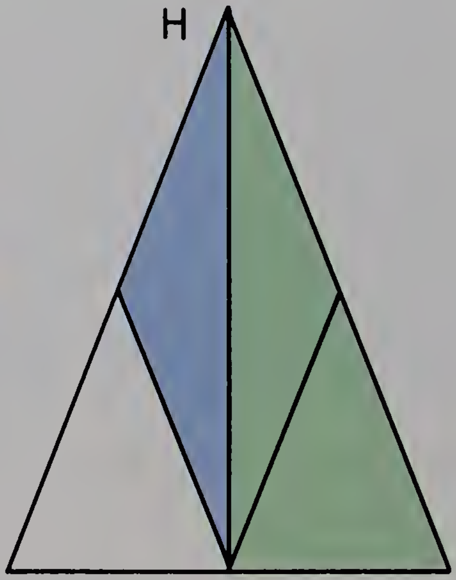
F



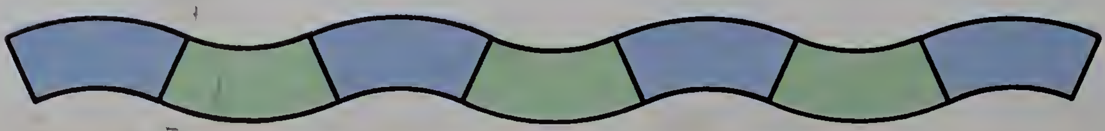
G



H



I



	Part shaded blue	Part shaded green
A	$\frac{1}{2}$	$\frac{1}{2}$
B	$\frac{1}{3}$	$\frac{2}{3}$
C	$\frac{2}{4}$ or $\frac{1}{2}$	$\frac{1}{4}$
D	$\frac{1}{4}$	$\frac{3}{4}$
E	$\frac{3}{8}$	$\frac{1}{8}$
F	$\frac{1}{5}$	$\frac{3}{5}$
G	$\frac{5}{6}$	$\frac{1}{6}$
H	$\frac{1}{4}$	$\frac{2}{4}$ or $\frac{1}{2}$
I	$\frac{4}{7}$	$\frac{3}{7}$

Tasks: Pupils record in the table the fraction of each figure shaded blue and the fraction shaded green.

Purpose: To introduce pupils to the notation for fractions.

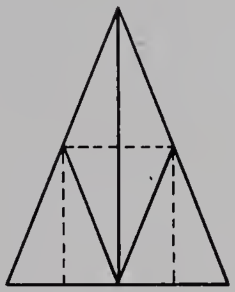
Unifying Ideas: Numeration; geometry; measurement.

Materials for Teacher: Tagboard models of the figures sectioned as shown on pupil page 74; a floor number line.

Materials for Pupils: Desk-size copies of the teacher's models.

The Lesson:

Let pupils dissect their models of the figures shown on pupil page 74 and compare the parts. The only figure that may prove to be difficult is that in exercise H. The problem can be resolved by adding three more dissection lines:



If you dissect the figure by cutting along all six dissection lines, you obtain 8 triangles of the same size and shape.

The Fraction Idea

The pupils already have some understanding of such ideas as a half, a third, a fourth, or a quarter of some-

more

Headline for the Day: It's 12:10 P.M. How much time has passed since 11:15 A.M.?

thing. To find out the extent of their understanding, ask pupils such questions as the following:

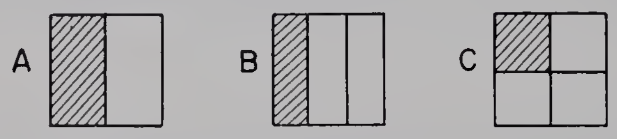
What is a half of 6? Of 8? Of 10? Of a dozen? What does a half of a pie look like? A half of a square? A half of a dollar bill? How much money would you have if you had a half of a half-dollar?

How many letters of the alphabet are easy to cut in two parts that are the same size? Some are more evident than others. I, O, X, and H do not leave much question, A, C, N, S and T are still easy. But many depend on the way they are drawn—B, E, K, etc.

If Bill had 12 marbles, would you rather he gave you 1 out of every 2, 1 out of every 3, or 1 out of every 4? How many would you have if he gave you 1 out of every 2? 2 out of every 4? 3 out of every 6?

How many is a third of 3? A third of 6? How many is a fourth of 4? A fourth of 12? How many is a half of 100? A fourth of 100?

Put examples like the following on the board:

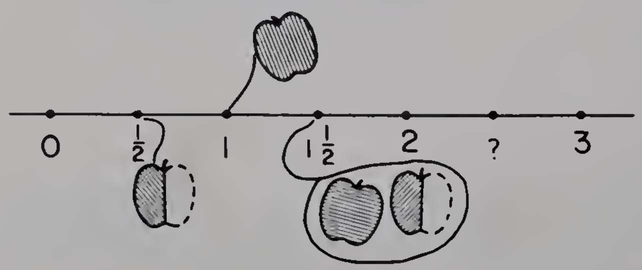


Into how many parts of the same size is A divided? What part is shaded? Not shaded? Are there other ways to divide the square into two parts of the same size? Ask similar questions about B and C. Discuss the examples on pupil page 74 in the same way. Now pupils may complete the table on pupil page 74.

(See the Notes on Pupil Page 111 for more suggestions for preliminary work with fractions.)

Fractions on the Number Line

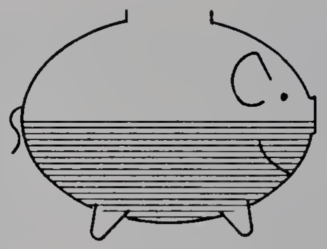
If the idea of one apple has already been associated with the number 1 on the number line, then how can ½ apple be logically labeled on the number line? The label ½ must be at a point halfway between 0 and 1. What about an apple and a half?



Using a floor number line, ask pupils to find the point that is halfway between 2 and 3. Between 3 and 6. Between 5 and 8. Between 4 and 10.

If-Then

Here is a sketch of a piggy bank. It is half full of pennies:

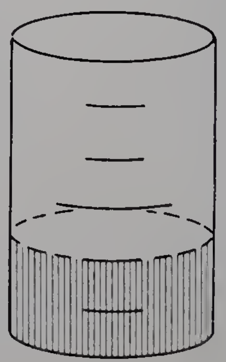


Can you tell me how many pennies there are in the bank? What do you need to know before you can tell me the number? If I told you how many pennies were needed to fill the piggy bank, could you tell me how many pennies the bank holds when it is half full?

Value when full	No. of pennies when half full
one dollar	
a half-dollar	
three dollars	
five dollars	
ten dollars	

If a full piggy bank holds 150 pennies then how many pennies does it hold when it is half full?

Here is a water tank. The marks on it help to indicate how full it is:



No. of gallons when full	No. of gallons in shaded area
6	
60	
3	
1	
10	

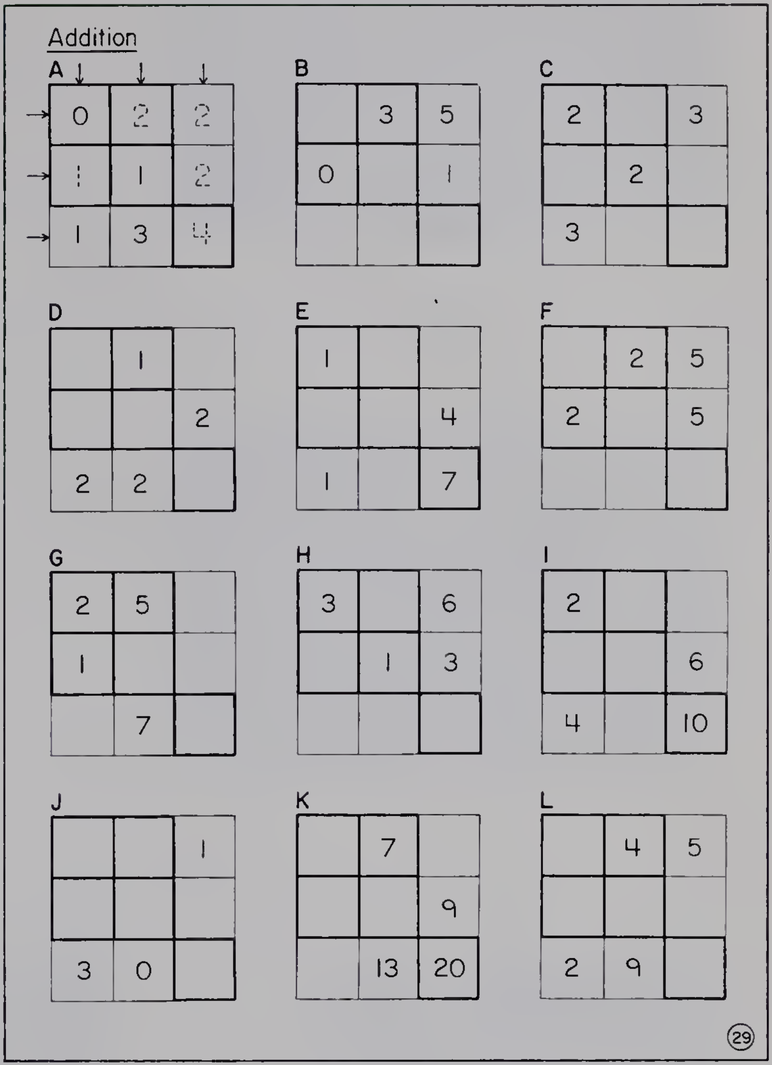
Drill

Here is a suggested drill sheet:

Add.

12	23	57	46	39
+70	+30	+20	+40	+50
27	50	70	65	30
+40	+43	+19	+20	+30
53	80	27	38	50
+10	+19	+40	+20	+50

Also consider using as a drill sheet page 29 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Addition

A

0	2	2
1	1	2
1	3	4

B

	3	5
0		1

C

2		3
	2	
3		

D

	1	
		2
2	2	

E

1		
		4
1		7

F

	2	5
2		5

G

2	5	
1		
	7	

H

3		6
	1	3

I

2		
		6
4		10

J

		1
3	0	

K

	7	
		9
	13	20

L

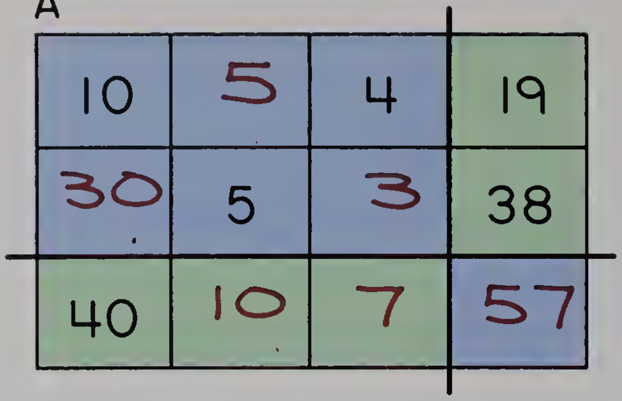
	4	5
2	9	

㉙

Here are two stories for drill purposes:

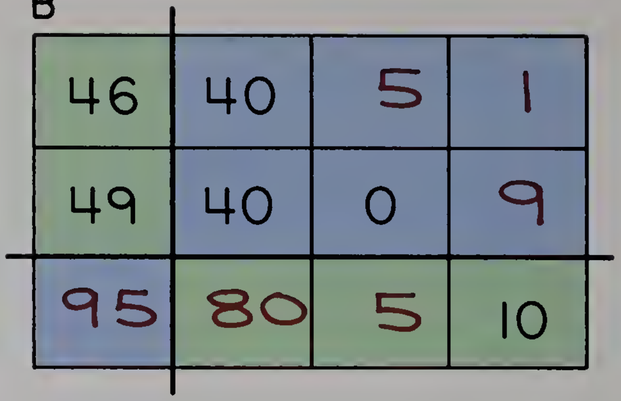
1) Francis has 7 toy cars. Ozzie has 10 toy cars.
2) There were 26 teachers at a meeting. 24 more came to the meeting.

A



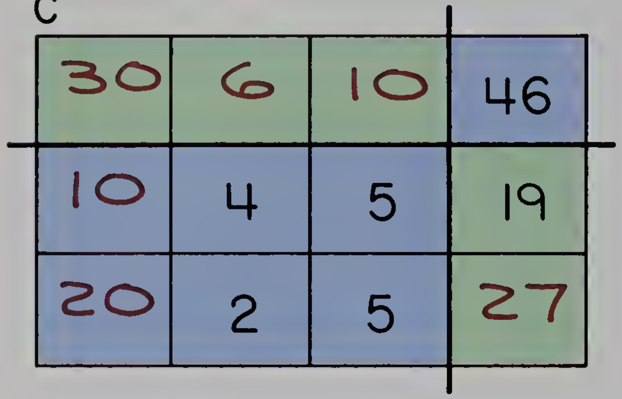
10	5	4	19
30	5	3	38
40	10	7	57

B



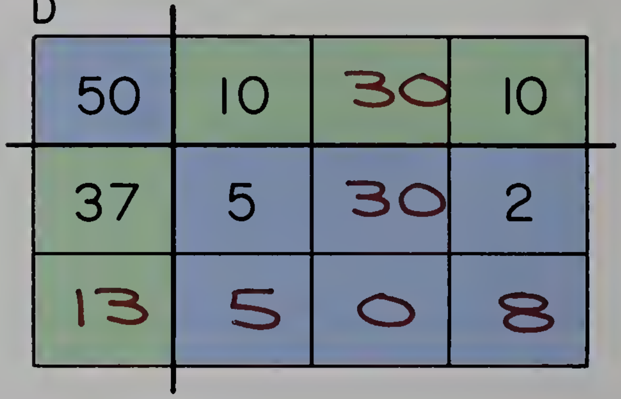
46	40	5	1
49	40	0	9
95	80	5	10

C



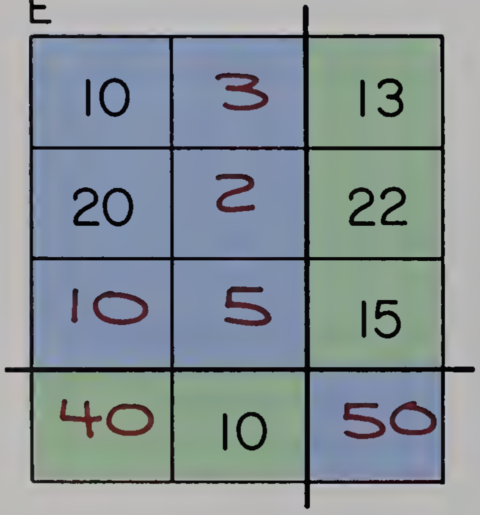
30	6	10	46
10	4	5	19
20	2	5	27

D



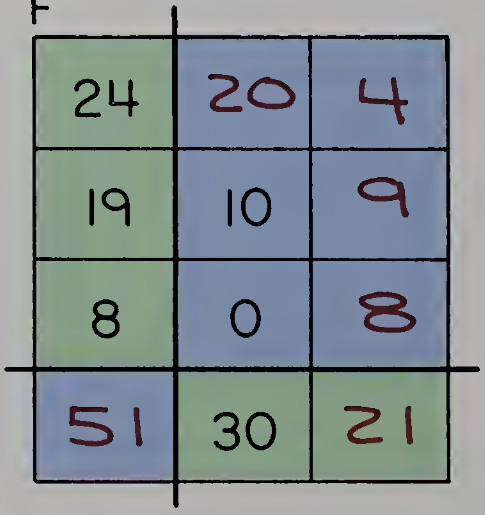
50	10	30	10
37	5	30	2
13	5	0	8

E



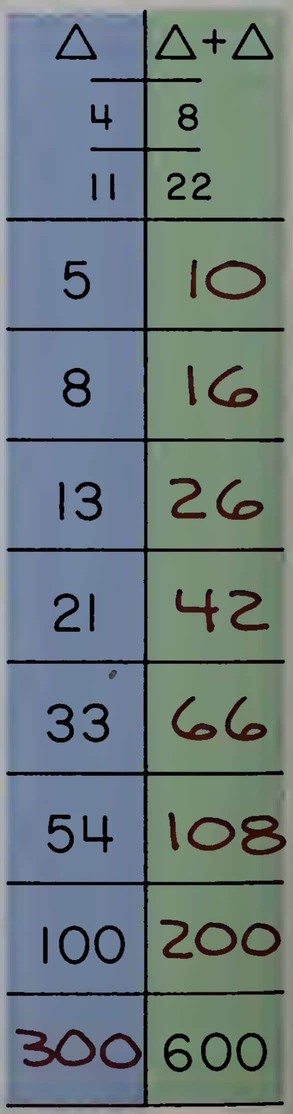
10	3	13
20	2	22
10	5	15
40	10	50

F



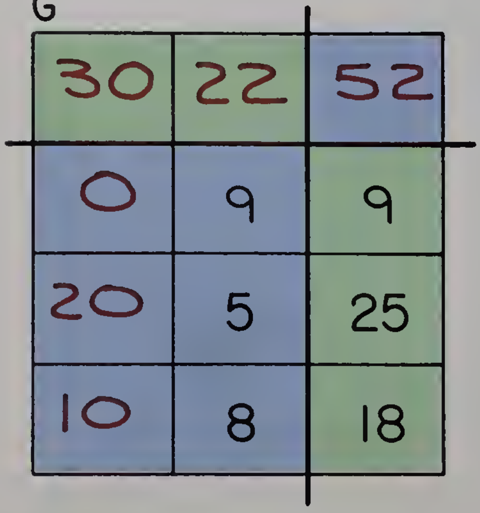
24	20	4
19	10	9
8	0	8
51	30	21

What's My Rule?



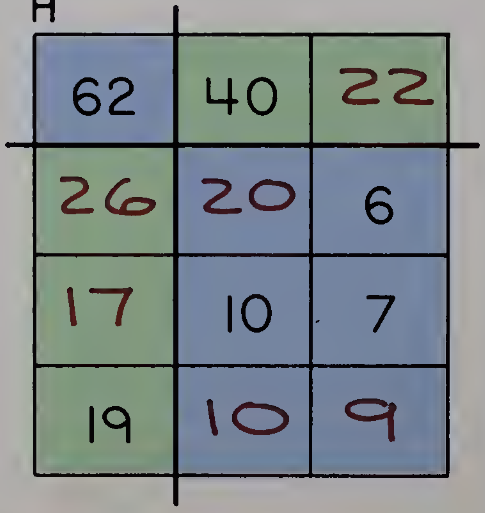
△	△+△
4	8
11	22
5	10
8	16
13	26
21	42
33	66
54	108
100	200
300	600

G



30	22	52
0	9	9
20	5	25
10	8	18

H



62	40	22
26	20	6
17	10	7
19	10	9

Tasks: Pupils reconstruct each Cross Number Puzzle. *They also complete the* What's My Rule? *table.*

Purpose: (a) To provide pupils with practice in addition and subtraction; (b) to help them develop skill in using the standard algorithm for finding the sum of pairs and of triples of 2-digit numbers; (c) to provide pupils with practice in doubling numbers.

Unifying Ideas: Structure; numeration; addition and subtraction; multiplication and division; functions and relations.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

The *Cross Number Puzzles* on this page illustrate the fact that one can partition a display into more than two parts. This is a page for which you need not give pupils a special buildup. They can readily transfer what they have learned about 3-by-3 *Cross Number Puzzles* to 3-by-4 and 4-by-3 *Cross Number Puzzles*.

Developing Independence

Whenever you feel that pupils are ready to tackle a page without any preparatory explanation, try it. You can always retreat and insert whatever class discussion and work you feel pupils need.

One very effective method is to partition the class into small work groups. Let each group talk among themselves about the page. Get them started with suggestions like: What work do you think needs to be done on this page? Talk about it together. If you reach an agreement, go ahead. I am ready to help if you are unable to get started on your own. Help each other as much as you can.

This kind of introduction to a new page provides a note of challenge, and suggests the idea of puzzle. These are two essential ingredients of creative mathematics.

In mathematics, as in other subject matter areas, there has been a long tradition which condones spoon-feeding. Traditionally, teachers are asked to point out clearly what is to be done next each step of the way. The result is that pupils become dependent on directions, instructions, and orders of procedure. This text provides many opportunities to encourage pupils to think. Have them consider a problem that may not be clear to them at first. Where does one begin? Is there a part of the problem that looks familiar? If so, can this be a clue to the less familiar sections? What's going on?

Almost all "secrets" within the pages of this text can be uncovered by pupils working in small groups and eager to prove their ability. Often a pupil working alone gets lost on a side road which does not help him toward the goal. But this is seldom the case with small groups.

It is essential, of course, that pupils stand a good chance of succeeding. For this reason, be quite sure that your class is prepared for any pages you may ask them to tackle on their own. Within a few minutes, you can easily tell whether success is in the wind. If it is not, offer what help you think will be most beneficial.

Drill

Here are two stories for drill purposes:

1) Stella made 23 sugar cookies, 24 nut cookies, and 17 plain cookies.
2) In a miniature bowling game Ted scored 97 points and Carmen scored 85 points.

Headline for the Day: Compare January and February, using a number sentence with one of these signs: $<, >$.

I

—	two ones	one ten	eleven ones	three ones	five ones	two tens	twenty-five ones	seven ones
25	23	15	14	22	20	5	0	18
30	28	20	19	27	25	10	5	23
40	38	30	29	37	35	20	15	33
55	53	45	44	52	50	35	30	48
33	31	23	22	30	28	13	8	26
42	40	32	31	39	37	22	17	35

II

+	three ones	five ones	one ten	twenty-five ones	two tens	seven ones	twenty-seven ones	seventeen ones
8	11	13	18	33	28	15	35	25
15	18	20	25	40	35	22	42	32
0	3	5	10	25	20	7	27	17
25	28	30	35	50	45	32	52	42
57	60	62	67	82	77	64	84	74
100	103	105	110	125	120	107	127	117

III What's My Rule?

5¢	7¢	3¢	10¢	25¢	9¢	15¢	18¢	50¢	75¢
10¢	14¢	6¢	20¢	50¢	18¢	30¢	36¢	$1.00	$1.50

Notes on Pupil Page 76

Tasks: Pupils compute and record sums and differences as required. They complete the table at the bottom of the page.

Purpose: (a) To provide pupils with practice in adding and subtracting 1-digit and 2-digit numbers; (b) to provide pupils with practice in doubling amounts of money.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; functions and relations; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

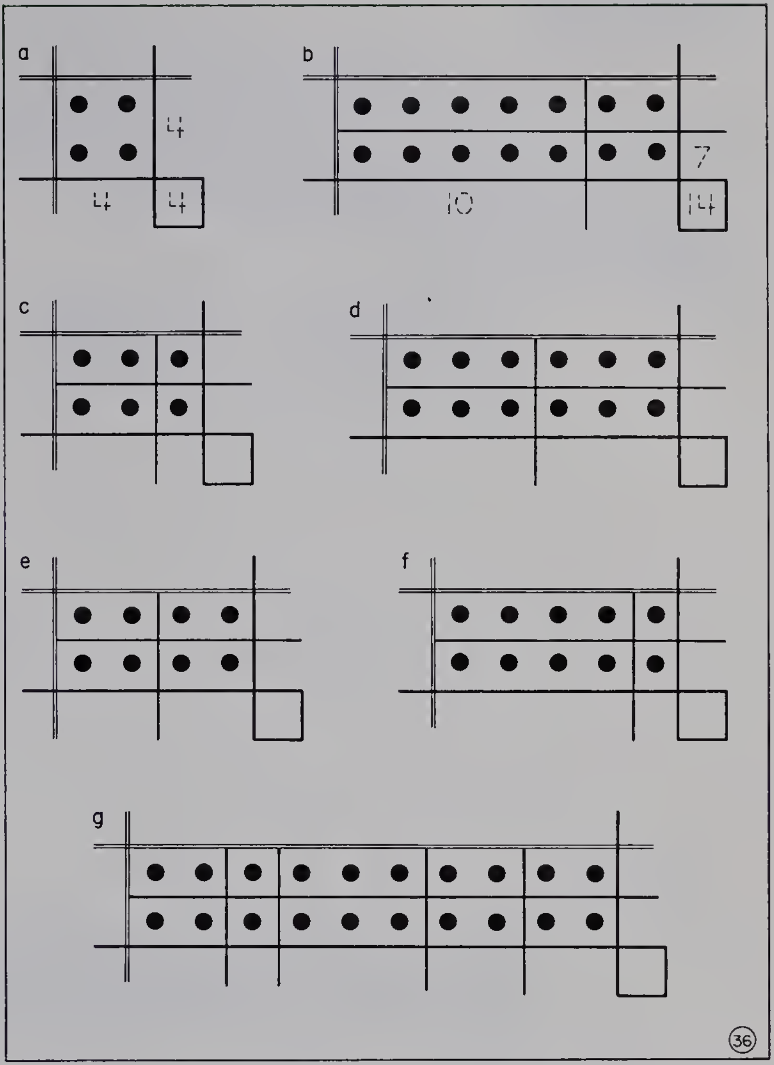
The Lesson:

Pupils can be set to work on this page without any special preparation.

Notice that the numbers selected for the top row in each of tables I and II help establish patterns for carrying out the computations. For example, subtracting 25 ones amounts to subtracting two tens and then subtracting five ones (or to subtracting five ones and then subtracting two tens).

Drill:

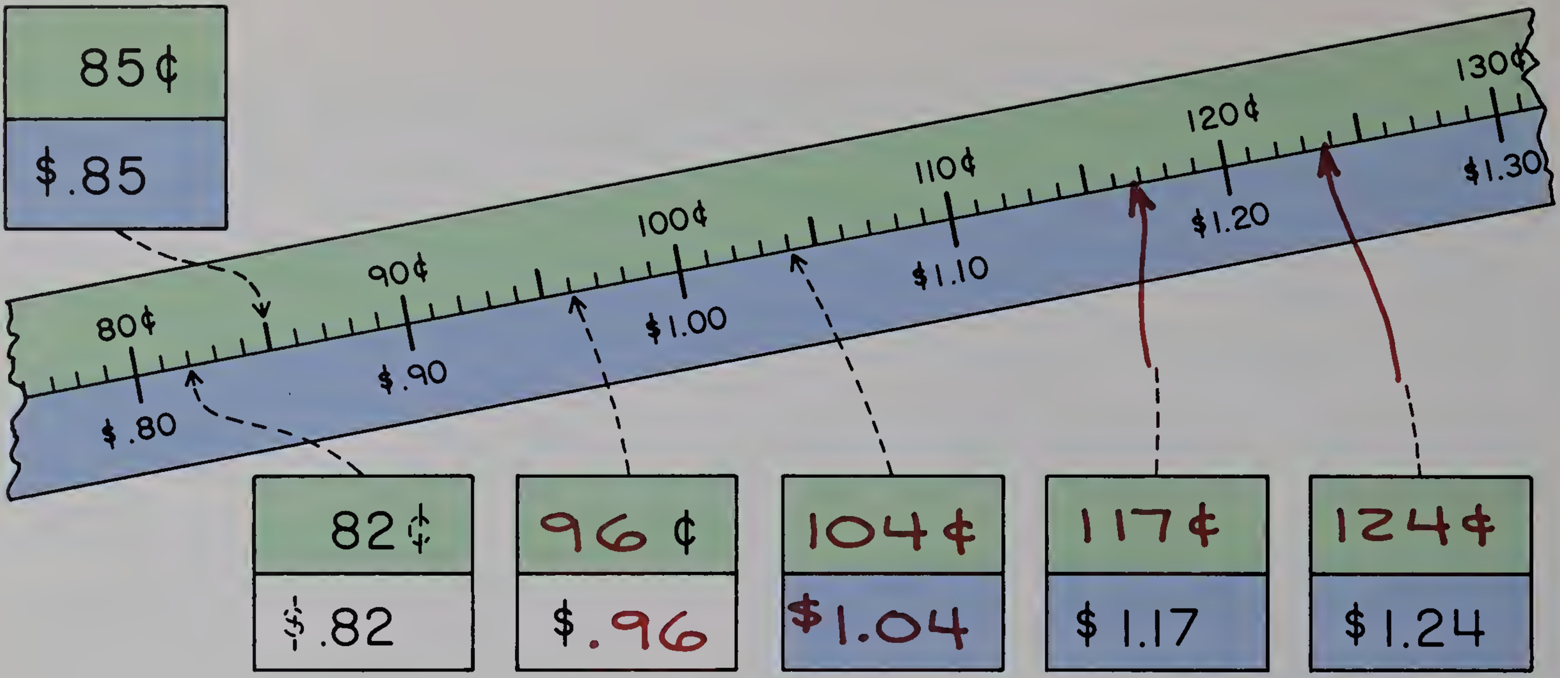
If an easy preparatory drill sheet is needed for pupil page 76, try page 36 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Here are two stories for drill purposes:

1) Doug bought 3 candy bars. Each candy bar cost 7 cents.
2) There are 94 pages in Sandra's story book. She has read 25 of them.

Headline for the Day: Candy bars cost 3 for 31¢. How much will you pay for 1 candy bar?


85¢
$.85
80¢
$.80
90¢
$.90
100¢
$1.00
110¢
$1.10
120¢
$1.20
130¢
$1.30
82¢
$.82
96¢
$.96
104¢
$1.04
117¢
$1.17
124¢
$1.24

In each box below, write the sums in both ways as shown.

The way you should write the sums in the blue boxes below is the way we usually write an amount of money over one dollar.

Add	5¢	25¢	8¢	$1.00	84¢	90¢
7¢	12¢ $.12	32¢ $.32	15¢ $.15	107¢ $1.07	91¢ $.91	97¢ $.97
15¢	20¢ $.20	40¢ $.40	23¢ $.23	115¢ $1.15	99¢ $.99	105¢ $1.05
50¢	55¢ $.55	75¢ $.75	58¢ $.58	150¢ $1.50	134¢ $1.34	140¢ $1.40
$1.16	121¢ $1.21	141¢ $1.41	124¢ $1.24	216¢ $2.16	200¢ $2.00	206¢ $2.06
$3.20	325¢ $3.25	345¢ $3.45	328¢ $3.28	420¢ $4.20	404¢ $4.04	410¢ $4.10

Notes on Pupil Page 77

Tasks: *Pupils determine the missing sums and record them in both cents notation and dollars-and-cents notation.*

Purpose: (a) To introduce pupils to decimal notation for United States money; (b) to provide pupils with practice in addition.

Unifying Ideas: Numeration; addition and subtraction; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

The number line shown at the top of pupil page 77 carries two notations—for example, one point is labeled 90¢ and $.90. Both labels refer to the same amount of money and both labels are pronounced the same way. On the other hand, for amounts of money in excess of one dollar, the labels are pronounced differently. We pronounce 120¢ as one hundred twenty cents and $1.20 as one dollar and twenty cents.

Work with the number line until pupils are able to give both pronunciations with ease.

Drill

Here are two stories for drill purposes:

1) Dominick saved 55 bottle tops. He gave 43 of them away.
2) Bolivar has 6 books about pets and 3 books about trucks.

Here are suggested drill sheets:

$35 + 24 = \square + 9 = \triangle$
$86 + 12 = \square + 8 = \triangle$
$16 + 31 = \square + 7 = \triangle$
$24 + 14 = \square + 8 = \triangle$
$32 + 47 = 70 + \square = \triangle$
$58 + 21 = 70 + \square = \triangle$
$43 + 34 = 70 + \square = \triangle$
$65 + 21 = \square + \bigcirc = \triangle$
$24 + 31 = \square + \bigcirc = \triangle$

$9 + \square = 12$
$3 + \square = 13$
$\square + 6 = 11$
$\square + 5 = 12$
$\square + 7 = 10$

$13 - 7 = \square$
$15 - \square = 5$
$12 - \square = 3$
$11 - 2 = \square$
$10 - 6 = \square$

$6 + 2 + \square = 11$
$5 + 3 + \square = 12$
$2 + 3 - \square = 13$
$5 + 2 + \square = 9$
$2 + 8 + \square = 17$

Headline for the Day: Mother had 3 bills—one each for $1.00, $5.00, and $10.00—and 3 coins—one each for 10¢, 25¢ and 50¢. She gave Jill one of the bills and one of the coins. How many possible combinations could Jill have got? How much is each combination worth? (9 combinations; $1.10, $1.25, etc.)

A.

33	25 + 8	13 + 20	15 + 15 + 3	10 + 20 + 3
− 15	10 + 5	5 + 10	7 + 5 + 3	4 + 8 + 3
18 =	15 + 3 =	8 + 10 =	8 + 10 + 0 =	6 + 12 + 0

B.

42	20 + 22	30 + 12	25 + 15 + 2	40 + 5 − 3
− 27	5 + 22	20 + 7	25 + 0 + 2	30 + 0 − 3
15 =	15 + 0 =	10 + 5 =	0 + 15 + 0 =	10 + 5 − 0

C.

60	29 + 31	10 + 50	20 + 25 + 15	35 + 35 − 10
− 26	15 + 11	6 + 20	10 + 15 + 1	35 + 1 − 10
34 =	14 + 20 =	4 + 30 =	10 + 10 + 14 =	0 + 34 − 0

D.

21	15 + 6	11 + 10
− 12	10 + 2	2 + 10
9 =	5 + 4 =	9 + 0

F.

53	13 + 40	43 + 10
− 35	5 + 30	30 + 5
18 =	8 + 10 =	13 + 5

E.

41	11 + 30	25 + 16
− 14	4 + 10	10 + 4
27 =	7 + 20 =	15 + 12

G.

62	30 + 32	70 − 8
− 26	20 + 6	30 − 4
36 =	10 + 26 =	40 − 4

H. What Are My Rules?

b − a	9	6	16	0	30	35	7	3	15	31	17
a	8	15	12	17	25	8	13	9	25	19	16
b	17	21	28	17	55	43	20	12	40	50	33
a + b	25	36	40	34	80	51	33	21	65	69	49

Notes on Pupil Page 78

Tasks: Pupils determine and record the missing addends and sums in exercises A-G. They also complete the table in exercise H.

Purpose: (a) To remind pupils of the general procedure for finding a difference by partitioning; (b) to provide pupils with practice in subtraction and addition.

Unifying Ideas: Structure; addition and subtraction; functions and relations.

Materials for Teacher: The chart described in the Notes on Pupil Page 27.

Materials for Pupils: Cuisenaire rods.

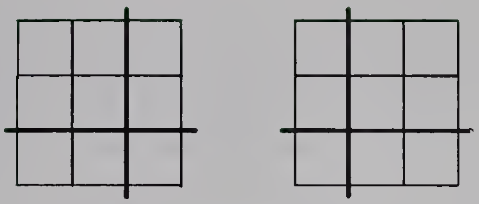
The Lesson:

See the remarks in the Notes on Pupil Page 72. They are readily adapted to pupil page 78. Also see the Notes on Pupil Page 46 for remarks applicable to exercise H.

How to Use a Cross Number Puzzle

You will find it helpful to experiment by yourself with the various ways of using the basic *Cross Number Puzzle* format in explaining why the procedures for addition and subtraction work. The "rules" given below are not to be stated formally for pupils.

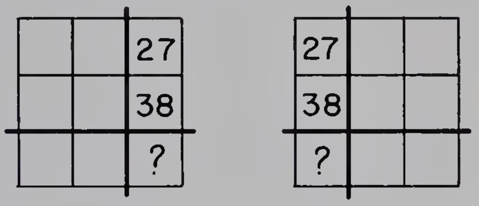
For an addition problem given in vertical format with the sum written below the addends, we can use either of these positions for the *Cross Number Puzzle:*



Let's say that our addition problem is:

$$\begin{array}{r} 27 \\ +38 \\ \hline \end{array}$$

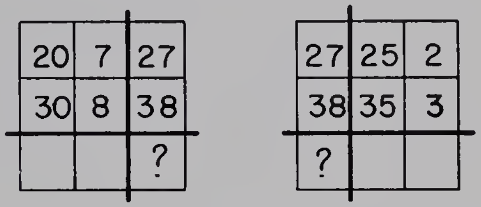
If we wish to find the sum by partitioning the addends, we write the problem on the narrow side of the heavy line:



		27
		38
		?

27		
38		
?		

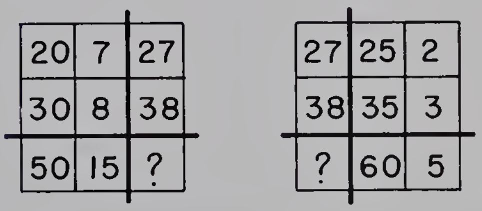
We then partition each addend, listing the parts in the row containing the addend. Examples:



20	7	27
30	8	38
		?

27	25	2
38	35	3
?		

Then we add the parts, column by column:



20	7	27
30	8	38
50	15	?

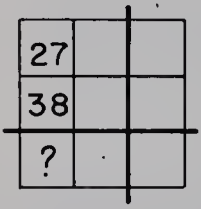
27	25	2
38	35	3
?	60	5

In view of the twist property of addition (see the Notes on Pupil Page 72), we can be assured that 50 + 15 is the sum of 27 and 38 and that so is 5 + 60.

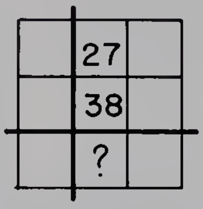
more

Headline for the Day: Make up a time-saver chart for the movie cashier. Admission prices are 35¢ for children and $1.25 for adults.

On the other hand, we may wish to find the sum of 27 and 38 by expressing each of these numbers as a difference. In this case, we write the problem in a column on the wide side of the heavy line:

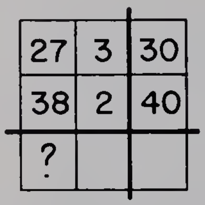


27		
38		
?		

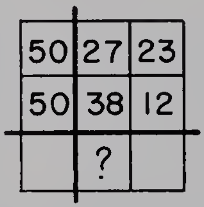


	27	
	38	
	?	

We then fill the cells in the rows containing 27 and 38. Examples:

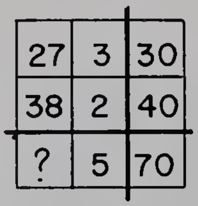


27	3	30
38	2	40
?		

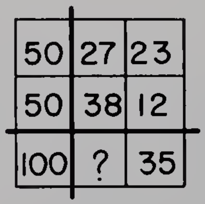


50	27	23
50	38	12
	?	

Then we add column by column:



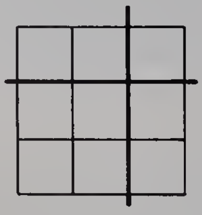
27	3	30
38	2	40
?	5	70

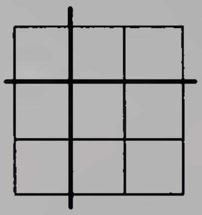


50	27	23
50	38	12
100	?	35

In view of the twist property of addition and the fact that adding a number and subtracting that number are inverses of each other, we know that 70—5 is the sum of 27 and 38 and that so is 100—35.

Now let's see how we might handle subtraction problems. Let's say that the problem is written in vertical format. This time we use positions for the *Cross Number Puzzle* in which the total-box is in the top row:

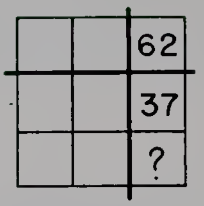




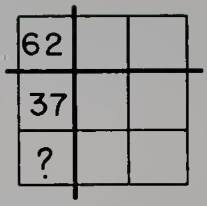
Let's say that our subtraction problem is:

$$\begin{array}{r}62\\-37\\\hline\end{array}$$

If we wish to find the difference by partitioning 62 and 37, we write the problem on the narrow side of the heavy line:

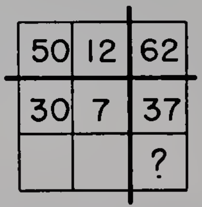


		62
		37
		?

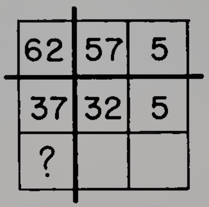


62		
37		
?		

We then partition 62 and 37, writing the parts in the rows containing 62 and 37. Examples:

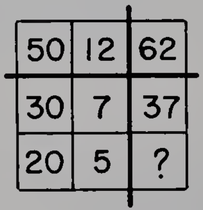


50	12	62
30	7	37
		?

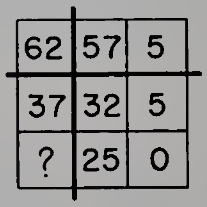


62	57	5
37	32	5
?		

(Of course, the partitioning used for 62 depends upon the partitioning used for 37.) Then we fill the empty cells by subtracting:



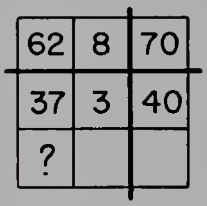
50	12	62
30	7	37
20	5	?



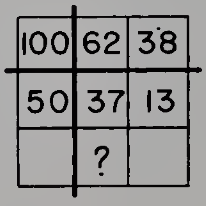
62	57	5
37	32	5
?	25	0

The twist property of addition and the inverse relationship between adding and subtracting a number assure us that 20 + 5 is 62—37 and that so is 0 + 25.

Finally, we may wish to find the difference by expressing 62 and 37 as differences. As in the corresponding case of addition, we write the problem in a column on the wide side of the heavy line and fill the cells in the rows containing 62 and 37:

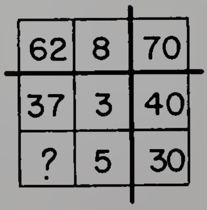


62	8	70
37	3	40
?		

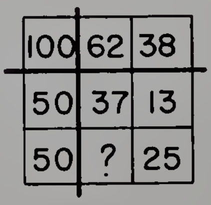


100	62	38
50	37	13
	?	

Then we fill the remaining empty cells by subtracting, column by column:



62	8	70
37	3	40
?	5	30



100	62	38
50	37	13
50	?	25

The properties of addition assure us that 30—5 is 62—37 and that so is 50—25.

Drill

Here are two stories for drill purposes:

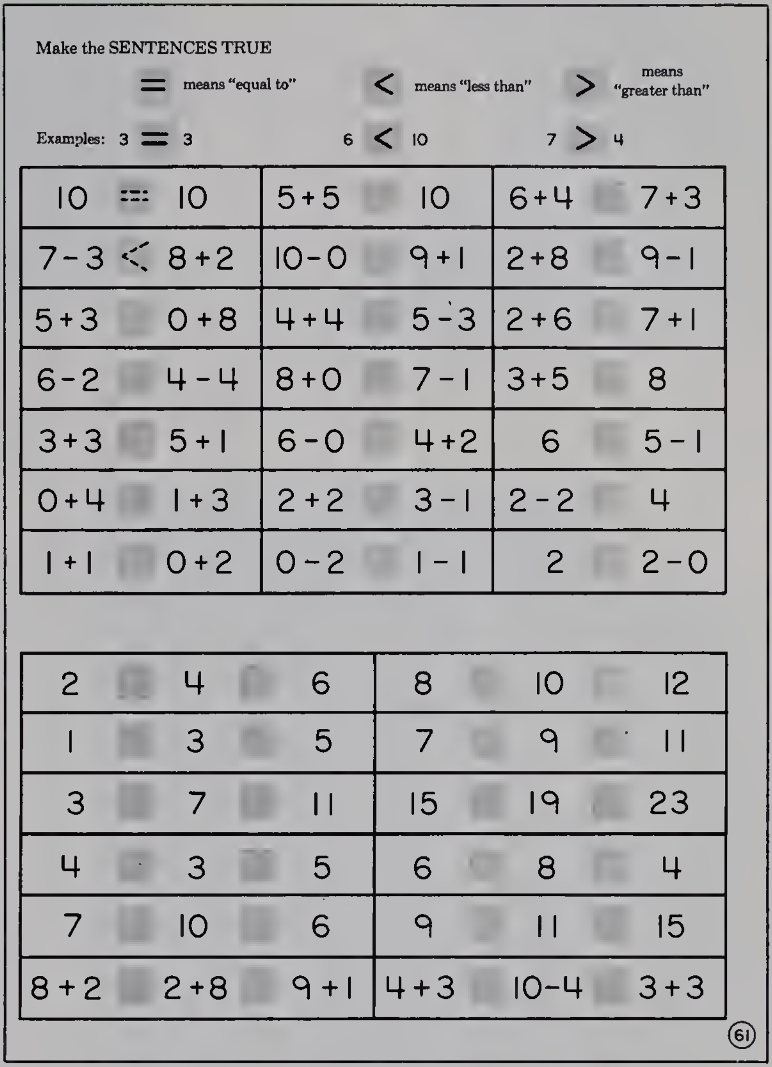
1) There were 700 children in one school. 74 of them were absent on Tuesday.
2) There were 32 boys and 17 girls on the play-ground.

Here is a suggestion for a drill sheet:

Add.

26 +31	62 +37	46 +42	30 +48	54 +13
85 +12	36 +53	56 +22	39 +20	40 +34
41 +28	83 +13	30 +59	26 +71	66 +33

Also consider using as a drill sheet page 61 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Make the SENTENCES TRUE

= means "equal to" < means "less than" > means "greater than"

Examples: 3 = 3 6 < 10 7 > 4

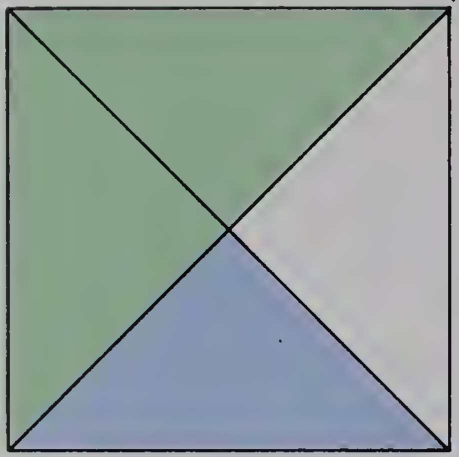
10	=	10	5+5		10	6+4		7+3
7-3	<	8+2	10-0		9+1	2+8		9-1
5+3		0+8	4+4		5-3	2+6		7+1
6-2		4-4	8+0		7-1	3+5		8
3+3		5+1	6-0		4+2	6		5-1
0+4		1+3	2+2		3-1	2-2		4
1+1		0+2	0-2		1-1	2		2-0

2		4		6	8		10		12
1		3		5	7		9		11
3		7		11	15		19		23
4		3		5	6		8		4
7		10		6	9		11		15
8+2		2+8		9+1	4+3		10-4		3+3

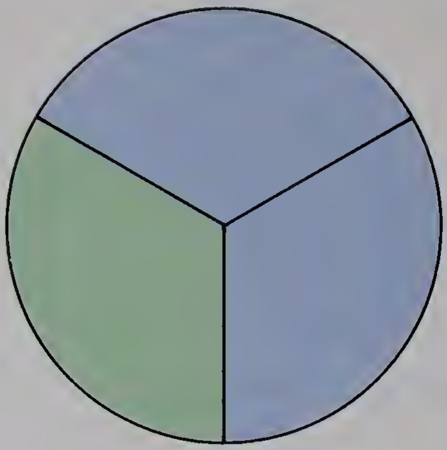
(61)

I.

(a)



(b)



	Part shaded green	Part shaded blue
(a)	$\frac{2}{4}$ or $\frac{1}{2}$	$\frac{1}{4}$
(b)	$\frac{1}{3}$	$\frac{2}{3}$

II. Add

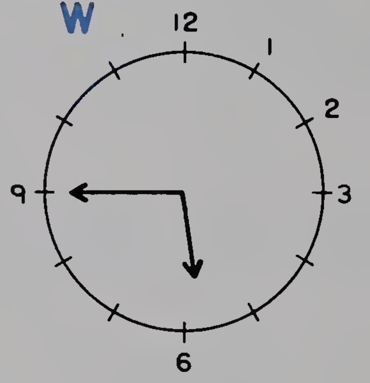
+	one ten	six ones	16
4	14	10	20
10	20	16	26
30	40	36	46
25	35	31	41
19	29	25	35

III. Subtract

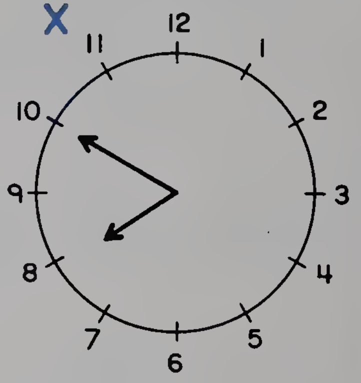
—	one dime	one nickel	15¢
25¢	15¢	20¢	10¢
50¢	40¢	45¢	35¢
29¢	19¢	24¢	14¢
43¢	33¢	38¢	28¢
31¢	21¢	26¢	16¢

IV.

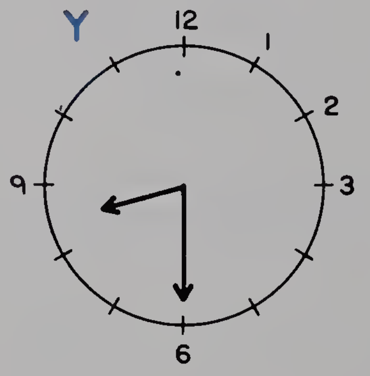
W



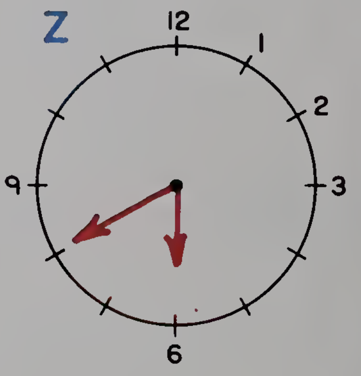
X



Y



Z



Draw the hands in clock Z so that each of the sentences below can be made into a true statement.

Clock W shows 5:45 o'clock.

Clock X shows fifty minutes after seven o'clock.

Clock Y shows half past eight.

Clock Z shows twenty minutes before six o'clock.

Clock W shows a quarter of six.

Notes on Pupil Page 79

Tasks: *Pupils record in the table for item I the fraction of each figure shaded green and the fraction shaded blue. For items II and III they compute and record sums and differences. For item IV they draw the hands in Clock Z so that the sentences can be completed correctly.*

Purpose: To test pupil achievement.

Unifying Ideas: Numeration; addition and subtraction; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Suggested point distribution for test:

I–2 points	III–5 points
II–5 points	IV–5 points

Pupils may be put to work without preparation for items I-III. You may have to read to them the information in item IV.

Drill

Here is a suggestion for a drill sheet.

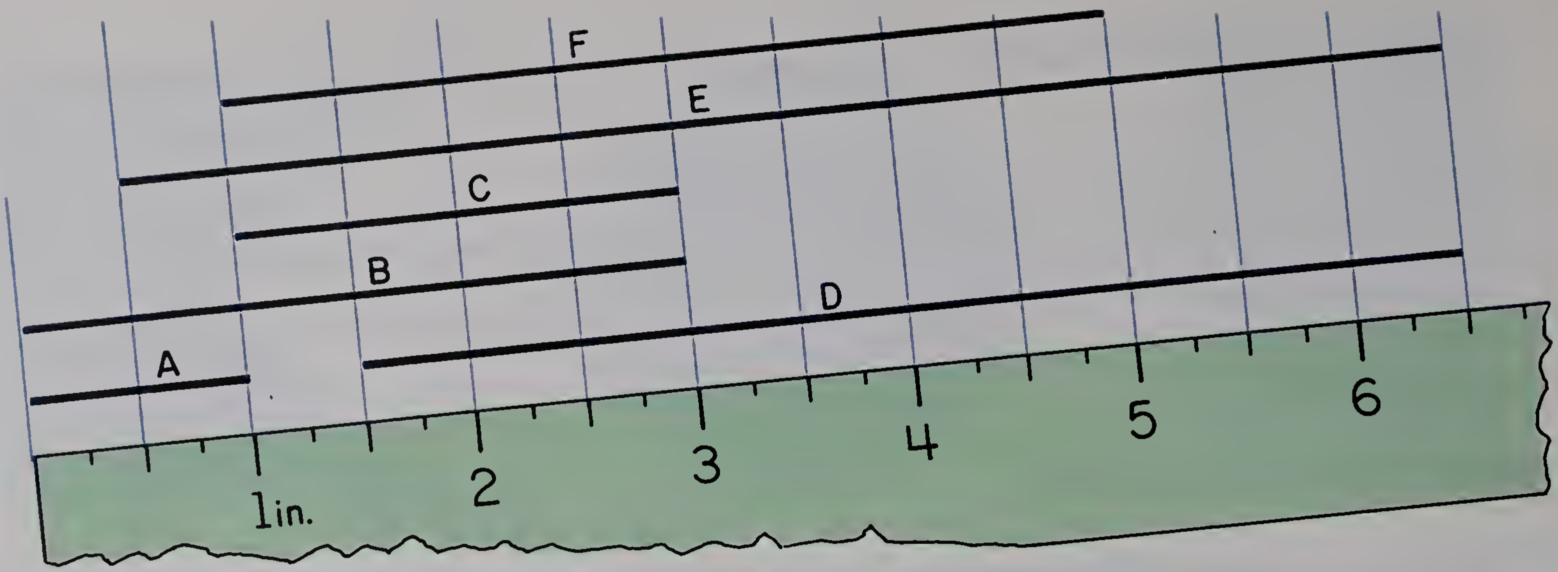
Add.

31	26	44	32	15
42	41	30	42	60
15	20	25	22	12
—	—	—	—	—
16	21	24	10	45
33	13	31	37	2
30	4	42	30	50
—	—	—	—	—
2	21	31	10	12
71	16	21	20	31
5	12	11	30	14
10	10	1	40	41
—	—	—	—	—

Here are two stories for drill purposes:

1) Arnold was looking at 5 cars. Each car had 4 tires and 1 spare tire.
2) 9 children went fishing on the lake. 3 children sat in each boat.

Headline for the Day: Saturdays fall on ___, ___, ___, and ___ this month. So Wednesdays fall on ___, ___, ___, and ___.



Lengths of lines:

A: 1 in. B: 3 in. C: 2 in.
D: 5 in. E: 6 in. F: 4 in.

Sums of lengths of lines:

A and D: 6 in. B and C: 5 in.
E and F: 10 in. D and F: 9 in.
A and F: 5 in. B and E: 9 in.

Differences between lengths of lines:

B and A: 2 in. F and C: 2 in.
E and C: 4 in. D and F: 1 in.
E and D: 1 in. E and F: 2 in.

A game called WATCH THE SIGNS!

I
12 - 9 = 3
+ + +
30 - 10 = 20
= = =
42 - 19 = 23

II
25 - 15 = 10
- - -
10 - 8 = 2
= = =
15 - 7 = 8

III
30 - 4 = 26
- + -
7 + 0 = 7
= = =
23 - 4 = 19

IV
10 + 16 = 26
- - -
10 + 8 = 18
= = =
0 + 8 = 8

V
14 + 6 = 20
+ - +
16 - 4 = 12
= = =
30 + 2 = 32

VI
36 - 8 = 28
- - -
15 - 5 = 10
= = =
21 - 3 = 18

Notes on Pupil Page 80

Tasks: *Pupils record the lengths of the line segments displayed above the ruler. They also determine the sums and differences of specified lengths. Finally, they record sums and differences in the* Watch the Signs! *exercises.*

Purpose: (a) To review and extend pupils' understanding of linear measure; (b) to provide pupils with practice in addition and subtraction.

Unifying Ideas: Addition and subtraction; measurement.

Materials for Teacher: Demonstration ruler; Ditto masters.

Materials for Pupils: Strips of paper each one foot long with a 1-inch mark drawn one inch from an end; Dittoed sheets.

The Lesson:

Let pupils use the paper strips to make rulers. By folding on the 1-inch mark, one can make a 2-inch mark. Fold on the 2-inch mark to make a 4-inch mark; fold on the 4-inch mark to make an 8-inch mark. Fold from the other end of the ruler to the 4-inch mark to check the accuracy of the placement of the 8-inch mark.

Let pupils experiment to find the positions for the 3-, 5-, 6-, 7-, 9-, 10-, and 11-inch marks. Then have them find the ½-inch mark, the 1½-inch mark, etc.

Distribute a Dittoed sheet showing segments labeled as on pupil page 80. Have pupils measure these segments in the ways shown on pupil page 80. We know that segment C is 2 inches long. If we put our ruler down along the segment so that one end of the segment is at the 3-inch mark, where will the other end be? (1-inch mark or 5-inch mark.) Etc.

Telling Stories and Fitting Headlines

Read brief stories such as the following. Pupils tell you (or write) mathematical sentences that fit. (Perhaps some of the stories might use the number sentences in the *Watch the Signs!* exercises on pupil page 80.)

I had 6 balloons. 4 of them got loose and floated away. There were only 2 left. ($6-4=2$)

It's the 10th of March. I have to wait 4 days until it's my birthday. ($10+4=14$)

I spent 16¢. I spent 6¢ for an ice cream cone and 10¢ for carfare. ($16=6+10$)

I had a quarter and spent some money. Now I've got 23¢; so I must have spent 2¢. ($25-2=23$)

My father is 30 years old and my mother is 3 years younger. She is 27 years old. ($27=30-3$)

Some responses might be more involved:

I had a dozen green cookies and 3 more blue ones than green ones. There were 9 of us; so we each had 3 cookies. ($12+15=27$)

24 astronauts were ready to go to the moon. Only 21 left, because 3 overslept. ($24-3=21$)

I traded a half-dollar for two quarters, and still had the same amount of money. ($50=25+25$)

One dozen more than three dozen is 2 less than 50. ($12+36=48$)

We've been gone exactly one day. It's now 3 o'clock in the afternoon. We are due to arrive tomorrow afternoon at 5 o'clock. If we arrive on time, we will have been gone 50 hours. ($24+26=50$)

Drill

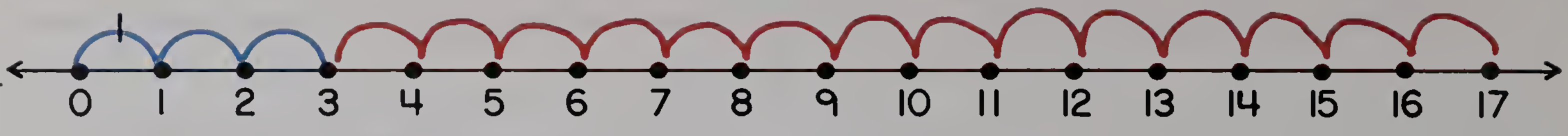
Here is a suggested drill sheet:

$11-3-2-1=\square$
$13-1-3-2=\square$
$15-2-4-3=\square$
$12-2+1-4=\square$
$18-3+3-5=\square$
$19-5+5-1=\square$
$23-9+9+1=\square$

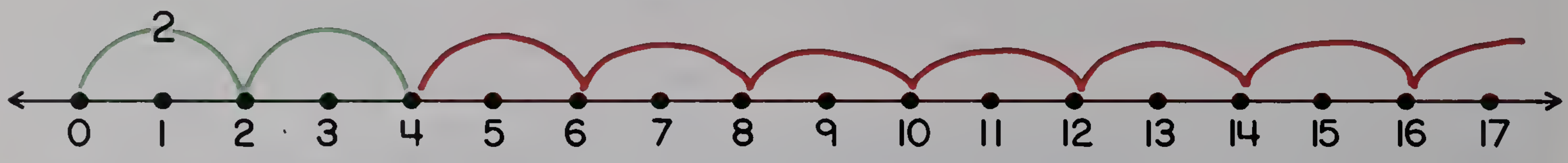
(See "Order of Operations" in the Notes on Pupil Page 5.)

Headline for the Day: Compare 2 feet with 1 yard, using < or > in the number sentence.

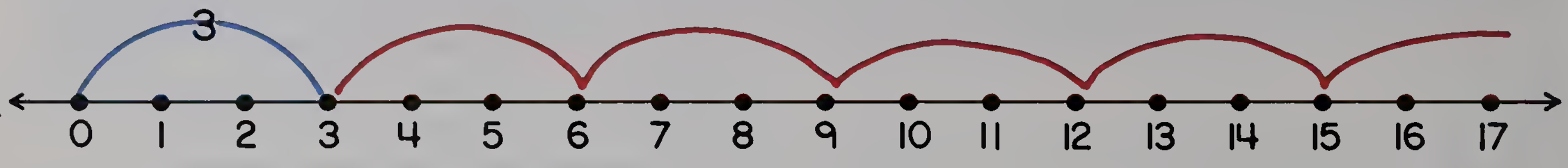
A.


1
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17

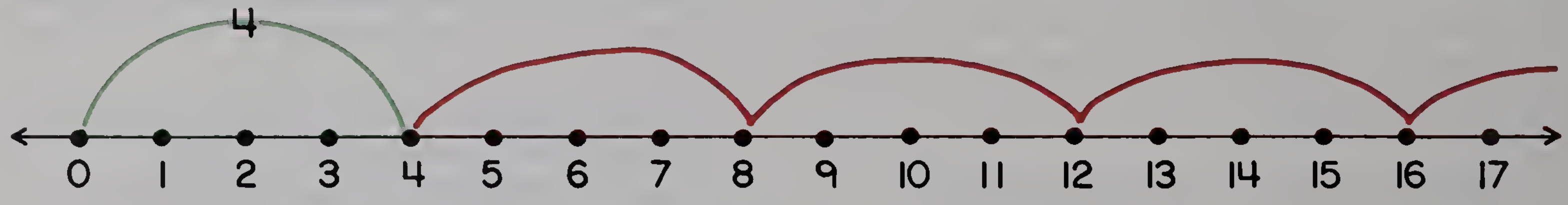
B.


2
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17

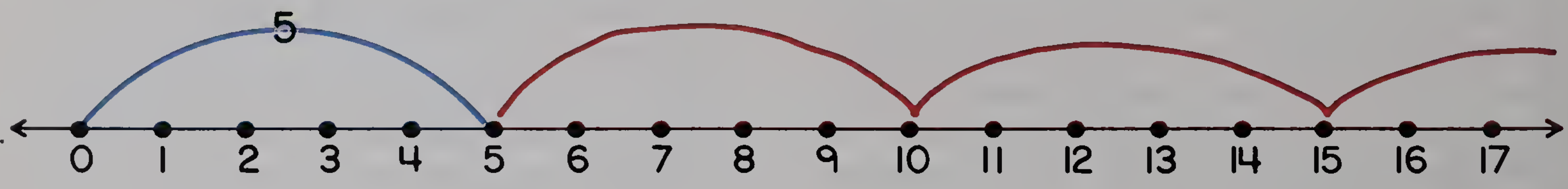
C.


3
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17

D.


4
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17

E.


5
0
1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17

	1 st	2 nd	3 rd	4 th	5 th	6 th	7 th	8 th	9 th	10 th
A. ↷ 1	1	2	3	4	5	6	7	8	9	10
B. ↷ 2	2	4	6	8	10	12	14	16	18	20
C. ↷ 3	3	6	9	12	15	18	21	24	27	30
D. ↷ 4	4	8	12	16	20	24	28	32	36	40
E. ↷ 5	5	10	15	20	25	30	35	40	45	50

Notes on Pupil Page 81

Tasks: *Pupils continue the pattern of jumps specified on each number line and record the landing points in the chart at the bottom of pupil page 81.*

Purpose: To anticipate the approach to multiplication as repeated addition.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

When pupils have completed pupil page 81, ask questions such as the following (all questions assume that the jumping starts from 0 and proceeds to the right):

1) If you make jumps of 1, how many jumps does it take to get to 8?
2) If you make jumps of 2, how many jumps does it take to get to 8?
3) If you make 4 jumps of 3, where do you finish?
4) If you make 3 jumps of 4, where do you finish?
5) Somebody made 5 jumps of 20 and finished at 100. Where would you finish if you made 20 jumps of 5?

Drill

Here are suggested exercises for a drill sheet:

Subtract.

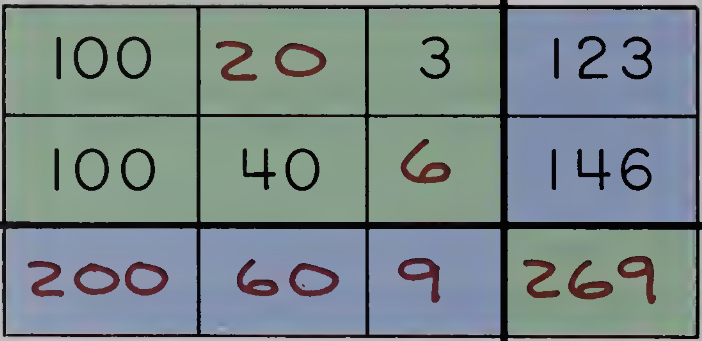
84	75	58	37	48
−21	−33	−24	−27	−21
36	59	27	54	63
−21	−34	−27	−18	−24
42	53	84	17	83
−27	−35	−26	−11	−33

Here are two stories for drill purposes:

1) Olive and Branch worked in the garden. Each child filled 4 baskets with leaves.
2) Jack has 10 marbles. He won some from Alan and now has 17 marbles.

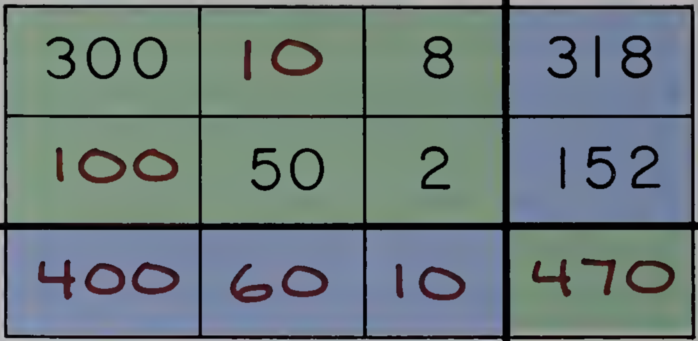
Headline for the Day: Bob's dad is three times as old as Bob. If Bob is ___ years old, his dad is . . . years old. Make a chart.

A



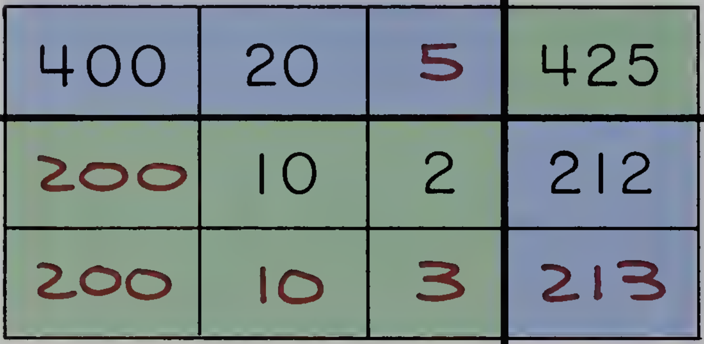
100	20	3	123
100	40	6	146
200	60	9	269

B



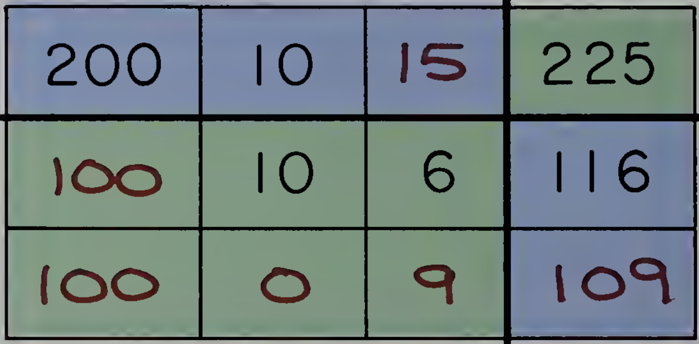
300	10	8	318
100	50	2	152
400	60	10	470

C



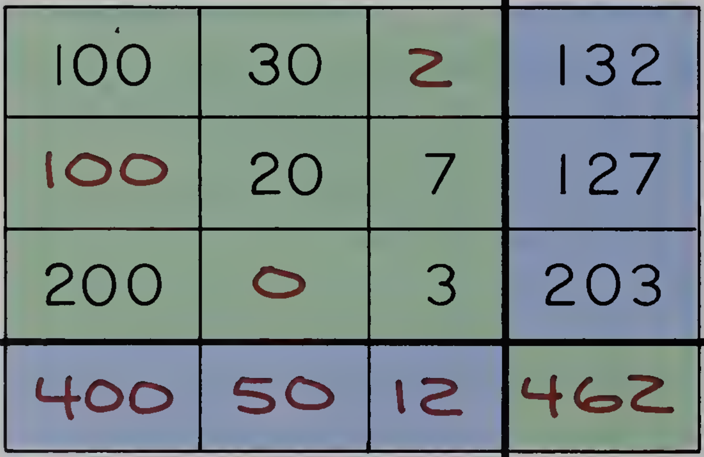
400	20	5	425
200	10	2	212
200	10	3	213

D



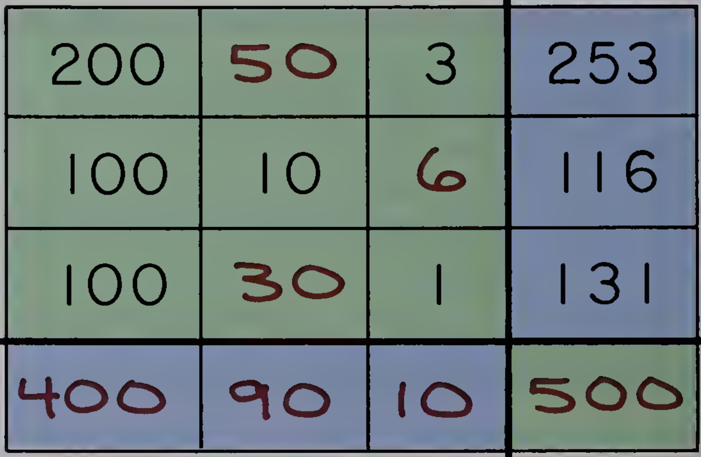
200	10	15	225
100	10	6	116
100	0	9	109

E



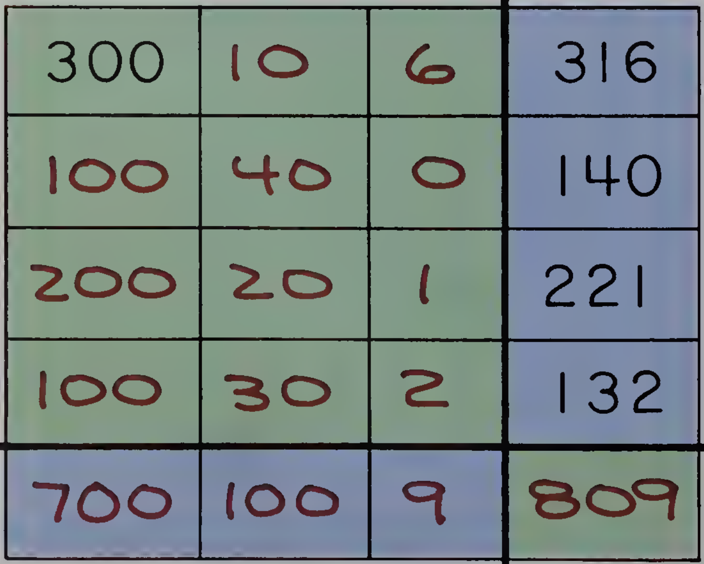
100	30	2	132
100	20	7	127
200	0	3	203
400	50	12	462

F



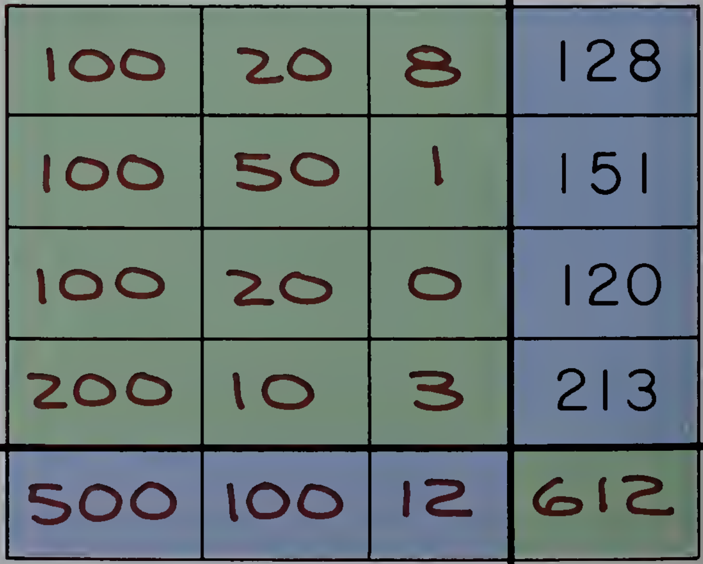
200	50	3	253
100	10	6	116
100	30	1	131
400	90	10	500

*G



300	10	6	316
100	40	0	140
200	20	1	221
100	30	2	132
700	100	9	809

*H



100	20	8	128
100	50	1	151
100	20	0	120
200	10	3	213
500	100	12	612

* MANY CORRECT ANSWERS POSSIBLE.

Notes on Pupil Page 82

Tasks: Pupils reconstruct each Cross Number Puzzle.

Purpose: To help pupils perfect their skills in adding and subtracting 3-digit numbers.

Unifying Ideas: Numeration; addition and subtraction.

Materials for Teacher: Ditto masters.

Materials for Pupils: Dittoed sheets.

The Lesson:

Here we are asking pupils to use *Cross Number Puzzles* to help them carry out fairly extensive problems in addition and subtraction. To make sure that they see the purpose of this work, distribute a Dittoed sheet with the following problems:

$$\begin{array}{llll} \text{A.} & \text{B.} & \text{C.} & \text{D.} \\ \begin{array}{r} 123 \\ +146 \\ \hline \end{array} & \begin{array}{r} 318 \\ +152 \\ \hline \end{array} & \begin{array}{r} 425 \\ -212 \\ \hline \end{array} & \begin{array}{r} 225 \\ -116 \\ \hline \end{array} \end{array}$$

$$\begin{array}{llll} \text{E.} & \text{F.} & \text{G.} & \text{H.} \\ \begin{array}{r} 132 \\ 127 \\ +203 \\ \hline \end{array} & \begin{array}{r} 253 \\ 116 \\ +131 \\ \hline \end{array} & \begin{array}{r} 316 \\ 140 \\ 221 \\ +132 \\ \hline \end{array} & \begin{array}{r} 128 \\ 151 \\ 120 \\ +213 \\ \hline \end{array} \end{array}$$

These are, of course, the eight problems which are to be solved on pupil page 82. Tell pupils to record their answers on the Dittoed sheets as they work each problem on pupil page 82.

Drill

Here is a drill sheet you may wish to use at this point:

Watch The Signs!

$$\begin{array}{ccccc} \begin{array}{r} 77 \\ +12 \\ \hline \end{array} & \begin{array}{r} 58 \\ -27 \\ \hline \end{array} & \begin{array}{r} 49 \\ -23 \\ \hline \end{array} & \begin{array}{r} 80 \\ +17 \\ \hline \end{array} & \begin{array}{r} 49 \\ -22 \\ \hline \end{array} \\ \begin{array}{r} 66 \\ -25 \\ \hline \end{array} & \begin{array}{r} 74 \\ -43 \\ \hline \end{array} & \begin{array}{r} 95 \\ -41 \\ \hline \end{array} & \begin{array}{r} 30 \\ +29 \\ \hline \end{array} & \begin{array}{r} 52 \\ -30 \\ \hline \end{array} \\ \begin{array}{r} 86 \\ -45 \\ \hline \end{array} & \begin{array}{r} 53 \\ +21 \\ \hline \end{array} & \begin{array}{r} 82 \\ -50 \\ \hline \end{array} & \begin{array}{r} 21 \\ -21 \\ \hline \end{array} & \begin{array}{r} 41 \\ +41 \\ \hline \end{array} \end{array}$$

Here are two stories for drill purposes:

1) Lola saved 10 dollars. She spent 8 dollars.
2) Mrs. Sanchez made 2 dresses for Connie. She put 9 buttons on each dress.

Headline for the Day: Write stories that fit some of the number sentences on page 82.

A

300	30	7	337
100	0	4	104
200	30	3	233

B

200	0	13	213
100	0	5	105
100	0	8	108

C

500	20	10	530
100	10	4	114
400	10	6	416

D

400	60	12	472
100	10	6	116
300	50	6	356

E

600	130	5	735
100	40	0	140
500	90	5	595

F

500	90	13	603
200	0	7	207
300	90	6	396

G

400	100	17	517
300	80	8	388
100	20	9	129

H

300	120	12	432
100	90	4	194
200	30	8	238

I What's My Rule?

x	20	17	27	19	25	37	50	39	34	46	100
x − 17	3	0	10	2	8	20	33	22	17	29	83

Notes on Pupil Page 83

Tasks: Pupils reconstruct each Cross Number Puzzle *and complete the* What's My Rule? *table.*

Purpose: (a) To help pupils perfect their skills in subtracting 3-digit numbers; (b) to provide pupils with practice in adding 17 and subtracting 17.

Unifying Ideas: Numeration; addition and subtraction; functions and relations.

Materials for Teacher: Ditto masters.

Materials for Pupils: Dittoed sheets.

The Lesson:

This page is a continuation of pupil page 82. Distribute a Dittoed sheet with the following problems:

A. 337 −104

B. 213 −105

C. 530 −114

D. 472 −116

E. 735 −140

F. 603 −207

G. 517 −388

H. 432 −194

New Problems

I. 426 −239

J. 618 −427

K. 300 −299

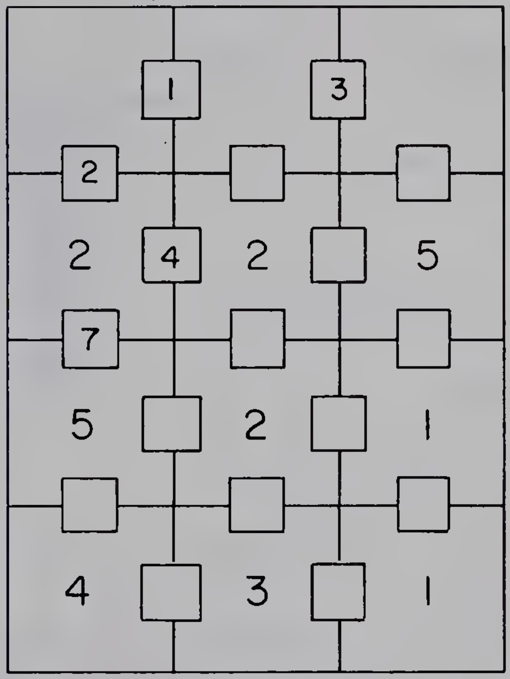
L. 724 −386

Notice that problems A-H are precisely those on pupil page 83. Tell pupils to record their answers on the Dittoed sheet after they have reconstructed the *Cross Number Puzzles.* Then challenge them to do problems I-L any way they please.

Drill

Here is a suggested drill sheet:

What's My Rule ?



Here are two stories for drill purposes:

1) Jake bought 13 pieces of candy. He ate some and had 2 pieces left.
2) 13 children were playing. 7 of them went home.

Headline for the Day: Which number doesn't belong in the pattern? 3, 7, 1, 8, 14, 21, 27

Label the largest amount with an A in the blue box; next largest with a B; next largest with a C; etc. Write the differences in each blank space.

1

Four days	B
Twenty-four hours	C
One week	A

Difference between:

A and B: 3 days

B and C: 3 days

A and C: 6 days

2

Eighteen cents	C
One dollar	A
A half-dollar	B

Difference between:

A and B: 50¢

B and C: 32¢

A and C: 82¢

3

Sixteen inches	C
One foot	D
One foot and seven inches	B
One yard	A

Difference between:

A and B: 17 in.

B and C: 3 in.

A and C: 20 in.

B and D: 7 in.

C and D: 4 in.

A and D: 24 in.

4

One year and two months	C
Half of a year	D
A year and a half	B
Two years	A

Difference between:

A and B: 6 mos.

B and D: 12 mos.

A and D: 18 mos.

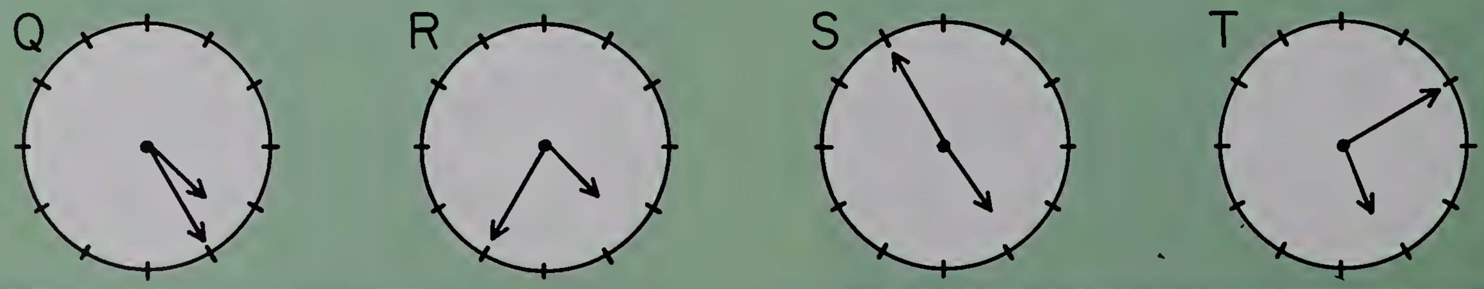
B and C: 4 mos.

C and D: 8 mos.

A and C: 10 mos.

5

Billy was waiting for his favorite TV program to begin at 5:15. He looked at the clock four different times before the program started.



How long was it before the program started in each case?

Q: 50 minutes to wait

R: 40 minutes to wait

S: 20 minutes to wait

T: 5 minutes to wait

How long was it between the times he looked at the clock?

Q and R: 10 minutes

R and S: 20 minutes

S and T: 15 minutes

Q and T: 45 minutes

Notes on Pupil Page 84

Tasks: *In sections 1-4 pupils record the required information in accordance with directions given at the top of the page. In section 5 pupils record the various time measurements.*

Purpose: To provide practice in working with measurements and in reading word names for measurements.

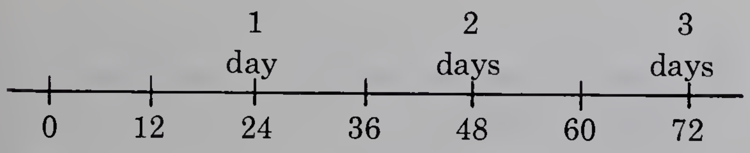
Unifying Ideas: Addition and subtraction; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Prepare pupils for pupil page 84 by using number lines as scales for various quantities. For example, here is an hour number line:



Let pupils help you mark more points on it.

Make a scale for pennies, dimes, and dollars; for inches, feet, and yards; for months and years.

When pupils become acquainted with these scales, they can easily handle the ranking problems posed in sections 1-4.

Notice in each of the sections that, since B is between A and C (points on a number line),

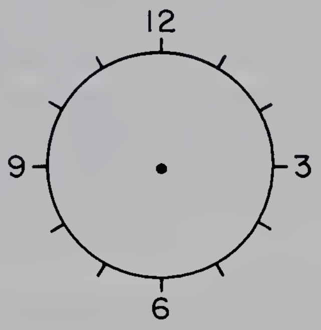
$$(A-B)+(B-C)=A-C.$$

Also, since C is between B and D,

$$(B-C)+(C-D)=B-D,$$
$$(A-B)+(B-D)=A-D,$$
$$(A-B)+(B-C)+(C-D)=A-D.$$

The Clock Game

Draw a clockface on the board, labeled as below:



Where would the big and little hands be at one o'clock? Have a volunteer draw the hands. Where

more

Headline for the Day: Suppose that I travel 10 miles every 20 minutes. Set up a table to show how far I will travel in 20 minutes, 40 minutes, 60 minutes, etc. How can I use the table to tell how far I would go in 30 minutes, 50 minutes?

would they be an hour later? Another volunteer erases the previously drawn hands and redraws them in the new position.

Continue the exercise asking: Where would the hands be two hours later? Where would they be half an hour later? Where would they be another half-hour later? Where would they be three quarters of an hour later? A quarter of an hour later? Half an hour later? Three quarters of an hour later? Three quarters of an hour later? One quarter of an hour later? Two quarters of an hour later? Four quarters of an hour later?

Then change the form of your questions to: 30 minutes later? 30 minutes later? 15 minutes later? 45 minutes later? Etc.

To write down the time appearing on a clock, we follow these rules: on the left of the two dots (:) we write the number reached by the small hand (the hour hand); on the right of the two dots we write the number of minutes past the hour reached by the large hand (the minute hand). Now, let's keep score as we draw the hands of the clock. We will start with one o'clock:

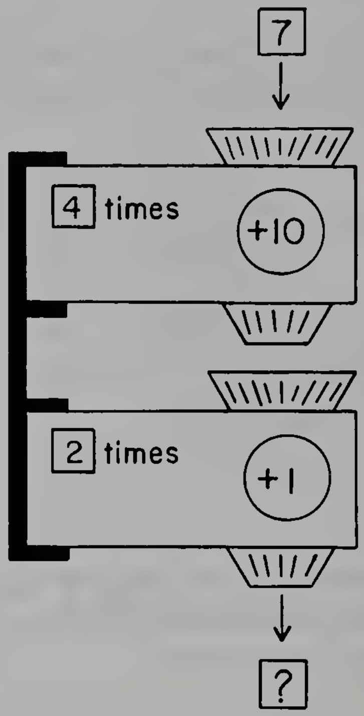
one o'clock	1:00
an hour later	2:00
30 minutes later	2:30
half an hour later	3:00
a quarter of an hour later	3:15
15 minutes later	3:30
30 minutes later	4:00
three quarters of an hour later	4:45
1 hour and three quarters of an hour later	6:30
1 hour and three quarters of an hour later	8:15
1 hour and 45 minutes later	10:00

Flash Cards

You may wish to prepare pupils for the work on multiplication and division which begins on pupil page 85 and continues through pupil page 105. We suggest that you have the pupils work with the rectangular array flash cards described in the Notes on Pupil Pages 12 and 22.

Adding Machines

An interesting class activity is to use a hooked-up pair of imaginary machines for adding 10 and adding 1. Billy sets the "times dial" on the adding-10 machine and Susan sets the times dial on the adding-1 machine. The machines are now ready to add 42:



Fred picks the starting number, 7. Charles describes the sequence of steps:

7, 17, 27, 37, 47, 48, 49

Amy writes the long number sentence:

$$7+10+10+10+10+1+1=49$$

Neal writes the short number sentence:

$$7+40+2=49$$

Michael writes the very short number sentence:

$$7+42=49$$

This activity can be varied by interchanging the positions of the two machines, by putting a 0 on a times dial, and by using subtracting machines.

Some questions that arise during this activity are:

What times dial settings should you make if you wish to add 62? 26? 33?

Suppose that the times dials can't go higher than 9. How would you add 100 to a given number? (First set both dials at 9 and let the machines work on the given number. Then reset the adding-10 machine dial at 0 and the adding-1 machine dial at 1, and let the machines work on the previous output. See the Notes on Pupil Page 118 for more on activities of this type.

Drill

Here are suggested exercises for a drill sheet:

Subtract.

65	38	47	29	87
−20	−15	−30	−17	−20
56	38	55	42	42
−43	−32	− 5	−21	−24
74	61	28	51	63
− 4	−50	−10	−21	−23

You may wish to combine the machine work with these drill problems. You would use a subtracting-10 machine onto which is hooked a subtracting-1 machine. Then to do the problem 56—43= ?, set the times dial on the subtracting-10 machine at 4 and the times dial on the subtracting-1 machine at 3, and drop 56 into the subtracting-10 machine. Here is the sequence of steps:

56, 46, 36, 26, 16, 15, 14, 13

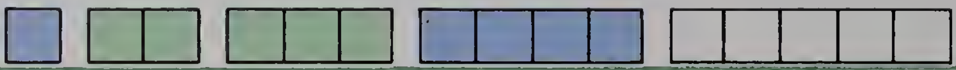
Here are suggestions for another drill sheet:

86 + 6 = 80 + □ = 90 + △ = ○
59 + 2 = 50 + □ = 60 + △ = ○
35 + 7 = 30 + □ = 40 + △ = ○
24 + 9 = 20 + □ = 30 + △ = ○
77 + 6 = □ + 13 = △ + 3 = ○
84 + 8 = □ + 12 = △ + 2 = ○
79 + 9 = □ + 18 = △ + ▽ = ○

Here are some stories for drill purposes:

1) Father had balloons for the boys and girls. He gave out 4 and mother gave out 8.
2) Bonnie had 7 toy chairs. Her mother gave her 5 more.
3) There were 3 rows of desks. There were 4 desks in each row.
4) Gene had 7 dollars in his bank. He took 3 dollars out to buy his friend a birthday present.

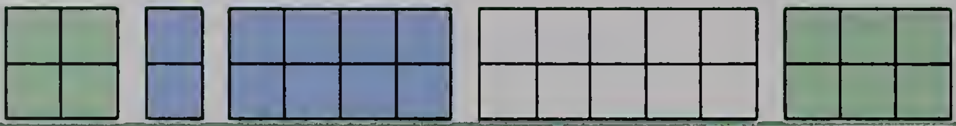
I



5 Green 5 White

5 Blue 15 Altogether

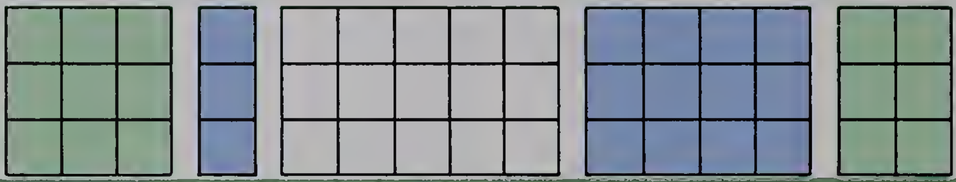
II



10 Green 10 White

10 Blue 30 Altogether

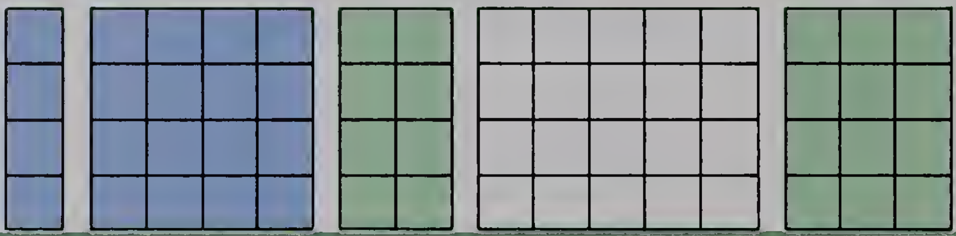
III



15 Green 15 White

15 Blue 45 Altogether

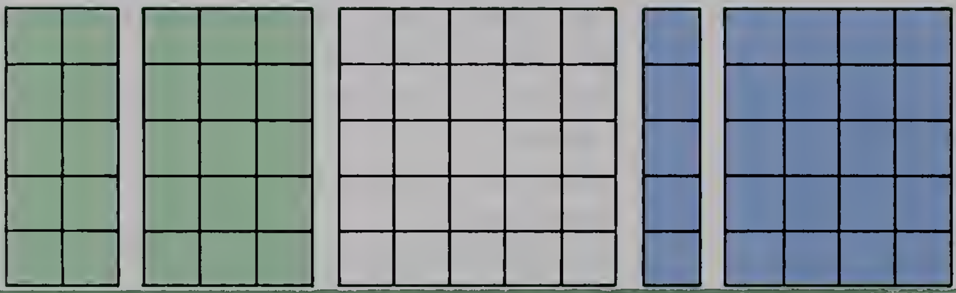
IV



20 Green 20 White

20 Blue 60 Altogether

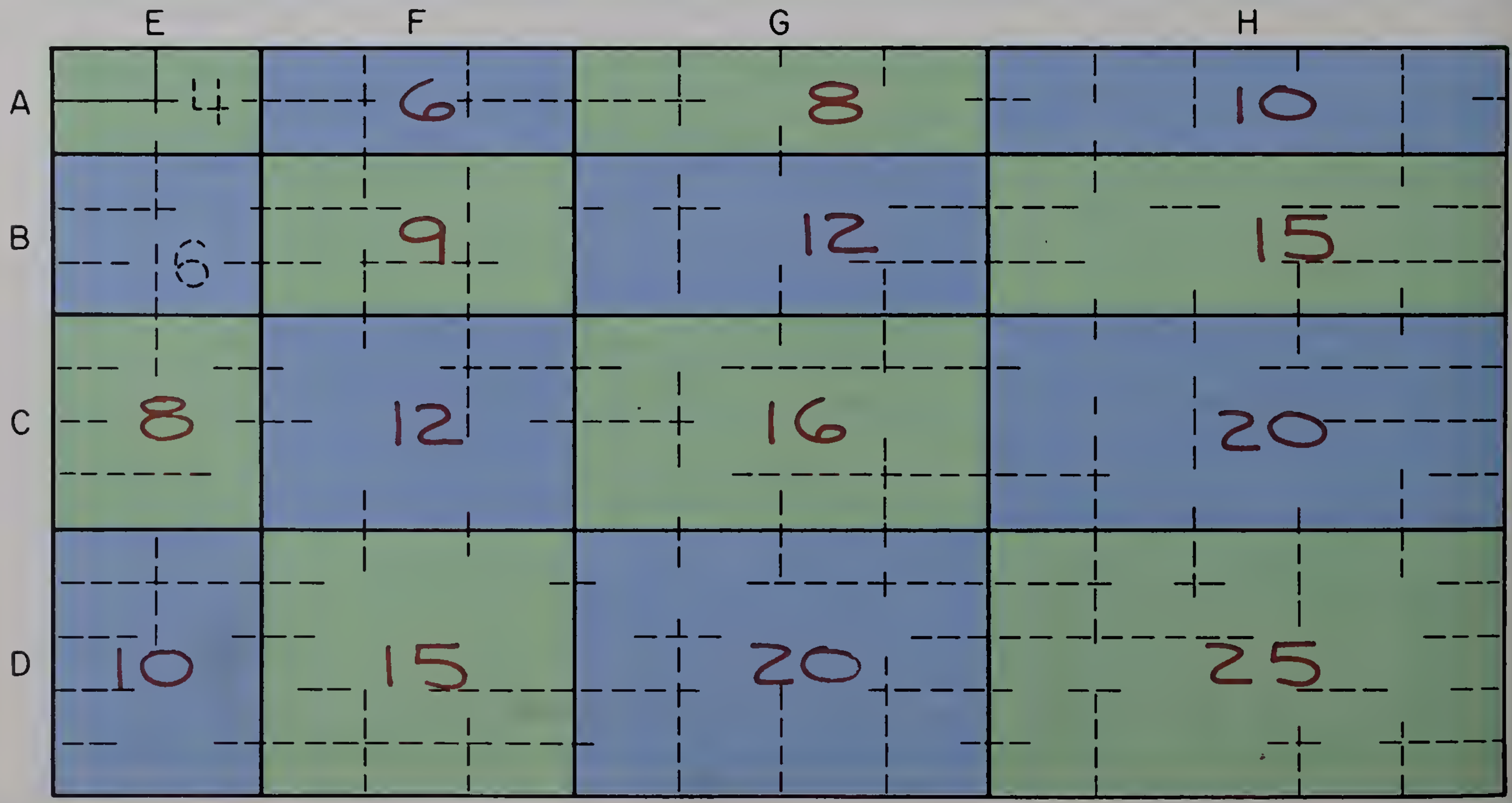
V



25 Green 25 White

25 Blue 75 Altogether

Write the number of bricks this size ▭ that would fit in each of the tinted boxes.



	E	F	G	H
A	4	6	8	10
B	6	9	12	15
C	8	12	16	20
D	10	15	20	25

Notes on Pupil Page 85

Tasks: *In exercises I-V pupils determine and record the total number of little squares and the number of little green, blue, and white squares. In the exercises at the bottom of the page pupils record the number of bricks required to fit each of the tinted boxes.*

Purpose: To give pupils practice in determining the counts for rectangular arrays in anticipation of multiplication and division.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division.

Materials for Teacher: Graph paper cards as described below.

Materials for Pupils: Cuisenaire rods.

The Lesson:

In exercises II-V each display consists of the preceding display augmented by the single strips of exercise I. For example, the number of little green squares in II is twice that in I; the number of little green squares in III is three times that in I; etc. The rectangles in exercise I represent the numbers 1, 2, 3, 4, and 5; those in II represent the doubles of these numbers; those in III, the triples, etc.

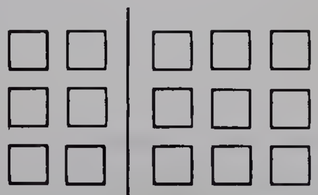
One may also see *the distributive principle for multiplication over addition* at work in these exercises. For example, in II the number of little blue squares is $(1 \times 2) + (4 \times 2)$, and this is the same as $(1 + 4) \times 2$.

The exercises at the bottom of the page present a physical representation of a portion of the multiplication table:

×	2	3	4	5
2	4	6	8	10
3	6	9	12	15
4	8	12	16	20
5	10	15	20	25

more

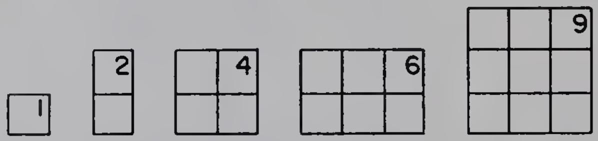
Headline for the Day: Write at least four number sentences which describe this picture:



Pupil page 85 can be assigned without any special preparation. You may wish to follow it with some of the Cuisenaire rod activities described in pages 62-65 of the booklet *Games and Enrichment Activities.*

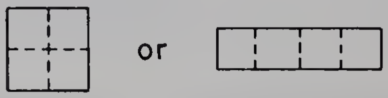
A Guessing Game

Prepare a set of cards from sheets of graph paper (1″ by 1″ squares), the cards to look like the 25 rectangles in I-V on pupil page 85. Each card has a graph paper side and a blank side. Now count the little squares on each card and record the results in the upper right-hand square:



By turning the cards face down, the stage is set for a guessing game. How many little squares? Turn the card up to check results. Then the game shifts to: I'm thinking of one of the cards. It has 4 little squares on it. Can you point to it?

This leads to a problem with a pair of rectangles, both of which contain 4 little squares.

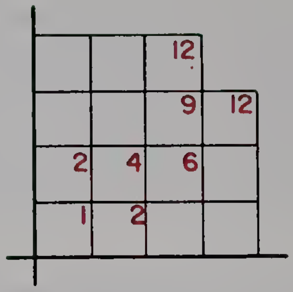


Attention is then focused on the lengths of the sides. And now the game shifts to guessing which one by talking about the edges.

A record can be made to summarize the results of problems dealing with rectangles that have the same area but different perimeters.

No. of little squares on an edge		No. of little squares
along the top	along the side	
1	1	1
1	2	2
2	2	4

Don't throw away the rectangular cards. Stack them, with the smaller cards on top of the larger cards, as shown at the top of the next column.



Now it is possible to answer the question whether or not all the possible shapes smaller than the largest are present. Where there is a little square showing without a number in it, a card is missing and ought to be included, because as the structure grows it becomes a multiplication table.

Reading a Grid

After agreeing that N(E,B) means, for the lower section of pupil page 85, the number of bricks both in column E *and* row B (6) and that N(F, BC) means the number of bricks both in column F *and* in either row B or row C (21), have pupils complete these sentences:

N(E,B) = 6

N(G,C) = 16

N(H,AB) = __

N(EF,B) = __

N(,) = 10

N(,) = 45

etc.

Drill

Here are two worksheets for maintaining skill in addition:

Add.

35	57	68	73	41
+ 5	+ 4	+ 5	+ 8	+ 7
52	38	27	38	56
+ 9	+ 6	+ 9	+ 2	+ 6
37	52	17	6	18
+ 0	+ 2	+ 5	+15	+ 3

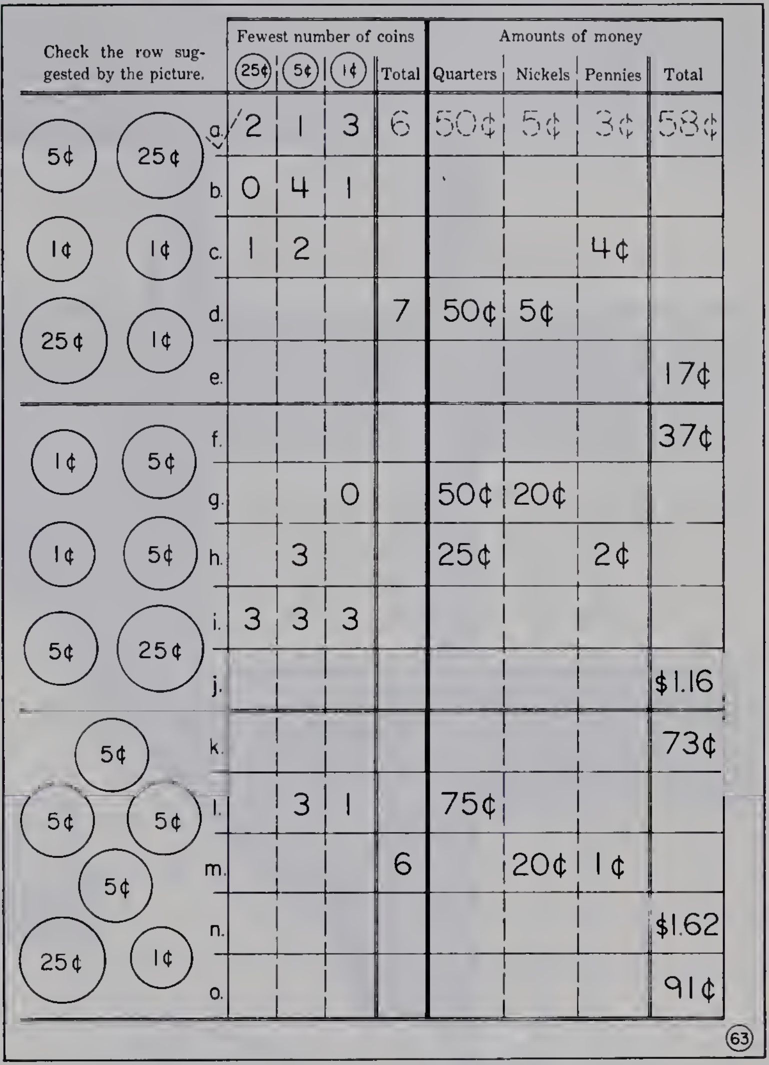
Add.

7 +5	6 +6	4 +7	3 +4	9 +9
5 +7	6 +5	7 +7	9 +4	8 +3
9 +2	7 +4	8 +8	5 +3	4 +8
4 +5	2 +9	5 +5	8 +5	5 +8
9 +0	7 +6	5 +6	4 +4	9 +3

Here are two stories for drill purposes:

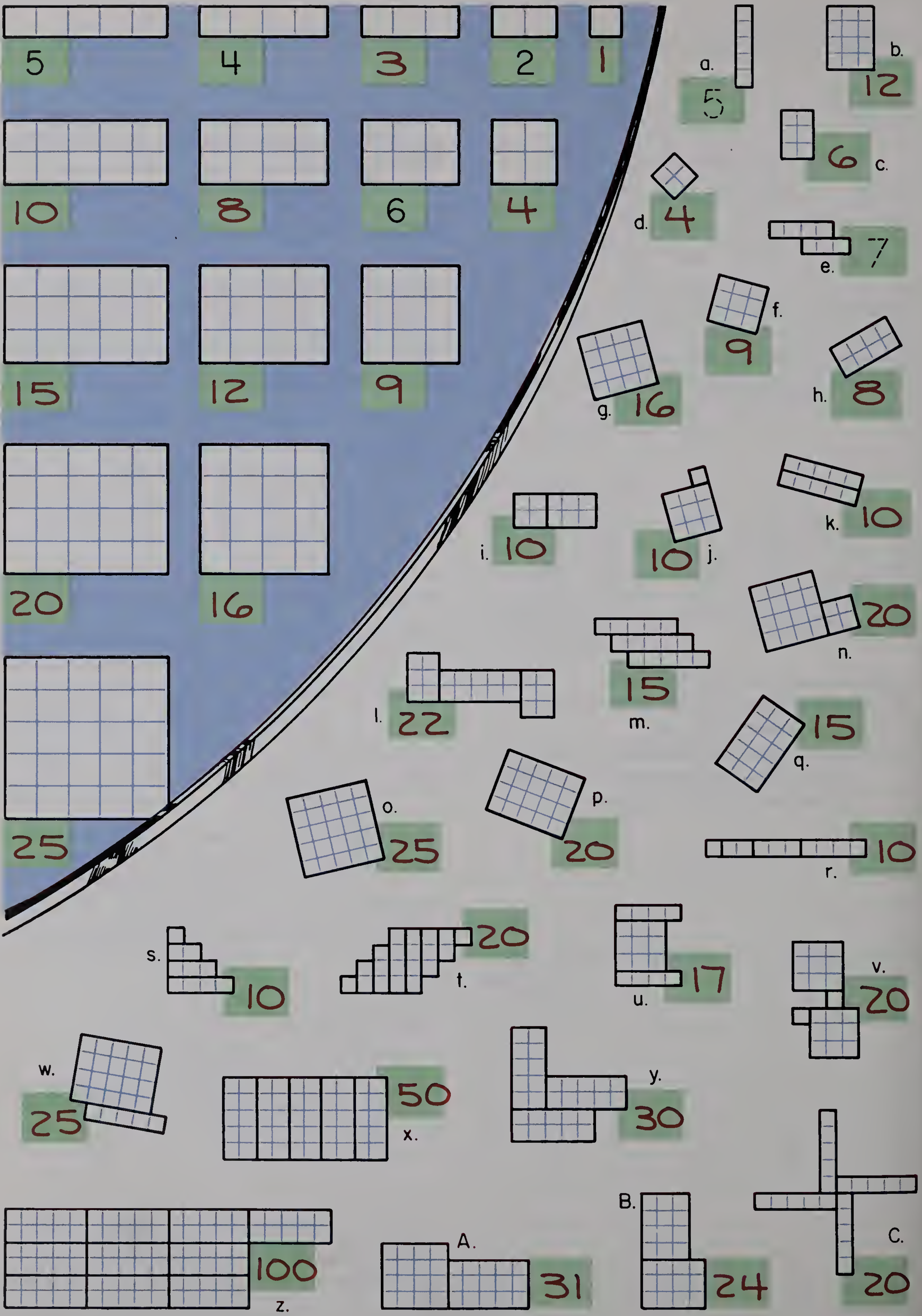
1) Yesterday Simon sold 31 glasses of lemonade. Today he sold only 5 glasses.
2) Sam and Clara sold lemonade on a hot summer day. They made 2 gallons of lemonade. They planned to sell it by the cup.

Also consider using as a drill sheet page 63 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Check the row suggested by the picture.		Fewest number of coins				Amounts of money			
		25¢	5¢	1¢	Total	Quarters	Nickels	Pennies	Total
5¢ 25¢	a. ✓	2	1	3	6	50¢	5¢	3¢	58¢
	b.	0	4	1					
1¢ 1¢	c.	1	2					4¢	
25¢ 1¢	d.				7	50¢	5¢		
	e.								17¢
1¢ 5¢	f.								37¢
	g.			0		50¢	20¢		
1¢ 5¢	h.		3			25¢		2¢	
5¢ 25¢	i.	3	3	3					
	j.								$1.16
5¢	k.								73¢
5¢ 5¢	l.		3	1		75¢			
5¢	m.				6		20¢	1¢	
25¢ 1¢	n.								$1.62
	o.								91¢

63


5
4
3
2
1
10
8
6
4
15
12
9
20
16
25
a. 5
b. 12
c. 6
d. 4
e. 7
f. 9
g. 16
h. 8
i. 10
j. 10
k. 10
n. 20
l. 22
m. 15
q. 15
o. 25
p. 20
r. 10
s. 10
t. 20
u. 17
v. 20
w. 25
x. 50
y. 30
z. 100
A. 31
B. 24
C. 20

Notes on Pupil Page 86

Tasks: *Pupils record in each green box the number of tiny squares in the figure adjacent to the green box.*

Purpose: To demonstrate to pupils the need for recognizing rectangular arrays and knowing their counts.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: Tagboard models of the arrays shown under the magnifying glass, including 10 copies of the 2-by-5 array; flannel board; Counting Board.

Materials for Pupils: None.

The Lesson:

The purpose of the exercises on pupil pages 86 and 87 is to motivate pupils to look for familiar arrays. Unless pupils use shortcuts, the exercises will be tedious and fail in their main purpose. Use the tagboard models as flash cards. Use the 2-by-5 arrays to form figures such as *x*, *y*, and *z*. Other figures that can be formed with your supply of tagboard models are *e*, *i*, *j*, *n*, *r*, *s*, *w*, and A.

Practice with the tagboard models on the flannel board, asking pupils to give the total number of tiny squares without counting them one by one. Then turn them loose on pupil page 86. Post times on the chalkboard (1 minute, 2 minutes, etc.) and tell pupils to record how many minutes it took them to complete the page. This emphasis on time will help convince pupils of the need for speed.

A pupil looking at *i* might determine the count in two ways. He might see a 2-by-2 array and a 2-by-3 array and say: 4 + 6 is 10. Or he might notice that each of the 2 rows contains 2 + 3 tiny squares. So: 5 + 5 is 10.

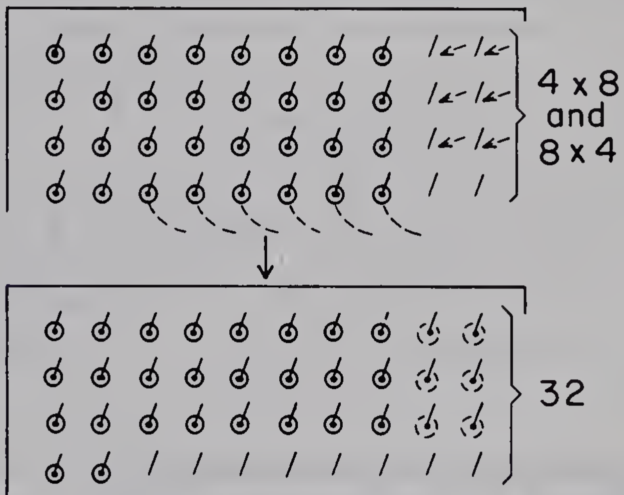
In the case of exercise *l*, a pupil might react with: 6 + 6 is 12 and 12 + 10 is 22.

In *t* a pupil should use what he discovered in *s* and say: 10 + 10 is 20.

In *v* 9 + 1 is 10 and 10 + 10 is 20.

The Counting Board

The board described in the Notes on Pupil Page 33 can be used to represent all the basic facts of multiplication and division. For example:



8 x 4 = 32   4 x 8 = 32

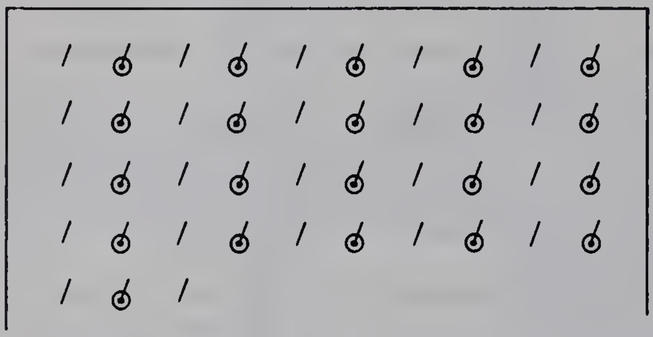
32 ÷ 4 = 8   32 ÷ 8 = 4

more

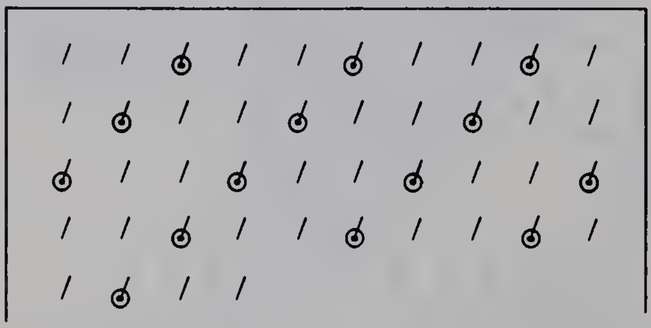
Headline for the Day: Apples cost 3 cents each or 2 for a nickel. How much will it cost to buy ____ apples? Make a chart.

If we imagine that the nails are labeled, from left to right, 1 through 10 in the top row, 11 through 20 in the second row, 21 through 30 in the third row, etc., a series of interesting patterns can be developed.

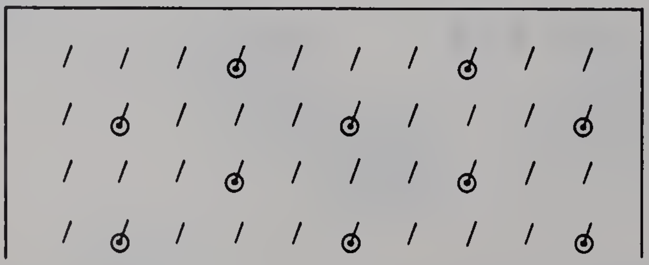
Put a washer on each nail labeled by a multiple of 2:



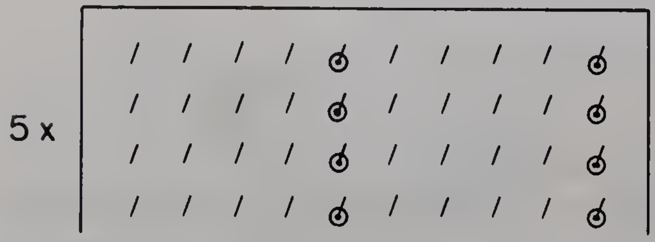
Put a washer on each nail labeled with a multiple of 3:

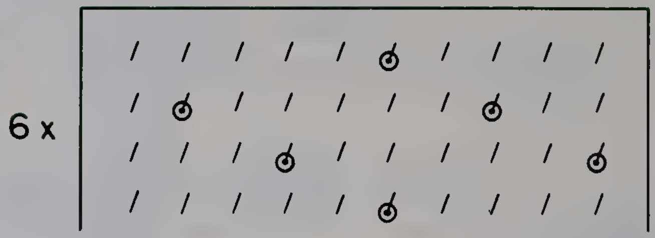


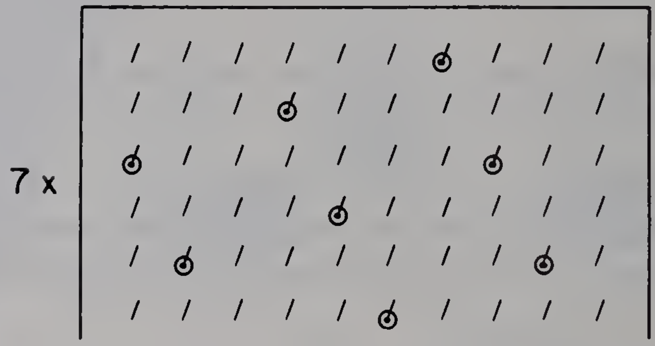
On nails labeled with multiples of 4:

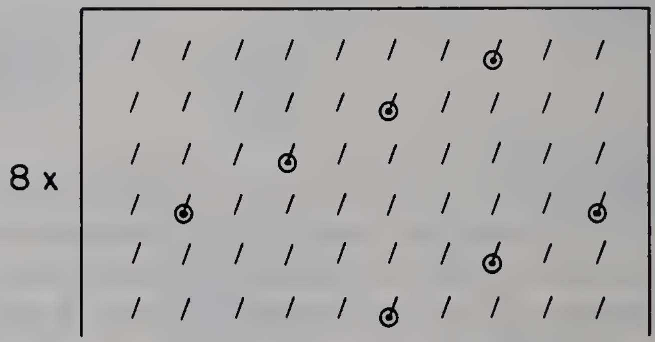


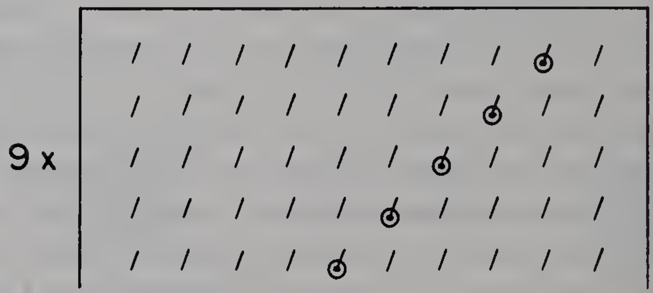
On nails labeled with multiples of 5, then 6, then 7, then 8, then 9, then 10, then 11, then 12:

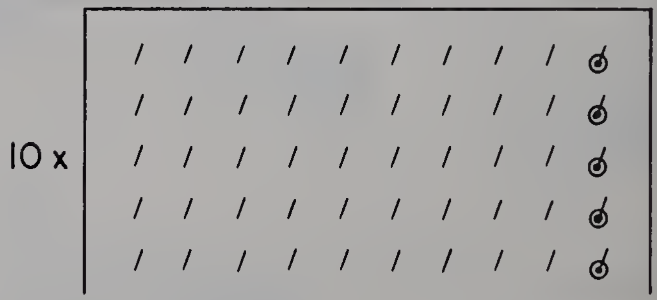


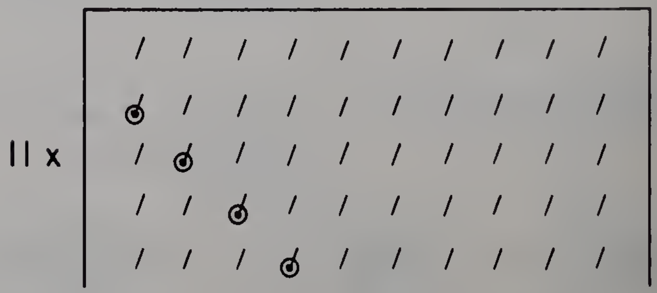


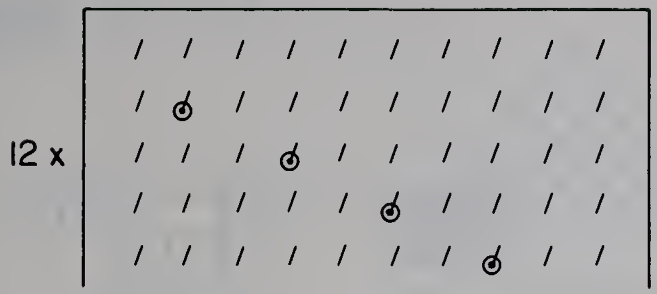




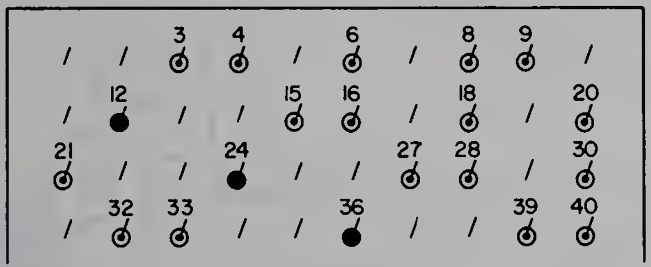






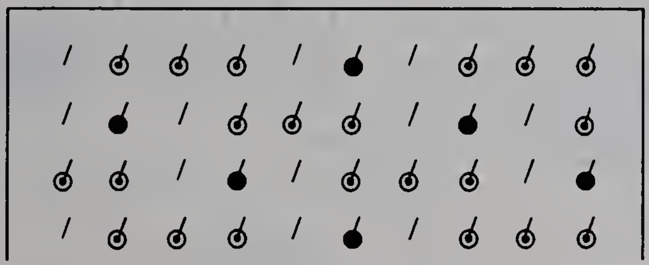


Next, combine two of these activities. Put a washer on each nail labeled with a multiple of 3 and, without removing any washers, put a washer on each nail labeled with a multiple of 4:



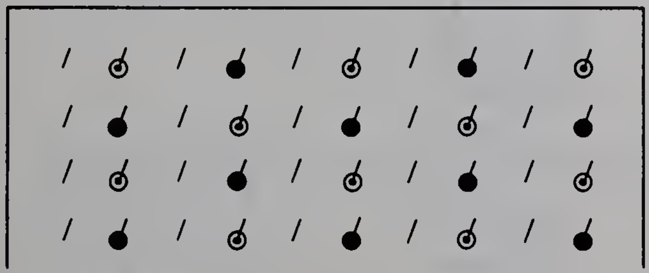
On which nails do we now find two washers? On 12, 24, 36, etc.—all multiples of 12. We also notice that 3×4=12. Is that a coincidence?

Repeat with multiples of 2 and multiples of 3:



Multiples of 2—2, 4, 6, 8, 10, 12, 14, 16, 18 etc.
Multiples of 3—3, 6, 9, 12, 15, 18, 21 etc.
In both lists—6, 12, 18, etc.—all multiples of 6, and 2×3=6.

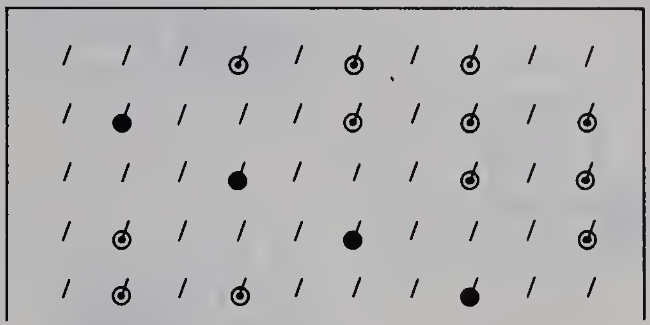
Repeat with multiples of 2 and 4:



Multiples of 2—2, 4, 6, 8, 10, 12, 14, 16, . . .
Multiples of 4—4, 8, 12, 16, . . .
In both lists—4, 8, 12, 16, . . .

Previous experiments might lead one to expect that those in both lists would all be multiples of 8 because 2×4=8. Why is this not so?

Repeat the experiment with multiples of 4 and 6:



Multiples of 4—4, 8, 12, 16, 20, 24, 28, 32, 36, . . .
Multiples of 6—6, 12, 18, 24, 30, 36, . . .
In both lists—12, 24, 36, . . .

How can the results be anticipated? This is a research project worth considerable study. The results of some experiments are summarized in the following table:

List of multiples of	2	3	2	4	6	3	6
List of multiples of	3	4	4	6	8	8	9
In both lists are multiples of	6	12	4	12	24	24	18

The pattern leads us to the notion of the *least common multiple* of two numbers. An important application of this idea is in the algorithm for adding fractions:

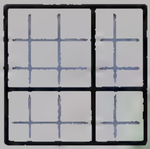
$$\frac{3}{4}+\frac{1}{6} = \frac{9}{12}+\frac{2}{12} = \frac{11}{12}$$

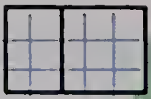
This example shows that 12 is the least common multiple of 4 and 6.

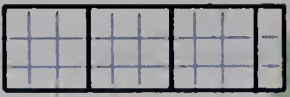
Drill

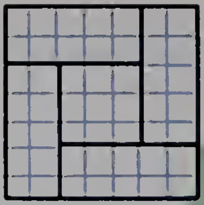
Here are two stories for drill purposes:

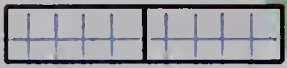
1) Mother gave Jane 91¢ for the picnic. She gave Josie only 77¢.
2) Lana bought 3 candy bars. Each candy bar cost 5 cents.

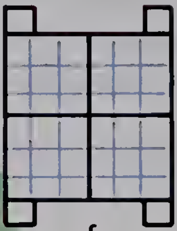

a. 25

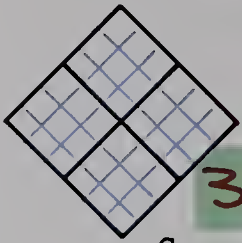

b. 15

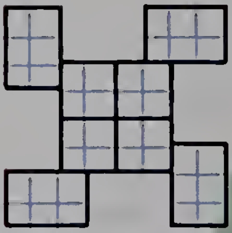

c. 30

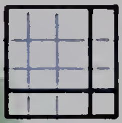

d. 49

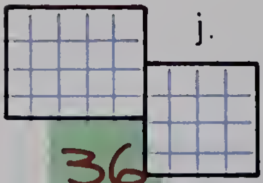

e. 20

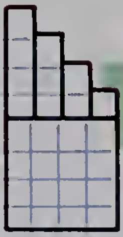

f. 40

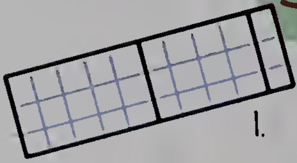

g. 36

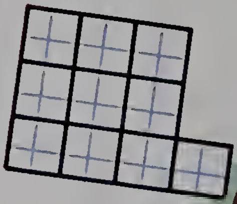

h. 40

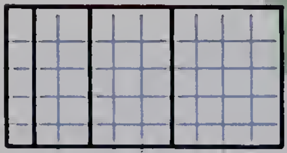

i. 16

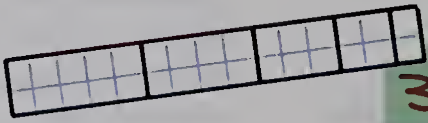

j. 36

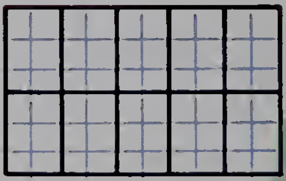

k. 26

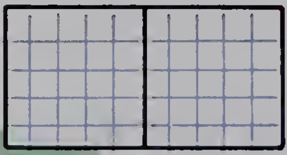

l. 30

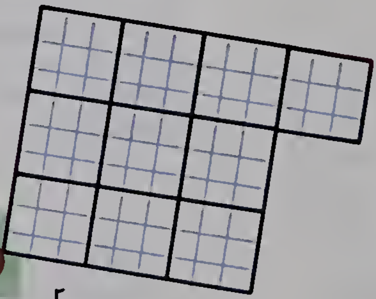

m. 40

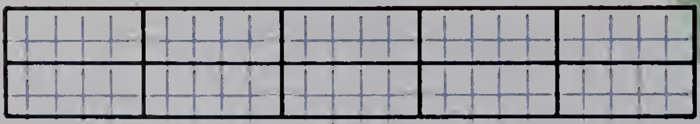

n. 50

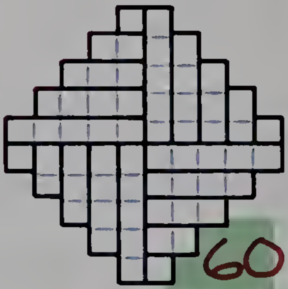

o. 30

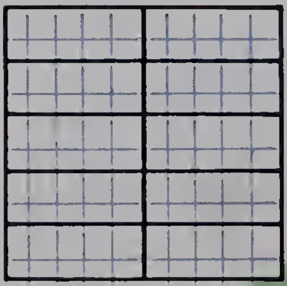

p. 60

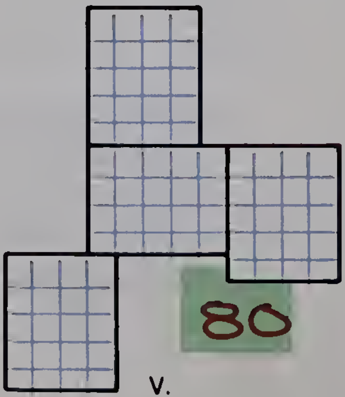

q. 50

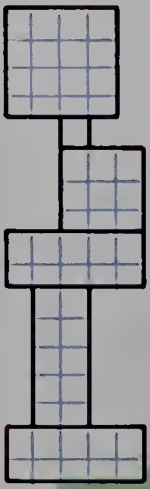

r. 90

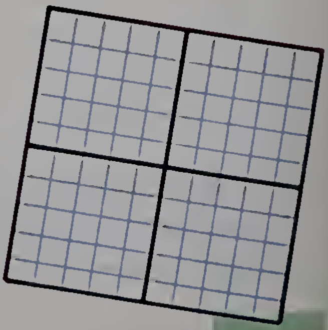

s. 100


t. 60


u. 100


v. 80


w. 56


x. 100

Notes on Pupil Page 87

Tasks: *Pupils record in each green box the number of tiny squares in the figure adjacent to the green box.*

Purpose: To demonstrate to pupils the need for recognizing rectangular arrays and knowing their counts.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: The models described in the Notes on Pupil Page 86 plus nine more 3-by-3 models; flannel board.

Materials for Pupils: None.

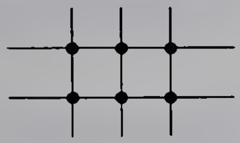
The Lesson:

With the materials suggested above you can form the figures in exercise *b-g*, *m*, *n*, *p*, *q*, *t*, *u*, and *w*. Let the pupils time themselves on pupil page 87 in the manner suggested for pupil page 86.

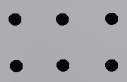
Representations for Multiplication

As with addition, we shall introduce the pupils to many representations for multiplication. All such representations can be manipulated to form rectangular arrays. Thus, it is the rectangular array which is the basis for multiplication. Here are some of the representations we shall use:

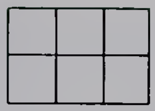
1. *Crossing points of horizontal lines and vertical lines*



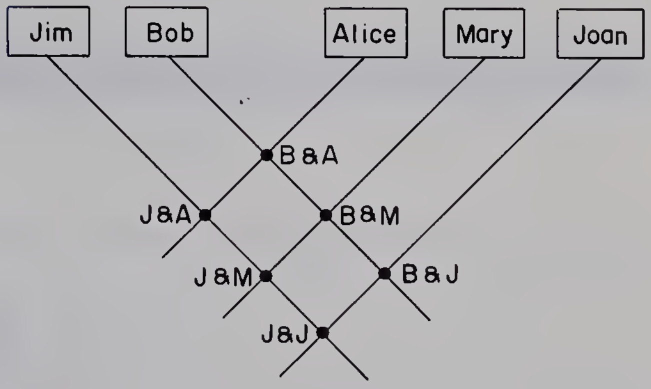
2. *Arrays of counters*



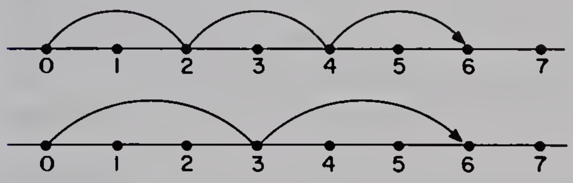
3. *Area*



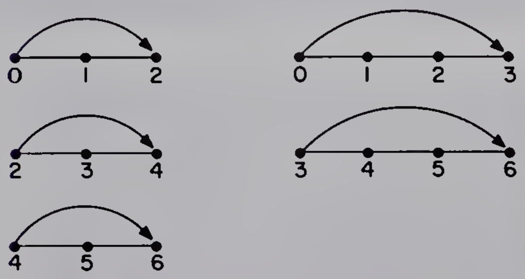
4. *Pairing*
   Consider two groups of children – 2 boys and 3 girls. In how many ways can a couple be chosen, one from the group of 2 boys and the other from the group of 3 girls?



5. *Number line jumps*

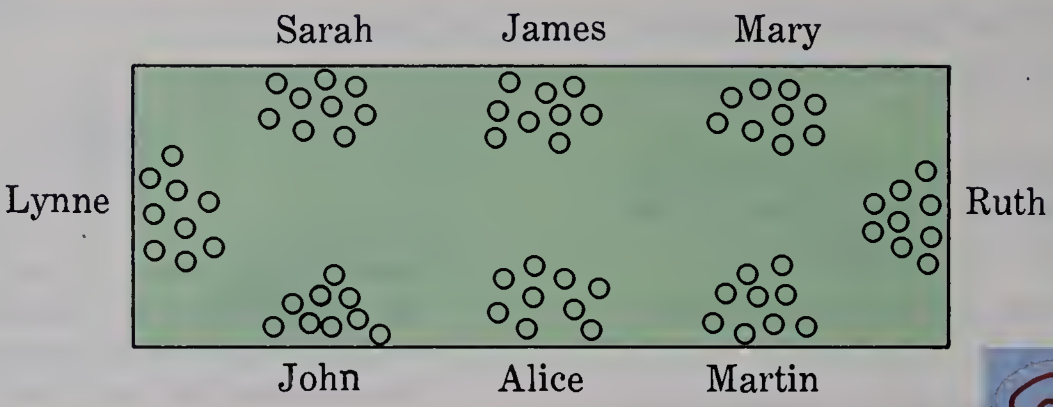


The rectangular array aspect of the number line jumps can be seen when the number line is chopped into chunks:



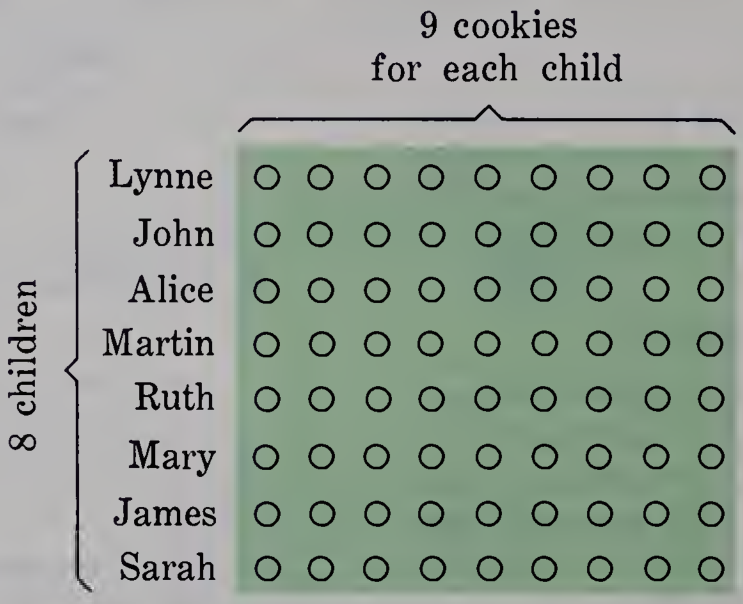
Headline for the Day: Tiles cost 12¢ each. Set up a time-saver chart to help the tile clerk.

There will be eight children at Lynne's party. Each of them will get nine cookies. How many dozens of cookies will Lynne need to buy?

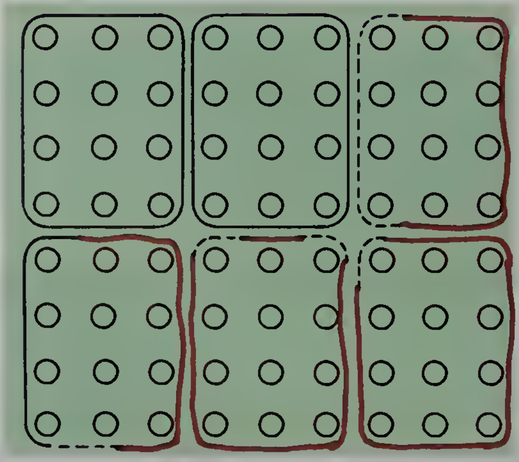

Sarah
James
Mary
Lynne
Ruth
John
Alice
Martin

6 dozens

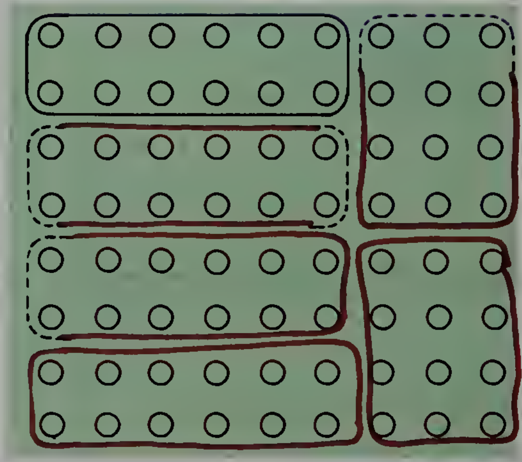
Here's an easy way to find out. Think of 8 rows of 9 cookies each.


9 cookies for each child
8 children
Lynne
John
Alice
Martin
Ruth
Mary
James
Sarah

Then group in dozens like this:



or like this:



Lynne will need to buy 6 dozens of cookies.

Notes on Pupil Page 88

Tasks: Pupils complete the partitioning diagrams and record the number of dozens of cookies.

Purpose: To help pupils appreciate that rectangular arrays help in determining counts.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: None.

Materials for Pupils: Counters.

The Lesson:

It is possible to draw loops around each dozen cookies in the first display, but it would be messy. Notice how easy it is to do this when the cookies have been displayed in a rectangular array. If one recognizes the 3-by-4 array and the 2-by-6 array, the job of counting by dozens is very easy.

Prepare pupils for this work by discussing the problem of arranging four groups of six counters each in such a way as to make it easy to see how many dozens we have.

Column and Row Arrangements

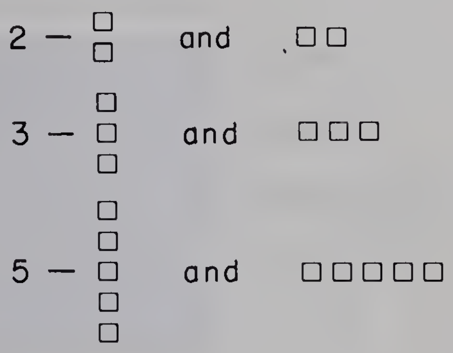
Can you arrange 12 chairs in rows and columns so that each row has the same number of chairs and each column has the same number of chairs?

Total chairs	In each row	In each column
12	3	4
12	4	3
12	2	6
12	6	2
12	1	12
12	12	1

If 24 chairs were used, it would be possible to illustrate 1×24, 24×1, 12×2, 2×12, 8×3, 3×8, 6×4, and 4×6. Other illustrations of multiplication can be shown by varying the number of chairs.

Prime Numbers

What numbers of chairs would have to be set up so that there could be only 1 row and more than 1 column, or only 1 column and more than 1 row? (2, 3, 5, 7, 11, 13, 17, 19, 23, etc.)



These numbers are different from others. They are called *prime numbers*. Can you find more prime numbers?

The above description of a prime number carefully excludes the number 1. This is not a matter for class discussion at this time. However, if a pupil asks about 1, simply explain that a prime number must have two different factors.

How many even numbers greater than 2 can be partitioned into two prime numbers?

$4 = 2 + 2$
$6 = 3 + 3$
$8 = 5 + 3$
$10 = 7 + 3$
$12 = 7 + 5$
$14 = 7 + 7$
$16 = 11 + 5$
etc.

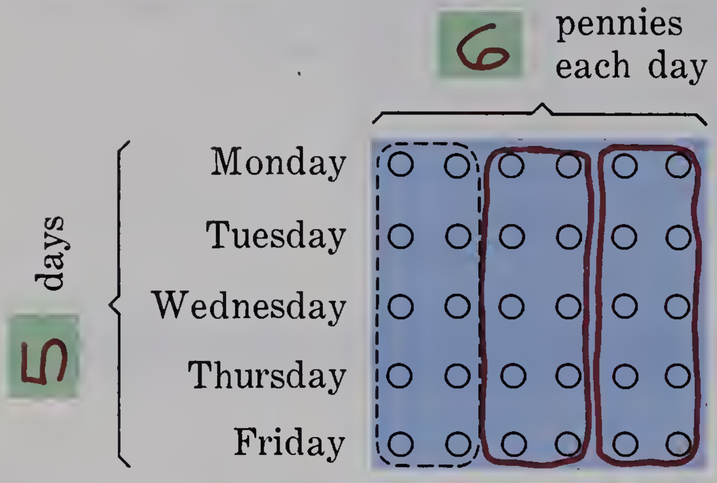
Mathematicians have never found an exception to the generalization that every even number greater than 2 is the sum of two prime numbers, but they are still trying to prove that there is no exception. He who finds an exception or can prove there is none will write an interesting chapter in the long history of mathematics. Pupils might be interested in knowing that such a seemingly simple problem has yet to be solved. The generalization is called *Goldbach's Conjecture.*

Headline for the Day: Write a story that fits: $25 + 8 = \square + 25$

Draw loops around groups to make it easy to find the answers.

I

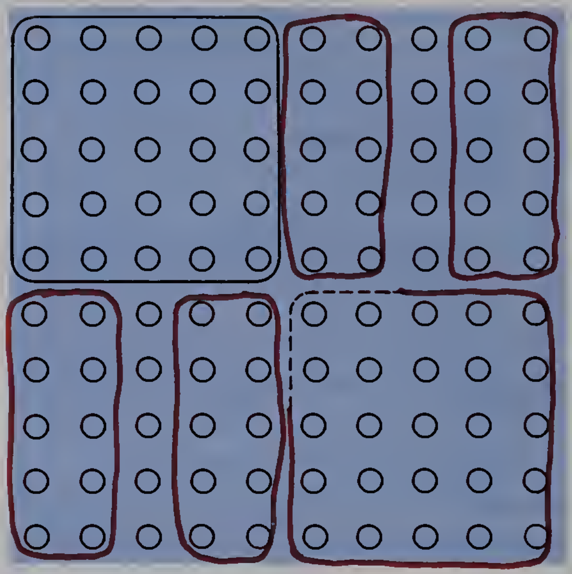
Nellie earns 6 pennies each day. How much does she earn in 5 days?


6 pennies each day
5 days
Monday
Tuesday
Wednesday
Thursday
Friday

3 dimes or 30 cents

II

Susan exchanged 100 pennies for quarters, dimes, and nickels.



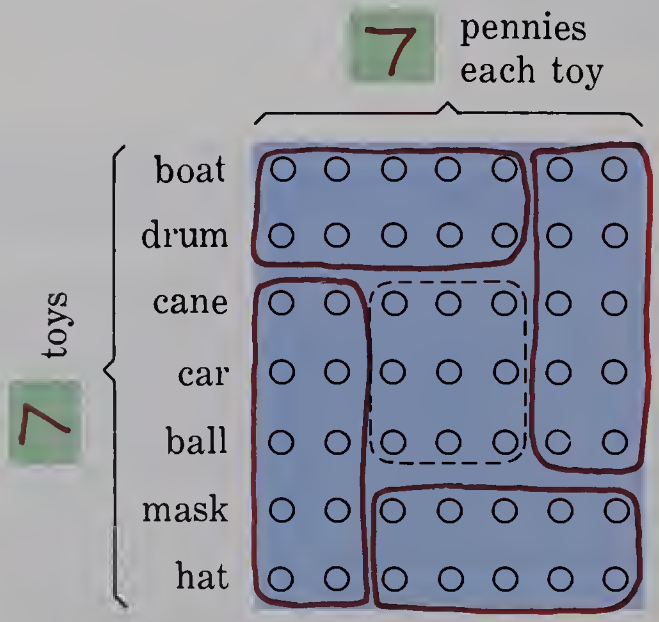
2 quarters

4 dimes and

2 nickels

III

Peter buys 7 toys and each costs 7 cents. How much does he spend?

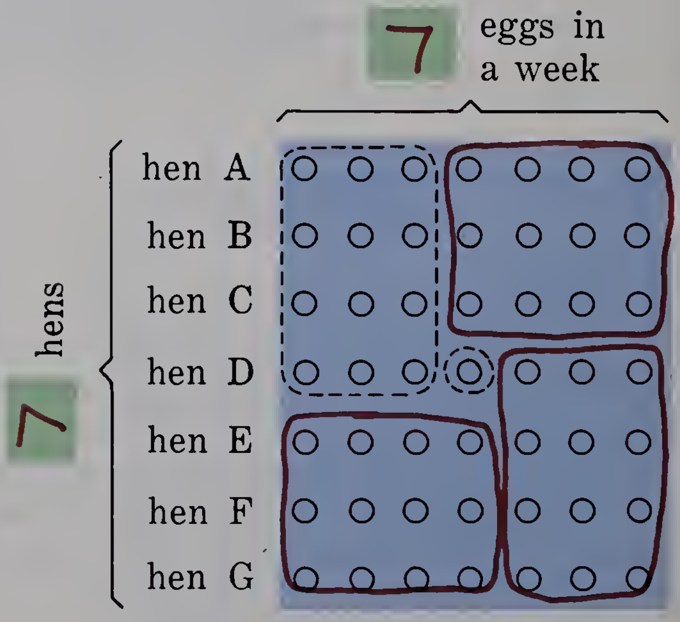

7 pennies each toy
7 toys
boat
drum
cane
car
ball
mask
hat

4 dimes and 9 pennies

or 49 ¢

IV

There are seven hens. Each hen lays an egg every day. How many eggs will the seven hens lay in a week?


7 eggs in a week
7 hens
hen A
hen B
hen C
hen D
hen E
hen F
hen G

4 dozens and 1 extra egg

or 49 eggs

Notes on Pupil Page 89

Tasks: *Pupils solve each of the four problems by partitioning according to the grouping pattern specified.*

Purpose: To demonstrate to pupils that rectangular arrays can represent certain quantitative situations.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: Flannel board; demonstration counters.

Materials for Pupils: Counters.

The Lesson:

Work each of these problems at the board using counters or X's. Exercises III and IV illustrate the fact that the partitioning used in reporting the number of members in a display depends upon custom. In III we ask for a decimal grouping while in IV we ask for a duodecimal (12) grouping. And in both problems the same 7-by-7 array is present.

A third variation for the 7-by-7 array would be to ask in III for the fewest coins (quarters, dimes, nickels, and pennies) needed to buy the toys.

Drill

Here are some suggested exercises for a drill sheet:

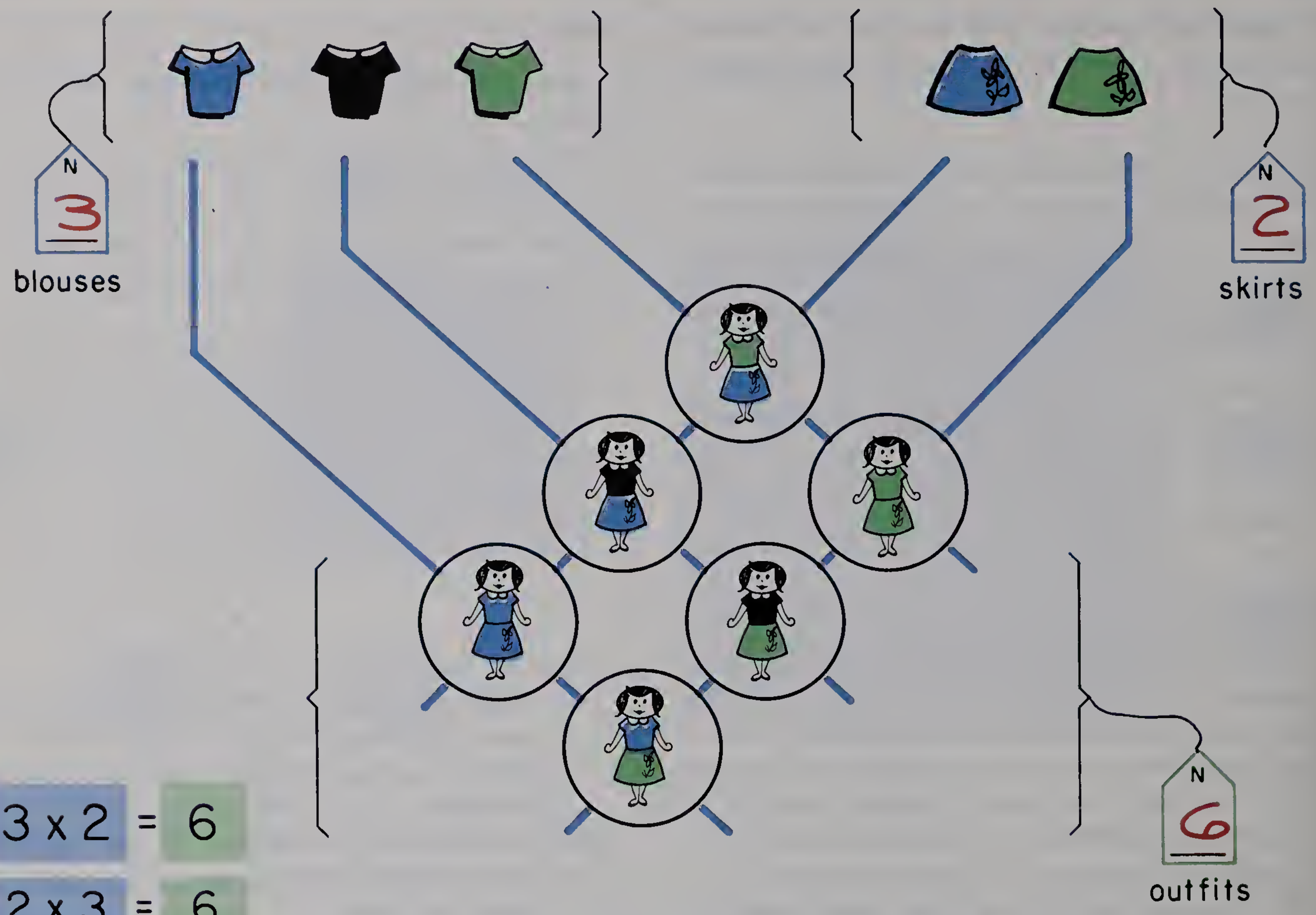
$$\begin{aligned}
(80 + 4) - 6 &= (70 + \square) - 6 = 70 + \bigcirc \\
(90 + 3) - 5 &= (80 + \square) - 5 = 80 + \bigcirc \\
(50 + 7) - 8 &= (40 + \square) - 8 = 40 + \bigcirc \\
(20 + 9) - 3 &= (10 + \square) - 3 = 10 + \bigcirc \\
(20 + 9) - 3 &= 20 + (\square - 3) = 20 + \bigcirc \\
(20 + 2) - 7 &= (10 + \square) - 7 = 10 + \bigcirc \\
(50 + 6) - 9 &= (40 + \square) - 9 = 40 + \bigcirc \\
(70 + 3) - 8 &= (\square + 13) - 8 = \bigcirc + 5
\end{aligned}$$

Here are some stories for drill purposes:

1) Mother sent Virginia to the store to buy ice cream. She bought 1 pint of chocolate and 3 pints of vanilla. The clerk had to use quart containers to pack the ice cream.
2) Bill had 3 pairs of mittens. He lost 1 mitten.
3) Ralph stacked 62 blocks to make a castle. 37 blocks fell down.
4) I have 12 spelling words to learn. Yesterday I learned 7 of them.

Headline for the Day: 2 dollars is the same as: ____ pennies; . . . nickels; . . . . dimes; _ _ quarters; _ _ _ half-dollars.

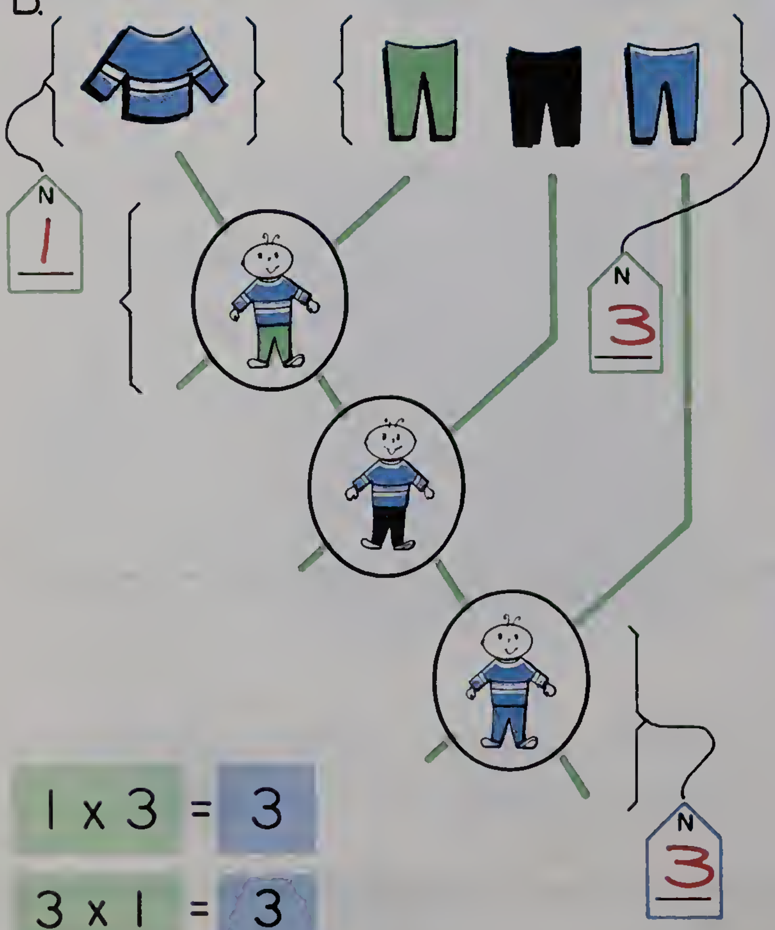
A. Sue has 3 blouses and 2 skirts. How many different outfits can she wear?


N
3
blouses
N
2
skirts
N
6
outfits

3 x 2 = 6

2 x 3 = 6

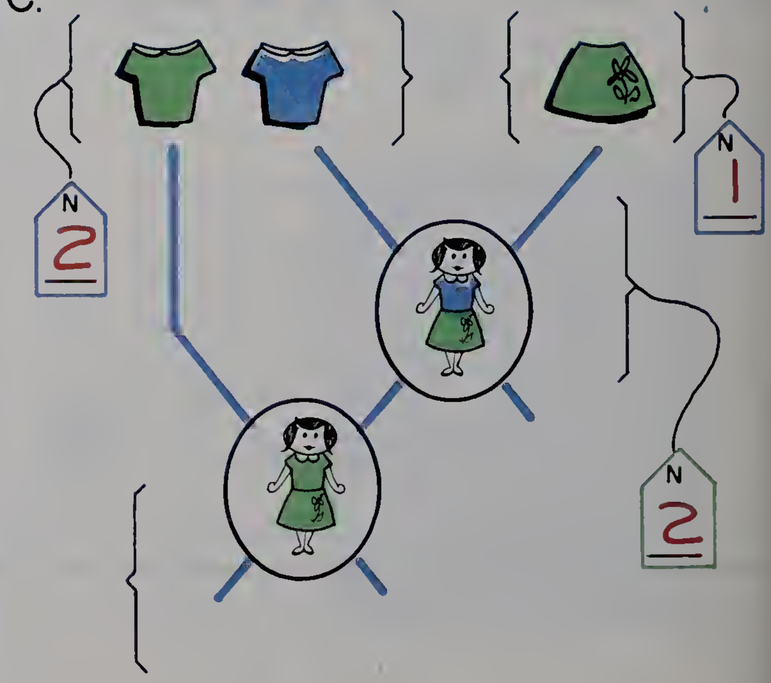
B.



1 x 3 = 3

3 x 1 = 3

C.



2 x 1 = 2

1 x 2 = 2

Tasks: *Pupils' record in the blanks in each of the exercises the number of objects in each of the three displays.*

Purpose: (a) To introduce the operation multiplication in terms of pairing; (b) to introduce the multiplication sign.

Unifying Ideas: Number and counting; multiplication and division.

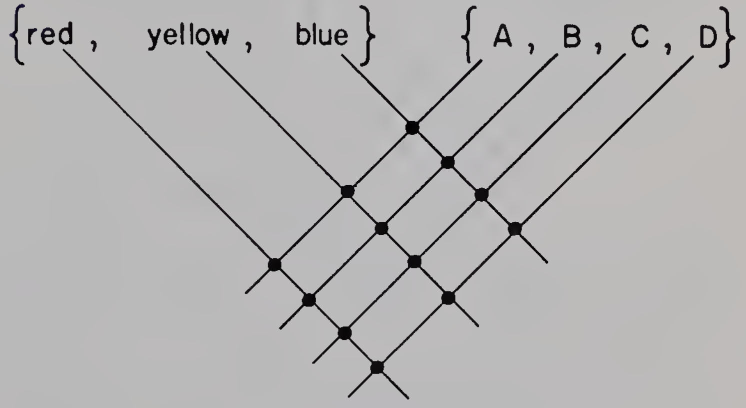
Materials for Teacher: Ditto masters.

Materials for Pupils: Dittoed sheets; cardboard dolls.

The Lesson:

Given a first number and a second number, consider two sets, one of which has the first number for its count and the other of which has the second number for its count. Consider next a third set whose members are all the pairs whose first components are members of the first set and whose second components are members of the second set. The third set is called *the Cartesian product* of the first set by the second set. The number of pairs in the third set is called *the product* of the first number by the second.

This abstract definition of multiplication is reflected in the illustrations on pupil pages 90 and 91. Let's consider an example. Suppose that we wish to find the product of 3 by 4. According to the definition, we should now consider a 3-set and a 4-set, and then consider a third set whose members are all the pairs each of which has a first component from the 3-set and a second component from the 4-set. For example, let the 3-set be the set consisting of the colors red, yellow, and blue, and let the 4-set be the set consisting of four cardboard squares, one with an A marked on one side, one with a B, one with a C, and one with a D. What are the possible pairs? Well, you could color the A-square red. Or you could color it yellow. Or you could color it blue. So far, we have three possible pairs. As you can see, we also have three possible pairs using the B-square. Similarly, we have three pairs for each of the C- and D-squares. We can represent the members of the third set as follows:



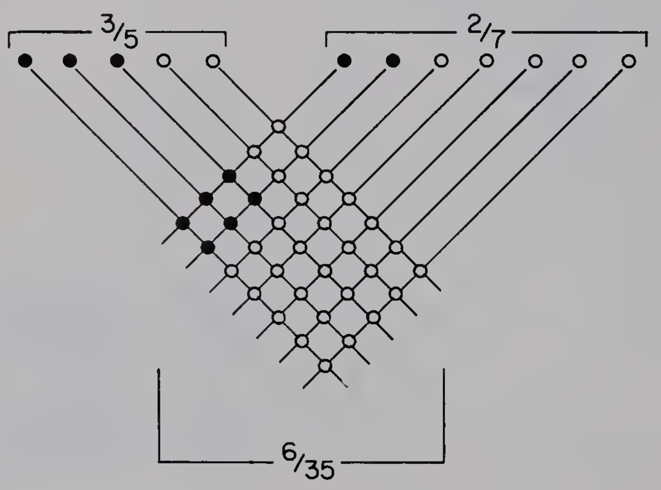
Each dot in the array represents one of the possible pairs in the third set. Since there are 12 such dots, it follows from the definition that 12 is the product of 3 by 4. The very same diagram also enables us to discover that the product of 4 by 3 is 12.

more

Headline for the Day: Mother has 2 blouses—1 red and 1 blue—and 3 skirts—black, white, and gray. If an outfit is a combination of a blouse and a skirt, how many different outfits are there? (6) Name them.

This definition of multiplication can also be used to show that the product of 0 by any number is 0 and that the product of any number by 0 is 0. In either of the cases it is obvious that the number of pairs in the Cartesian product is 0 since pairs cannot be formed if there is no supply of first components (first case) or no supply of second components (second case).

One of the reasons this definition for multiplication has been selected is that it can be easily extended to include multiplication of fractions. For example, suppose that we wish to find the product of $\frac{3}{5}$ by $\frac{2}{7}$. Here is a diagram that we might draw to help us discover this product:



The repeated addition approach to multiplication works well with whole numbers but cannot be reasonably extended to fractions.

It is not intended that pupils follow any verbal description of multiplication in terms of sets. If a pupil wishes to discover the product of 3 by 4, he might first draw 3 parallel lines, then 4 parallel lines that cross the first 3, and then count the crossing points. Or he might draw a 3-by-4 rectangle on graph paper and count the little squares. Or he might make 4 jumps of 3 on the number line, etc.

Pairing Skirts and Blouses

You might prepare pupils for the exercises on pupil page 90 by having them make outfits. Make a cardboard Numa for each girl in the class and a cardboard Numo for each boy. Give each girl a Dittoed sheet with pictures of three blouses and three skirts on it; give each boy a Dittoed sheet with pictures of three shirts and three pairs of trousers. Have pupils color each blouse (or shirt) and skirt (or pair of trousers) a different color, and then have the pupils cut out the costumes.

How many outfits can you make with 1 blouse and no skirt? Or with 1 shirt and no trousers? The answer, of course, is 0. Can you make any outfits with 1 skirt, or 1 pair of trousers, and no blouse or shirt?

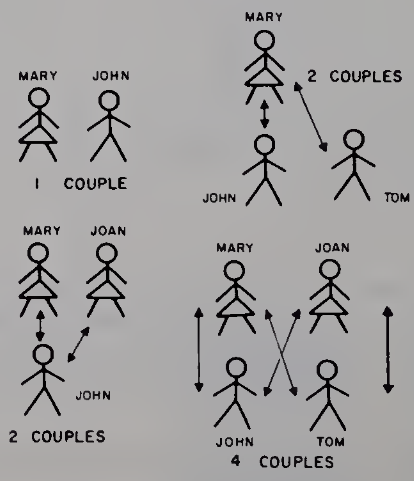
How many outfits can you make with 1 top and 1 bottom? (1) With 1 top and 2 bottoms? (2) With 2 tops and 1 bottom? (2) With 2 tops and 2 bottoms? (4) Let's make a record to show what we have discovered.

The record so far would look like this:

Tops	Bottoms	Outfits
1	0	0
0	1	0
1	1	1
1	2	2
2	1	2
2	2	4
2	3	6
3	2	6
3	3	9

Pairing Boys and Girls

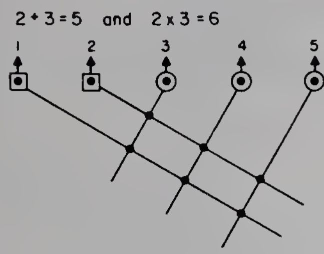
Have Mary come up. Is Mary a couple? (No) Ask John to come up, too. Now can we form a couple? (Yes) Ask 2 more girls and 2 more boys to come forward. There are 3 girls (Mary, Joan, and Sue) and 3 boys (John, Tom, and Jim). How many couples can we form with 2 girls (Mary and Joan) and 1 boy (John)? How many couples can we form using 2 girls (Mary and Joan) and 2 boys (John and Tom)? Draw on the board:

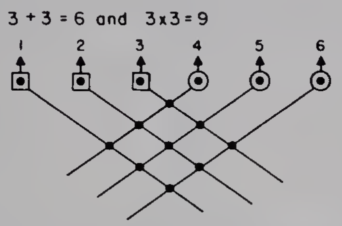


GIRLS	BOYS	COUPLES
1	0	0
0	1	0
1	1	1
2	1	2
1	2	2
2	2	4
3	2	?

As pupils act out situations like these, they get an increasingly clear picture of what we mean when we say that $3\times2=6$ and $3\times3=9$.

Pupils can come to grasp the difference between multiplication and addition if illustrations are used to show the difference:





A Shorthand Notation

Develop the agreement that a way to describe what happens in the pairing problems is to write multiplication number sentences:

$$3\times2=6$$

Read this as three times two is six or as three multiplied by two is six.

Factors and Product

None of the big words or complicated rules that usually accompany the ideas of multiplication or division are mentioned in MATH WORKSHOP. Pupils used to be told that there was a multiplicand and a multiplier, and that these led to a product. They were also told that in division there was a divisor that had to be divided into a dividend in order to get a quotient. No wonder pupils were so often confused. The terminology is enough to discourage most adults.

In MATH WORKSHOP, however, pupils are doing things without being burdened with big words. It is possible to say these things with two words, *factor* and *product*. These terms are not included in the pupil pages but there is no reason why you should not make use of them during class discussions.

In using the ÷ sign (first introduced on pupil page 98) it is not necessary to introduce any new terminology.

A FACTOR
ANOTHER FACTOR
2 X 3 = 6 ← THE PRODUCT

2 X __ = 6 – We have a factor and the product.
We can find the missing factor

2 X 3 = __ – We have two factors given.
We can find the product

__ X __ = 12 – We have the product given.
We can find several pairs of factors that would make this a true statement — (1 and 12, 2 and 6, 3 and 4).

THE PRODUCT
A FACTOR
6 ÷ 2 = 3 ← THE OTHER FACTOR

6 ÷ 2 = __ – We have the product and a factor.
We can find the missing factor

6 ÷ __ = 3 – (Same language).

__ ÷ 2 = 3 – We have both factors given
We can find the missing product

Automatic Mastery of Basic Facts

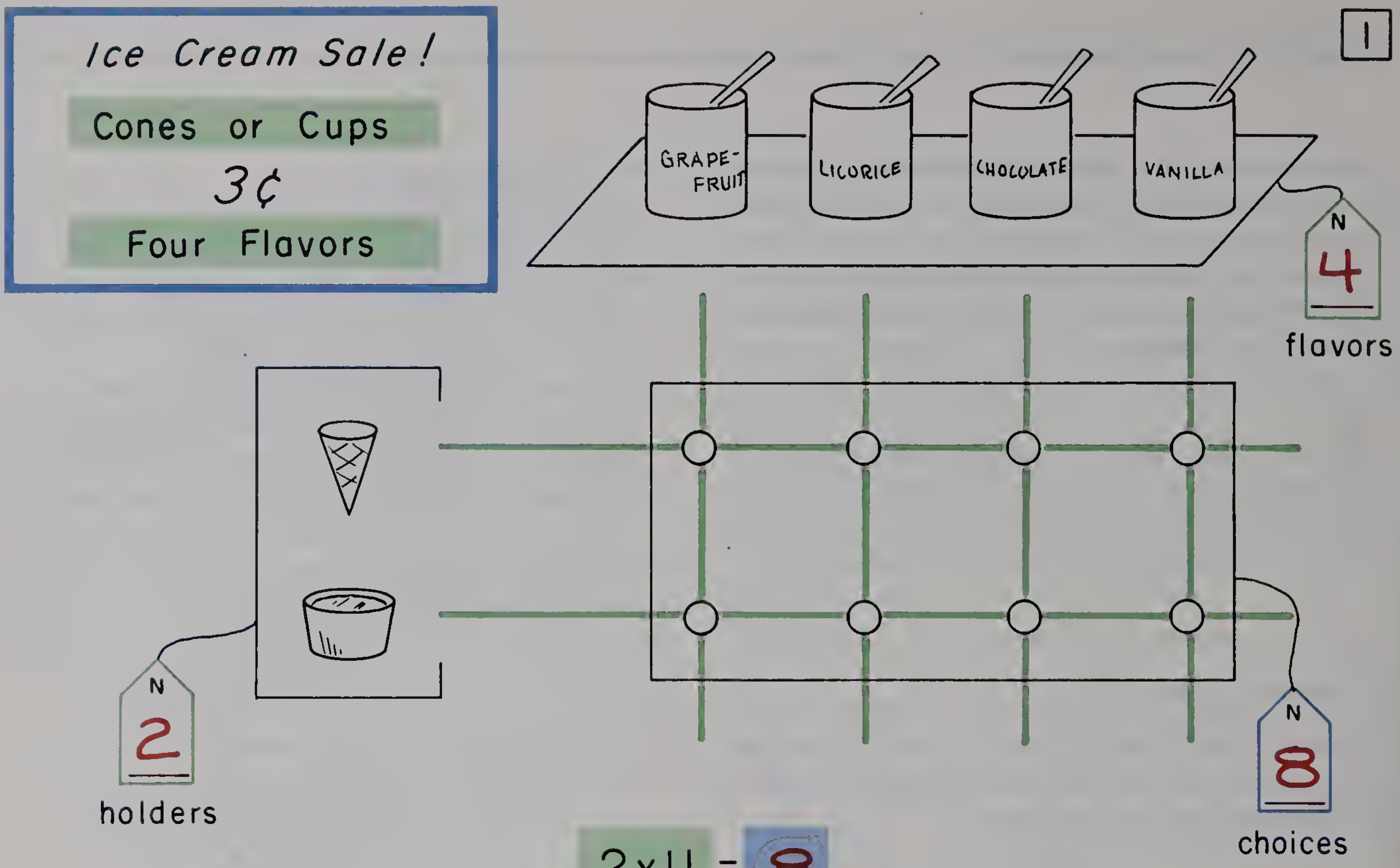
Regular sessions of 3 to 5 minutes should be spent reviewing the basic multiplication facts learned in connection with the various representations. Such drill will increase the automatic responses of pupils. Aim for automatic mastery of the multiplication facts up through 5×5.

Use flash cards like those suggested earlier for addition and subtraction. (See the Notes on Pupil Pages 28 and 30.) Pupils who miss facts should be asked to rediscover them via representations.

During drill sessions, as the facts are stated, keep a record in a table like the following:

	5	0	5	10	15	20	25
	4	0	4	8	12	16	20
Second factor	3	0	3	6	9	12	15
	2	0	2	4	6	8	10
	1	0	1	2	3	4	5
	0	0	0	0	0	0	0
	X	0	1	2	3	4	5
		First factor					

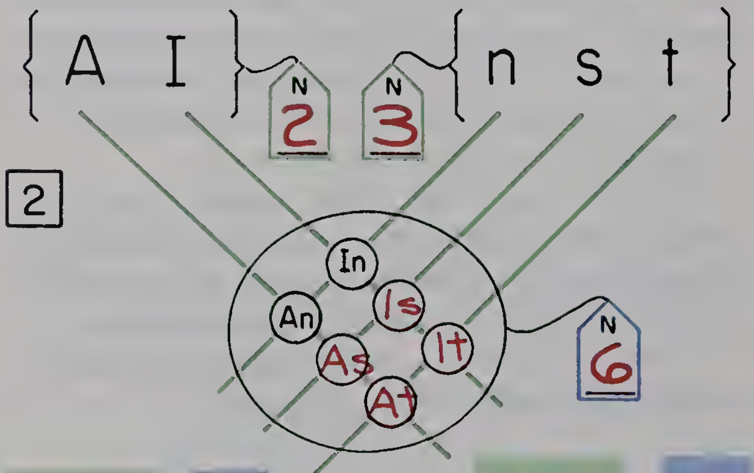
1


Ice Cream Sale!
Cones or Cups
3¢
Four Flavors
GRAPE-FRUIT
LICORICE
CHOCOLATE
VANILLA
N
4
flavors
N
2
holders
N
8
choices

2 x 4 = 8

4 x 2 = 8

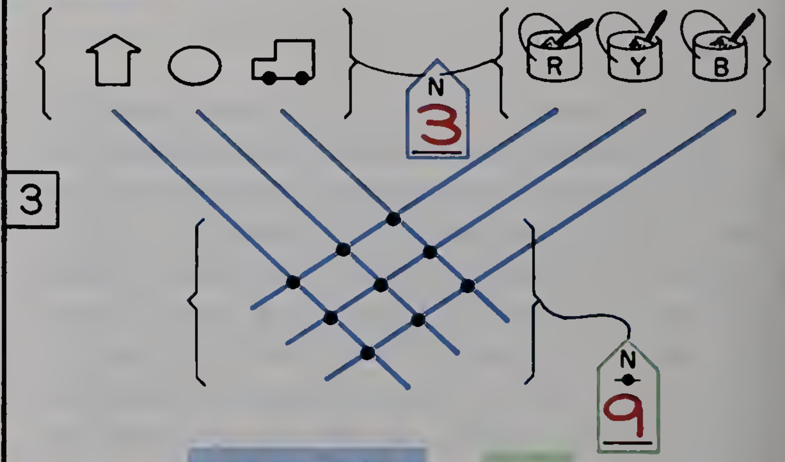
2


{A I}
N
2
N
3
{n s t}
In
An
Is
As
It
At
N
6

3 x 2 = 6

2 x 3 = 6

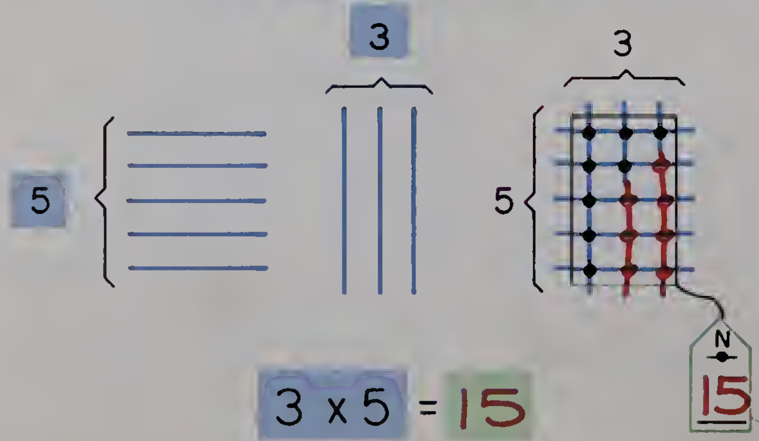
3



$\triangle 3 \times \triangle 3$ = 9

4

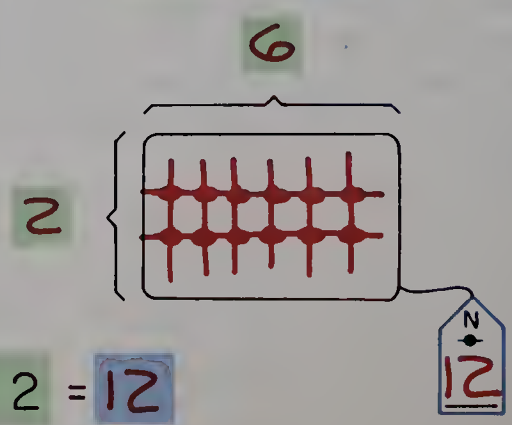
5 x 3 = 15



3 x 5 = 15

5

2 x 6 = 12



6 x 2 = 12

Notes on Pupil Page 91

Tasks: *In each of exercises 1-3 pupils record the number of members in each of the three displays and complete the multiplication sentences. In exercises 4 and 5 they complete the drawings in order to find the required products.*

Purpose: To give practice in using the pairing approach to multiplication.

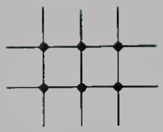
Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: Blank card like that shown below.

Materials for Pupils: Cuisenaire rods.

The Lesson:

Multiplication is not something that should be defined for pupils in a sentence or two or with one or two examples. An understanding of multiplication emerges from a host of experiences with problems such as those pictured on pupil pages 90 and 91 and those described in the Notes on Pupil Page 90. We talked about forming couples from a given number of boys and a given number of girls: how many couples can we form if there are 3 boys and 2 girls? A related problem is: if Mary has 1 skirt and 2 blouses, how many different skirt-blouse combinations can she wear? If she gets another skirt and another blouse for her birthday, how does this affect the number of combinations she can make? Answer: she will have as many skirt-blouse combinations as there were pairs in the problem involving 3 boys and 2 girls. And the same answer would apply to the number of crossing points formed when 2 horizontal lines cross each of 3 vertical lines as shown in the diagram in the next column.

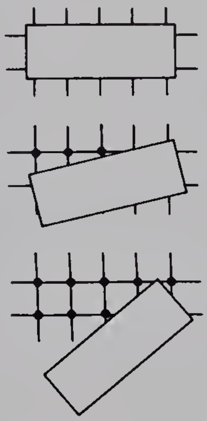


The exercises on pupil page 91 provide pupils with more examples of multiplication situations. Exercises 4 and 5 allow pupils to do their own representational work to determine products.

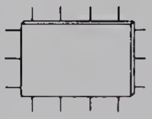
For classes equipped with Cuisenaire rods, activities like those described in pages 62-79 of *Games and Enrichment Activities* are suggested.

A Game

How many crossing points are under the card?



Now draw this picture:



How many points are there under the card?

Drill

Here are two stories for drill purposes (note the sample answers):

1) Bill collected the wheels from 4 tricycles.

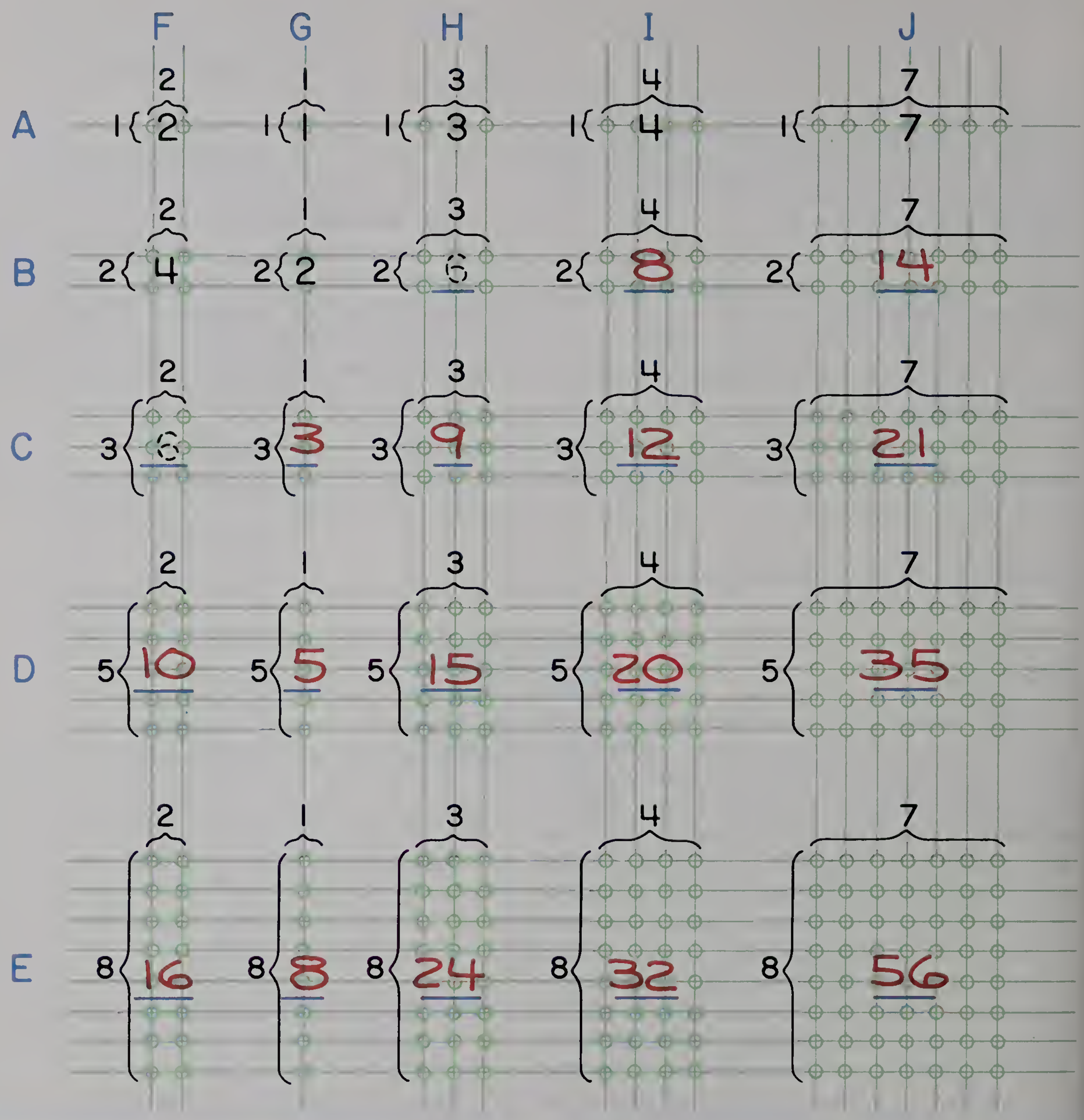
$4\times3=12 \qquad 3\times4=12$

$3+3+3+3=12 \qquad 4\times3=\square$

2) Mrs. Perez needs 12 peaches to make 3 pies.

Headline for the Day: If a couple is a boy-girl combination, in how many ways can we make couples with 4 boys (Jim, Michael, Bruce, Neil) and 3 girls (Bonnie, Karen, and Ruth)? (12) Name the possible couples.

Find my rule. Then complete all examples.



		F	G	H	I	J
		2	1	3	4	7
A	1	2	1	3	4	7
B	2	4	2	6	8	14
C	3	6	3	9	12	21
D	5	10	5	15	20	35
E	8	16	8	24	32	56

A shorthand way of writing some of the examples above.

1 x 2 = 2   1 x 3 = 3   1 x 7 = 7

2 x 2 = 4   2 x 3 = 6   2 x 4 = 8

3 x 2 = 6   3 x 3 = 9   3 x 4 = 12   3 x 7 = 21

5 x 2 = 10   5 x 3 = 15   8 x 2 = 16   8 x 3 = 24

5 x 1 = 5   5 x 4 = 20   8 x 1 = 8   8 x 4 = 32   7 x 8 = 56

5 x 3 = 15   5 x 7 = 35   8 x 3 = 24   8 x 7 = 56

Notes on Pupil Page 92

Tasks: *Pupils demonstrate their discoveries of various patterns by recording the products directly on the displays and, in the section at the bottom of the page, they record the products in the blanks in the number sentences.*

Purpose: (a) To give pupils practice in determining products; (b) to give pupils an opportunity to become aware of the distributive principle for multiplication over addition.

Unifying Ideas: Structure; number and counting; multiplication and division.

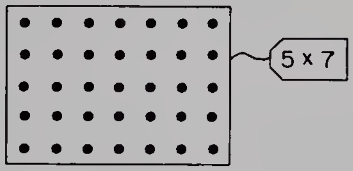
Materials for Teacher: A demonstration-size tagboard model of the upper section of pupil page 92; overhead projector; transparency of lattice points.

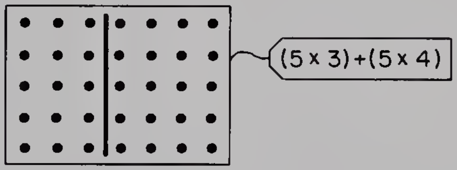
Materials for Pupils: Counters.

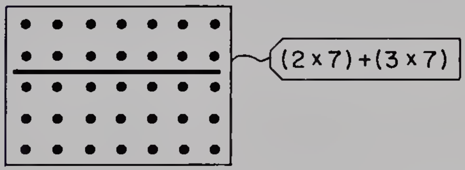
The Lesson:

Note that the clusters of horizontal lines and the clusters of vertical lines have been chosen in such a way that certain products can be determined by adding previously obtained ones. For example, the product 5×7 in row D can be obtained by adding the products 5×3 and 5×4 in row D or adding the products 2×7 and 3×7 in column J. These facts illustrate a general property of addition and multiplication, a property which depends upon the fact that displays (rectangular arrays of dots) can be partitioned without affecting the total number of objects in the display.

The following diagrams can be shown very nicely on an overhead projector:



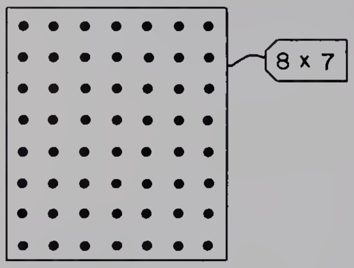


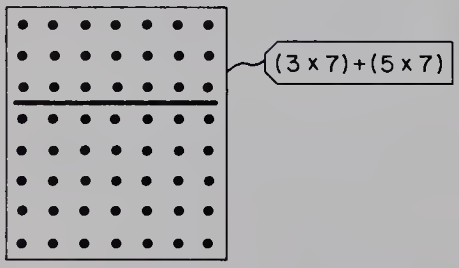


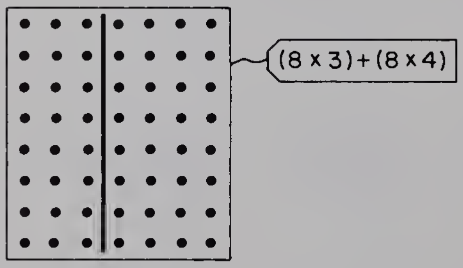
Headline for the Day: Bill helped dry the dishes each night for 8 nights in a row. His mother paid him 5 cents a night.

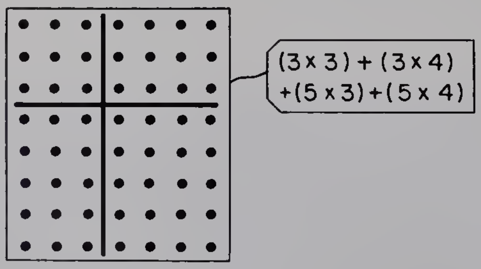
Thus, one can find larger products if he knows certain smaller products. (Of course, if the only product one knows is 1×1 then finding larger products amounts to counting one by one.) The *Column-Row Counting Game* described below will help pupils notice this property.

As a further application of the distributive principle of multiplication over addition, note some of the ways in which the largest product on the page can be found:









Pupils should not be told that one can find larger products this way because of the distributive principle for multiplication over addition. Reasons of this type are of no use to young pupils. It is much better for pupils to say (if they must say something) that larger rectangles can be chopped up into smaller ones or that smaller rectangles can be put together to form larger ones.

The Column-Row Counting Game

Use the demonstration-size tagboard model of the upper section of pupil page 92 for this game. N(F, B) stands for the number of dots (crossing points) which are in both column F and row B. Thus, N (F, B) = 4. Also, N (G, BC) stands for the number of dots which are in column G and in row B or row C. Thus, N (G, BC) = 5.

One way to call pupils' attention to the clustering of the horizontal and vertical lines is to give them problems in the following sequences:

N(F, B) =__
N(G, B) =__
N(H, B) =__

N(H, C) =__
N(I, C) =__
N(J, C) =__

N(G, A) =__
N(H, A) =__
N(I, A) =__

N(FG, B)=__
N(H, B) =__

N(HI,C)=__
N(J, C) =__

N(GH, A)=__
N(I, A) =__

N(F, C) =__

N(F, D) =__

N(F, CD)=__

You may also wish to give completion items such as the following:

	(answers)
1) N (H, __)=9	1) AB or C
2) N (__, D)=20	2) GH or I
3) N (__, D)=55	3) GHJ or IJ

Here are some other two-dimensional charts which will challenge your very best pupils.

Using any capital letters you wish, create a grid similar to the following:

	a	b	c	d	e	f	g
h	B	B	B	E	E	E	E
i	B	B	B	E	E	E	E
j	F	F	F	E	E	E	E
k	F	F	F	P	P	P	P
l	F	F	F	P	P	P	P
m	F	F	F	P	P	P	P
n	F	F	F	P	P	P	P

For this grid, N (ab, ijk) stands for the number of capital letters which are in column a or column b and in row i, row j, or row k. Thus, N (ab, ijk) = 6(2B's and 4 F's).

A chart based on the above grid could be drawn:

	Total number of letters	Number of			
		B's	E's	F's	P's
N (ab, ijk) →	6	2	0	4	0
N (cde, h) →	3	1	2	0	0
N (fg, mn) →	4	0	0	0	4
N (bcd, lmn) →	9	0	0	6	3

To vary the exercise, fill in the five columns with appropriate numbers and ask pupils to give the notation that could be written in the left-hand column. For example:

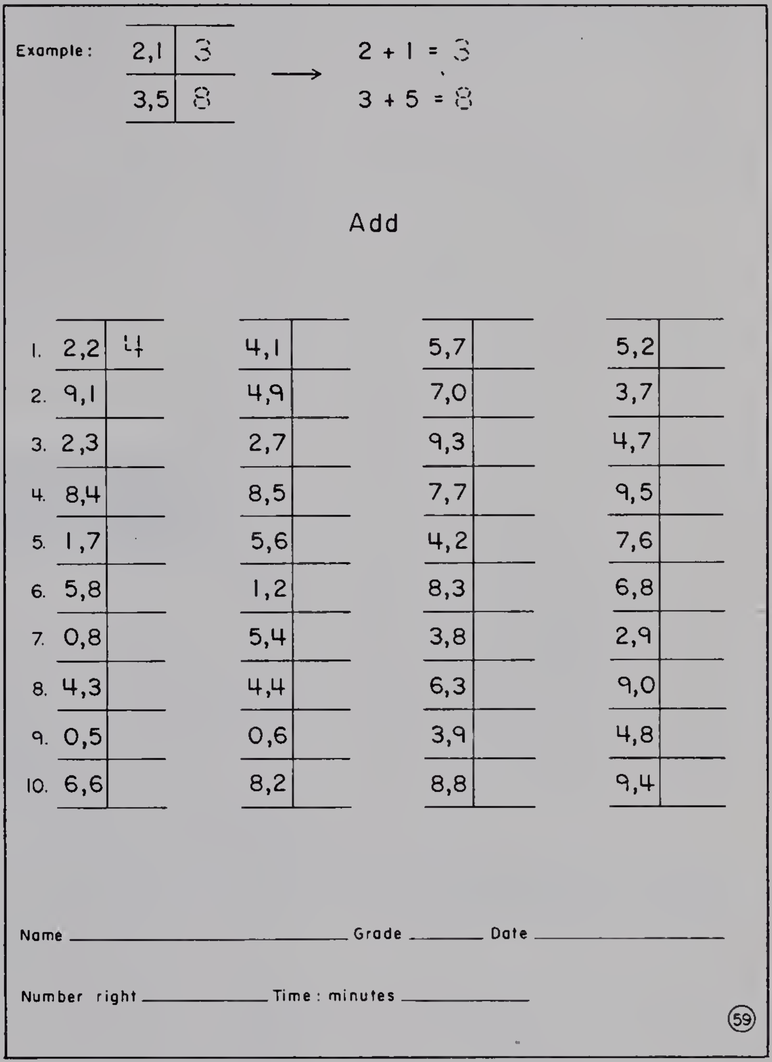
	All letters	B's	E's	F's	P's
N ( ) →	8	2	4	2	0
N ( ) →	4	1	1	1	1

Another grid that might be used consists of shapes:

	a	b	c	d	e	f
g	□	□	□	△	△	△
h	□	□	□	△	△	△
i	•	•	•	△	△	△
j	○	○	○	○	○	○
k	○	○	○	○	○	○

Drill

Consider using as a timed test on basic addition facts page 59 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Example:

2,1	3
3,5	8

→ 2 + 1 = 3
3 + 5 = 8

Add

1.	2,2	4	4,1		5,7		5,2	
2.	9,1		4,9		7,0		3,7	
3.	2,3		2,7		9,3		4,7	
4.	8,4		8,5		7,7		9,5	
5.	1,7		5,6		4,2		7,6	
6.	5,8		1,2		8,3		6,8	
7.	0,8		5,4		3,8		2,9	
8.	4,3		4,4		6,3		9,0	
9.	0,5		0,6		3,9		4,8	
10.	6,6		8,2		8,8		9,4	

Name ______________ Grade ______ Date ______________

Number right ______________ Time : minutes ______________

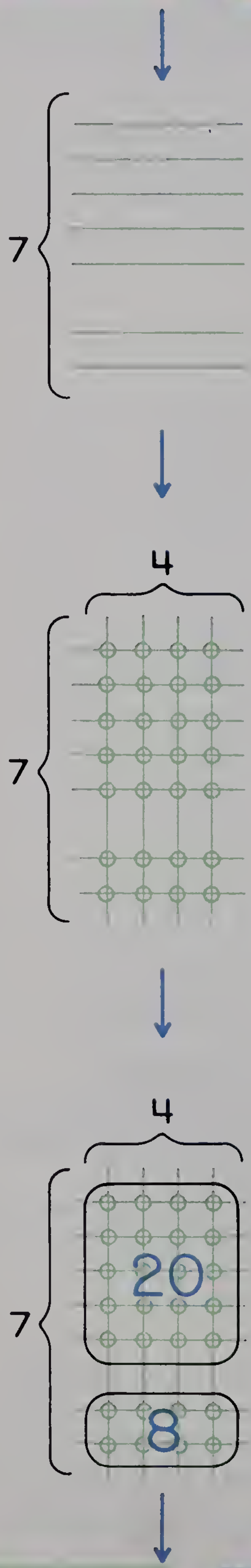
59

Here are two stories for drill purposes:

1) Matilda needed 1 quart of grapefruit juice, 2 quarts of orange juice, and 3 quarts of tomato juice.
2) Fred saw a toy boat for sale for 12 cents. He has a nickel.

Problem

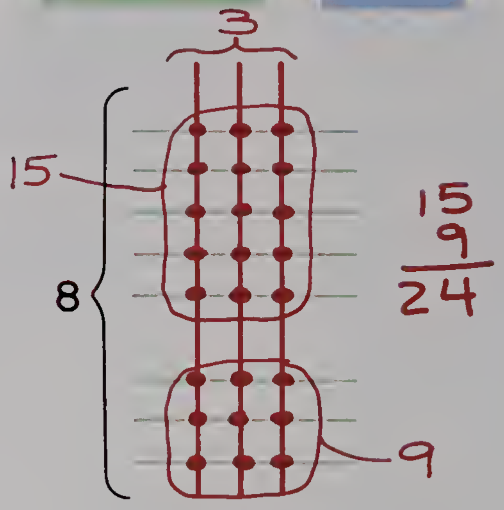
7 x 4 = 28



7 x 4 = 28

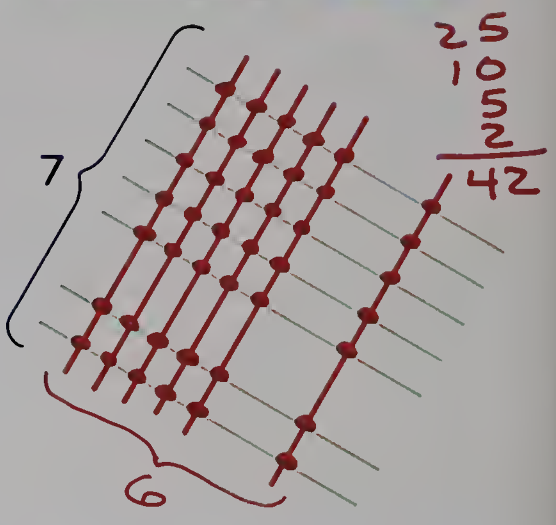
A

8 x 3 = 24


3
15
8
9
15
9
24

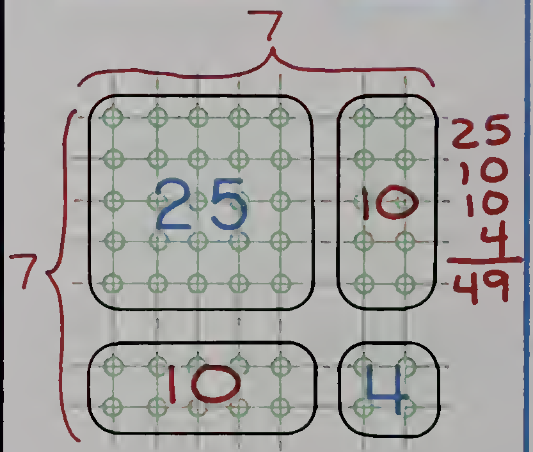
B

7 x 6 = 42


25
10
5
2
42
7
6

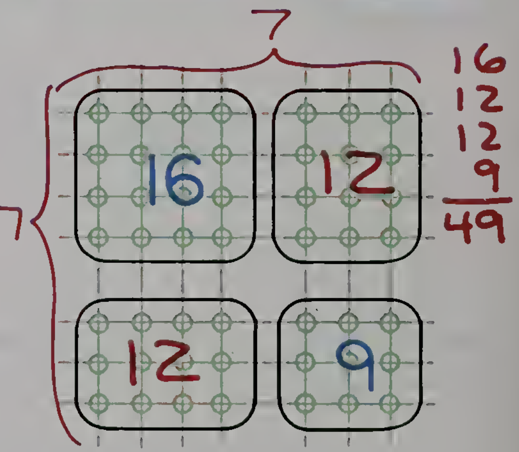
C

7 x 7 = 49


7
7
25
10
10
4
25
10
10
4
49

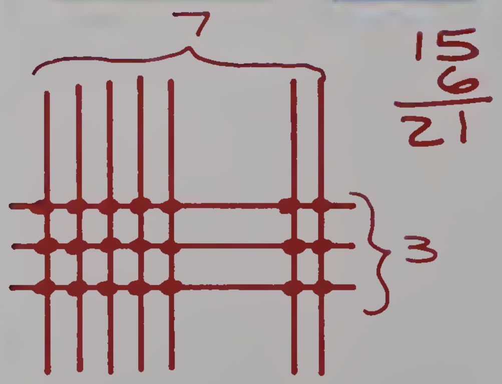
D

7 x 7 = 49


7
7
16
12
12
9
49
16
12
12
9

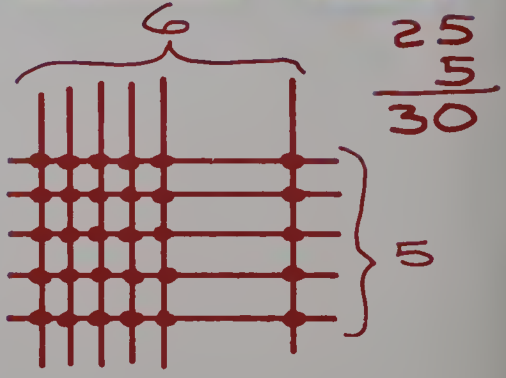
E

3 x 7 = 21



F

5 x 6 = 30



Notes on Pupil Page 93

Tasks: *Pupils study the sequence of steps in the left-hand column for clues in doing exercises A-F.*

Purpose: To demonstrate the advantage of partitioning an array as a shortcut in counting.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: A blank card.

Materials for Pupils: None.

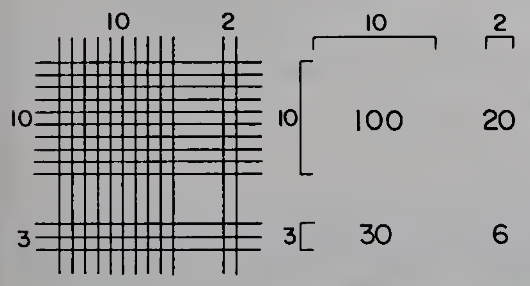
The Lesson:

The problem posed in the left-hand column is solved by clustering the horizontal lines in a group of 5 and a group of 2. One does this when he is familiar with multiplication combinations in which 5 and 2 are involved. In the case at hand, this particular clustering is useful to someone who has memorized: $5\times4=20$ and: $2\times4=8$. The product of 7 by 4 is then determined by adding 20 and 8.

In exercise A we suggest that the 8 horizontal lines be partitioned into a 5-group and a 3-group. We could have partitioned them into a pair of 4-groups and thus have seen that 8×3 is 2 dozen. This alternative should be discussed in class.

In exercise B both displays of lines are partitioned. This maneuver keeps all of the component arrays down to 5-by-5 or smaller.

Exercises C and D show two partitionings of the same basic array. Exercises E and F allow pupils to draw their own arrays.

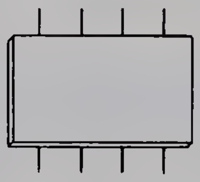


$$\begin{array}{r} 100 \\ 20 \\ 30 \\ +\ 6 \\ \hline 156 \end{array} \qquad \begin{array}{r} 13 \\ \times 12 \\ \hline 26 \\ 13 \\ \hline 156 \end{array}$$

Partitioning into four groups is the basis for the algorithm for finding the product of a pair of 2-digit numbers. Consider the problem $13\times12=?$ One can find the product by drawing 13 horizontal lines and 12 vertical lines, and counting the crossing points. The counting is facilitated by a decimal partitioning of the displays of lines as shown in the last column.

Missing Factors and Missing Products

A card illustration such as the following suggests the hunt for a missing factor or product:



Using the above drawing, explain that when the horizontal lines are shown, there will be 7 lines in all. Ask: How many horizontal lines must be drawn? How many crossing points will there be? (3 horizontal lines; 12 crossing points.) Still using the same drawing, alter the problem. Explain that there are 20 crossing points under the card. The ends of the horizontal lines are too short to show. How many horizontal lines are there? How many lines in all? (5 horizontal lines; 9 in all.)

You may wish to ask the questions in words rather than in pictures. Here is one way to do it:

Horizontal lines	Vertical lines	Number of crossing points	Number of lines

The first two columns contain the components of a pair. If the pair is a pair of factors then the product is noted under *Number of crossing points*. If the pair is a pair of addends then the sum is noted under *Number of lines*.

Headline for the Day: Write stories suggested by the number sentences on page 93.

13
+17

(A) 45 (B) 30 (C) 4 (D) 20

Mark the correct answer.

Mark the furthest from the correct answer.

A ○ B ● C ⊗ D ○ 26

Difference between the answers you marked.

1

One week Bill received his 50¢ allowance and he earned 65¢ shoveling snow. How much did he receive that week?

(A) $2.00 (C) $1.25

(B) $1.15 (D) 15¢

A ○ B ● C ○ D ⊗ $1.00

2

Mary saved $2.50 to spend at camp. She had 65¢ left when she returned home. How much had she spent?

(A) Nothing (C) 85¢

(B) $3.15 (D) $1.85

A ⊗ B ○ C ○ D ● $1.85

3

Harry's family drove 34 miles before lunch and 28 miles after lunch. How many miles did they drive altogether?

(A) 72 mi. (C) 62 mi.

(B) 6 mi. (D) 64 mi.

A ○ B ⊗ C ● D ○ 56 mi.

4

Which is longest?

(A) One yard

(B) Four feet

(C) Thirty-five inches

(D) Two and a half feet

A ○ B ● C ○ D ⊗ 18 in.

5

Which is two tens and eight ones?

(A) Eighty-two

(B) Eighteen

(C) Twenty-eight

(D) Fifty-six

A ⊗ B ○ C ● D ○ 54

6

Which is largest?

(A) 120

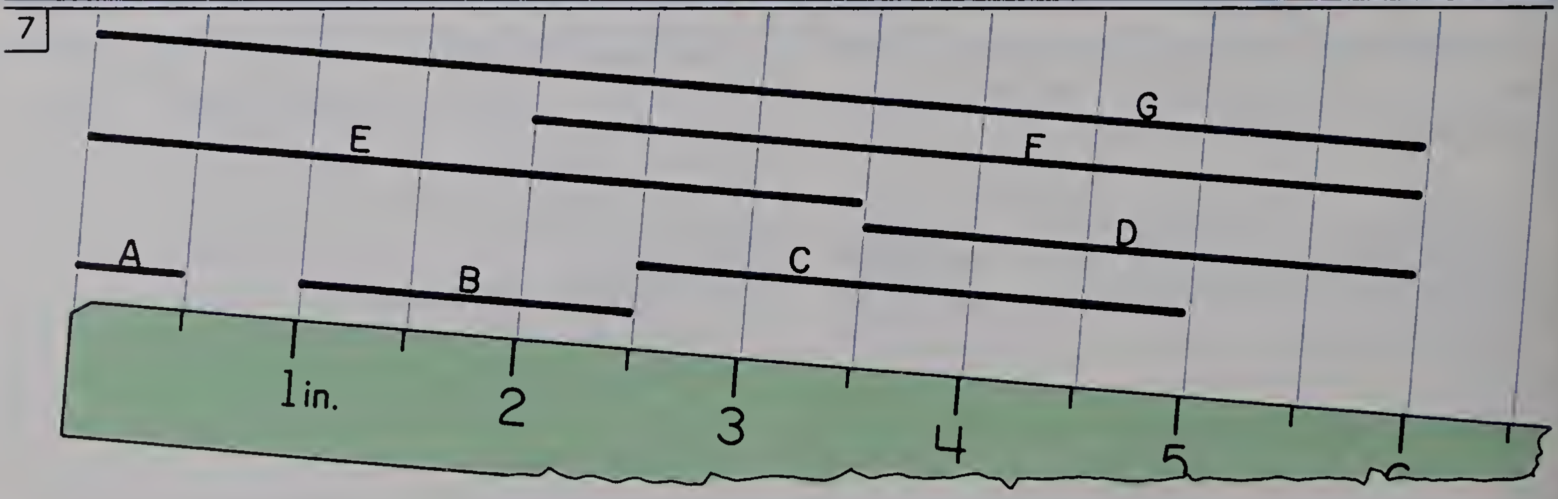
(B) 201

(C) 210

(D) 102

A ○ B ○ C ● D ⊗ 108

7



Sums of lengths:

A and B: 2 in.
G and F: 10 in.
A and D: 3 in.
B and C: 4 in.

A and C: 3 in.
E and B: 5 in.
E and D: 6 in.
E and C: 6 in.

Differences between lengths:

A and B: 1 in.
A and D: 2 in.
C and B: 1 in.
C and D: 0 in.

G and F: 2 in.
E and D: 1 in.
E and C: 1 in.
B and E: 2 in.

Notes on Pupil Page 94

Tasks: *In exercises 1-6 pupils read each statement and determine the correct answer, the answer furthest from the correct answer, and the difference between these answers. In exercise 7 pupils determine the length of segments A - G and record the various sums and differences of lengths as required.*

Purpose: To test pupil achievement.

Unifying Ideas: Numeration; addition and subtraction; geometry; measurement.

Materials for Teacher: See the Notes on Pupil Page 80.

Materials for Pupils: See the Notes on Pupil Page Page 80.

The Lesson:

Suggested point distribution for this test:

Exercises 1-6: 18 points
Exercise 7: 16 points

See the Notes on Pupil Page 80 for suggestions concerning using a ruler and measuring lengths.

Drill

Here are suggestions for a drill sheet:

Watch the Signs!

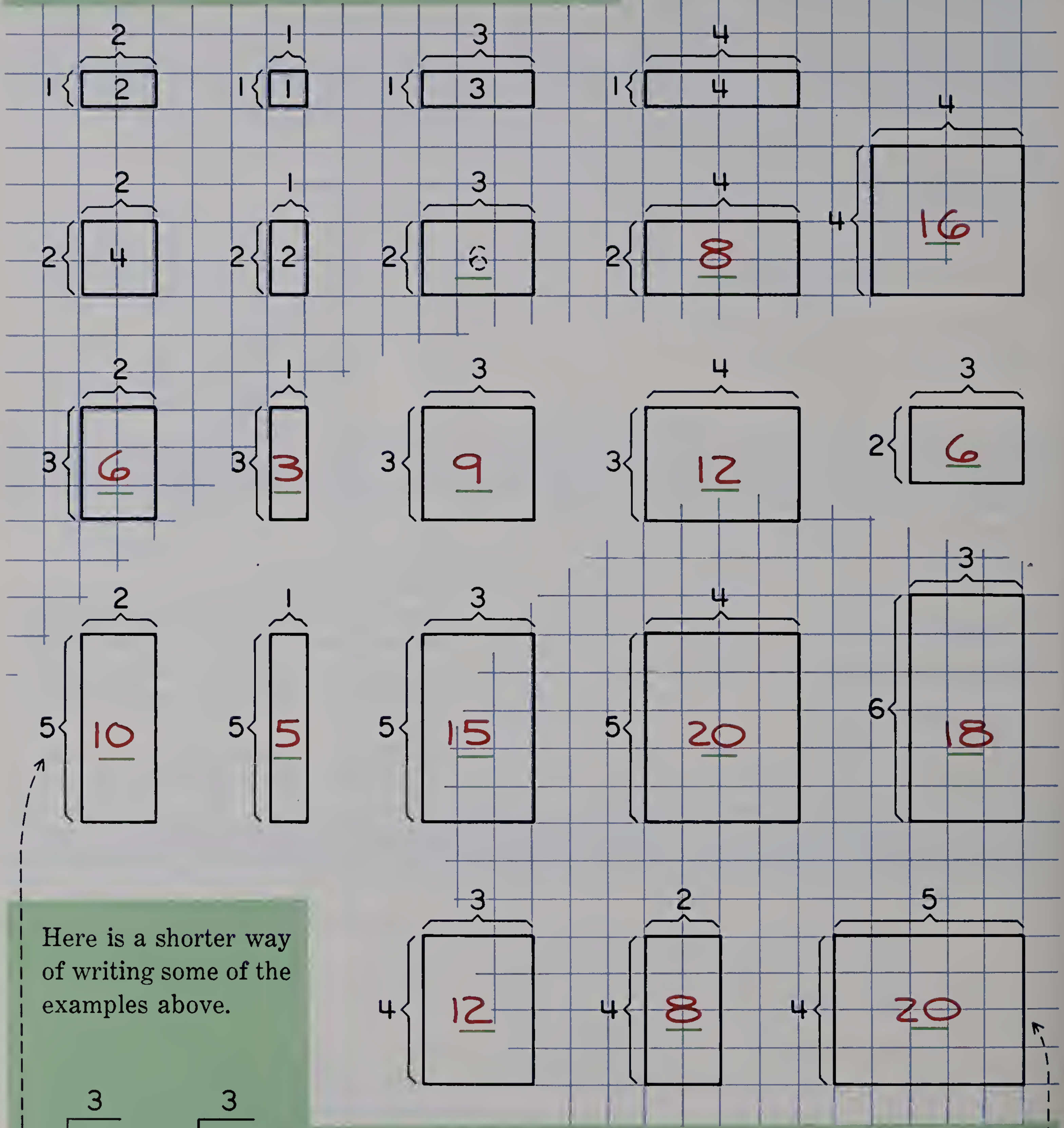
35 +7	69 +2	34 −7	29 +3	41 −6
55 −8	55 +8	63 −4	68 +5	51 −1
65 +5	48 −9	27 −8	52 +9	61 +3

Here are two stories for drill purposes:

1) Mitch had a yard of string. He wanted to tie up three boxes. For one he needed 11 inches of string, for another he needed 1 foot of string, and for the last he needed 13 inches of string.
2) Mildred bought 1 foot of ribbon to decorate her Mother's Day card. She only used 8 inches.

Headline for the Day: Compare 3 yards with 7 feet using < or > in a number sentence.

Find my rule. Then complete all the examples.


2
1
2
1
1
1
3
1
3
4
1
4
2
2
4
1
2
2
3
2
6
4
2
8
4
4
16
2
3
6
1
3
3
3
3
9
4
3
12
3
2
6
2
5
10
1
5
5
3
5
15
4
5
20
3
6
18
3
4
12
2
4
8
5
4
20

Here is a shorter way of writing some of the examples above.

$\begin{array}{r} 3 \\ 2\overline{)6} \end{array}$ $\quad$ $\begin{array}{r} 3 \\ 3\overline{)9} \end{array}$

$\begin{array}{r} 4 \\ 1\overline{)4} \end{array}$ $\quad$ $\begin{array}{r} 4 \\ 3\overline{)12} \end{array}$ $\quad$ $\begin{array}{r} 2 \\ 2\overline{)4} \end{array}$ $\quad$ $\begin{array}{r} 2 \\ 3\overline{)6} \end{array}$ $\quad$ $\begin{array}{r} 3 \\ 6\overline{)18} \end{array}$ $\quad$ $\begin{array}{r} 4 \\ 5\overline{)20} \end{array}$

$\begin{array}{r} 2 \\ 5\overline{)10} \end{array}$ $\quad$ $\begin{array}{r} 3 \\ 4\overline{)12} \end{array}$ $\quad$ $\begin{array}{r} 4 \\ 4\overline{)16} \end{array}$ $\quad$ $\begin{array}{r} 2 \\ 4\overline{)8} \end{array}$ $\quad$ $\begin{array}{r} 5 \\ 4\overline{)20} \end{array}$

Notes on Pupil Page 95

Tasks: *Pupils demonstrate their discovery of the rule by recording each of the required products. They also use some of the rectangular arrays to determine the required products at the bottom of pupil page 95.*

Purpose: (a) To give pupils practice in using the area representation for multiplication and division; (b) to introduce a sign for division.

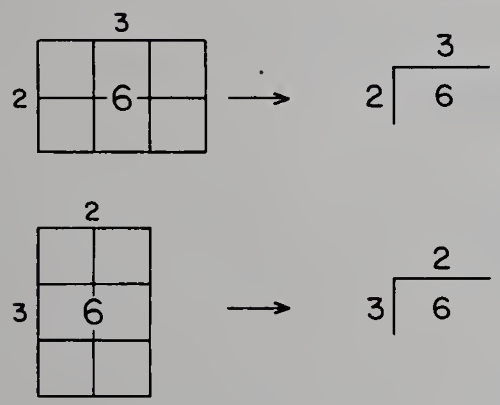
Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: None.

Materials for Pupils: Graph paper.

The Lesson:

Pupil page 95 introduces a division shorthand as another way to keep a record of columns and rows. In a sense this division sign can be looked at as a hasty sketch of a rectangular array:



This new shorthand should be compared with the multiplication shorthand to clarify two ideas:

1. Factors-Product
   The number of columns and the number of rows in an array are the factors, and the number of dots, crossing points, or small squares is the product.

2. Fact Teams
   Since it doesn't matter in which order we consider the columns and rows, any given picture can be the model for four statements:

$$3 \times 2 = 6 \qquad \begin{array}{r} 3 \\ 2\overline{)6} \end{array}$$

$$2 \times 3 = 6 \qquad \begin{array}{r} 2 \\ 3\overline{)6} \end{array}$$

Graph Paper Activities

Draw rectangles which bound 12 little squares. Each of the possible drawings reveals a team of multiplication and division facts:

$$1 \times 12 = 12 \qquad 3 \times 4 = 12$$

$$12 \times 1 = 12 \qquad 4 \times 3 = 12$$

$$\begin{array}{r} 12 \\ 1\overline{)12} \end{array} \quad \begin{array}{r} 1 \\ 12\overline{)12} \end{array} \qquad \begin{array}{r} 4 \\ 3\overline{)12} \end{array} \quad \begin{array}{r} 3 \\ 4\overline{)12} \end{array}$$

$$2 \times 6 = 12$$

$$6 \times 2 = 12$$

$$\begin{array}{r} 6 \\ 2\overline{)12} \end{array} \quad \begin{array}{r} 2 \\ 6\overline{)12} \end{array}$$

more

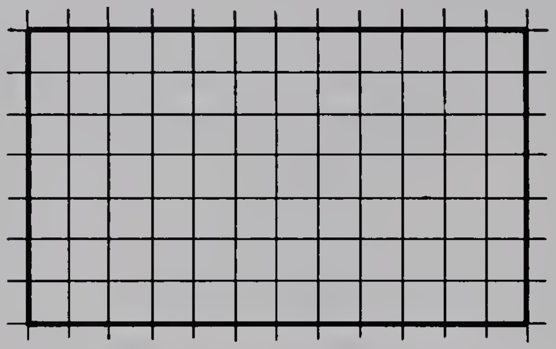
Headline for the Day: A team has 3 catchers and 4 pitchers:
Catchers: Jones, Robb, and Lewis
Pitchers: Kelly, Cohen, Burns, and Dunn
A battery in baseball is a catcher-pitcher combination. How many different choices of batteries does the manager have? (12) Name the batteries.

For each drawing, the pupil should write all four facts.

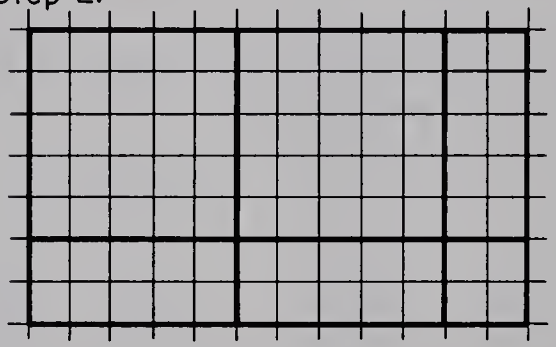
Repeat this activity for other numbers—18, 20, 25, etc.

Use graph paper to discover multiplication facts. For example, let's find 7 × 12:

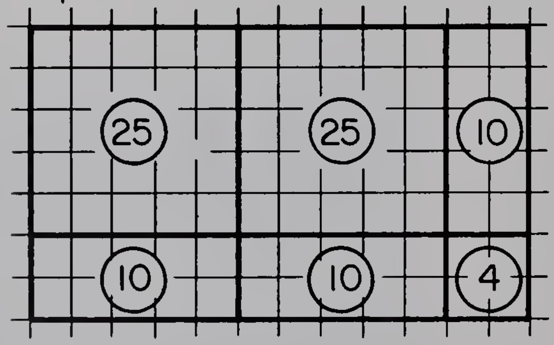
Step 1.



Step 2.



Step 3.



Step 4.
Count: 25, 50, 60, 70, 80, 84

Step 5.
Fact: 7 × 12 = 84

Multiplication and Pairing

Have pupils try some hard illustrations of pairing. Let us start with 3 groups of letters. There are 2 letters in each group. How many 3-letter combinations can we make if we take 1 letter from each group? (8) If Mary has 4 hats, 3 blouses, and 2 skirts, how many hat-blouse-skirt combinations can she devise? (24) If there are 4 sopranos, 4 contraltos, and 4 altos, how many trios can be arranged with a soprano, an alto, and a contralto in each? (64)

Drill

Here are suggested drill sheets:

(30 + 6) + (20 + 9) = □ + (6 + 9) = △ + 5
(20 + 3) + (40 + 9) = □ + (3 + 9) = △ + 2
(30 + 5) + (20 + 8) = □ + (5 + 8) = △ + 3
(40 + 3) + (30 + 8) = □ + (3 + 8) = △ + 1
(60 + 4) + (20 + 8) = □ + (4 + 8) = △ + 2
(30 + 5) + (40 + 7) = 70 + (5 + □) = △ + 2
(20 + 8) + (60 + 7) = 80 + (□ + 7) = 90 + △

Subtract.

10 − 4	7 − 3	12 − 5	13 − 9	8 − 3
13 − 8	11 − 8	12 − 3	10 − 7	11 − 0
7 − 4	7 − 3	13 − 5	9 − 5	12 − 7
9 − 6	10 − 3	13 − 7	8 − 5	12 − 4
13 − 6	12 − 8	13 − 4	12 − 6	12 − 9

Subtract.

15	23	11	34	28
− 6	− 8	− 9	− 7	− 9
26	42	59	37	62
− 7	− 3	− 9	− 8	− 6
53	21	34	28	46
− 7	− 8	− 9	− 2	− 0

Here are some stories for drill purposes:

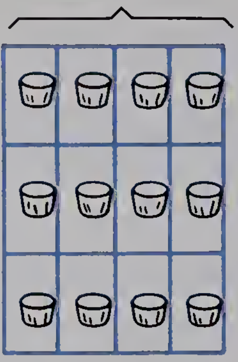
1) The class library has 3 animal books and 5 adventure books. Susan is allowed to take home one book of each kind.
2) Theodore's birthday is March 17. Today is March 12.
3) Sol walks 11 blocks to school. Carmen walks 13 blocks.
4) Tina packed 18 books in boxes. She packed 6 books in each box.

This is about sharing one dozen chocolates.

A.

4 for each child

3 children



Bill

Mary

Fred

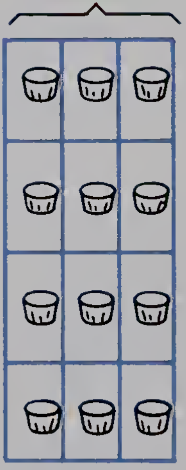
$\begin{array}{r} 4 \\ 3\overline{)12} \end{array}$ $\begin{array}{r} 3 \\ 4\overline{)12} \end{array}$

3 x 4 = 12 = 4 x 3

B.

3

4



Bill

Mary

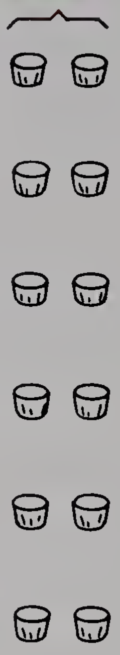
Fred

Sarah

C.

2

6



Bill

Mary

Fred

Sarah

Jose

Ruth

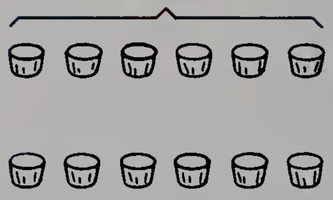
$\begin{array}{r} 2 \\ 6\overline{)12} \end{array}$ $\begin{array}{r} 6 \\ 2\overline{)12} \end{array}$

6 x 2 = 12 = 2 x 6

D.

6

2



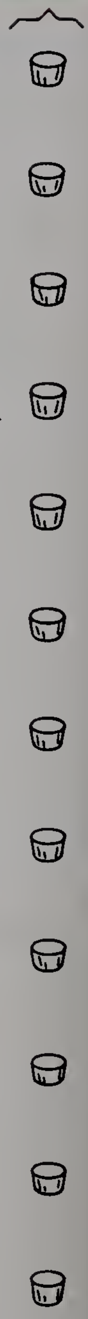
Bill

Mary

E.

1

12



Bill

Mary

Fred

Sarah

Jose

Ruth

Dino

Bob

Milt

Ellen

Carla

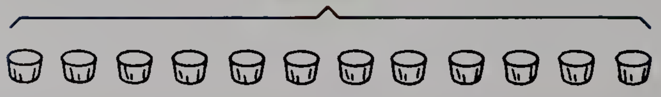
Dora

F.

12

1

Bill



$\begin{array}{r} 12 \\ 1\overline{)12} \end{array}$ $\begin{array}{r} 1 \\ 12\overline{)12} \end{array}$

1 x 2 = 12 = 12 x 1

G.

23	55	15	27	68	77	90	83	84	92
+54	+35	+68	+57	+24	−23	−35	−15	−57	−68
77	90	83	84	92	54	55	68	27	24

Notes on Pupil Page 96

Tasks: *Pupils use the displays of objects in rectangular arrays to determine factors and products as required. Pupils also compute and record at the bottom of pupil page 96 the sums and differences as required. Hopefully, they will notice that the addition and subtraction problems in exercise G are related.*

Purpose: (a) To further extend pupils' understanding of multiplication and division; (b) to provide pupils with further opportunities to perfect their skills and techniques with the addition and subtraction algorithms.

Unifying Ideas: Number and counting; addition and subtraction; multiplication and division.

Materials for Teacher: Demonstration counters; flannel board.

Materials for Pupils: Counters.

The Lesson:

This page deals with problems of sharing a dozen chocolates among 3 children, among 4 children, among 6 children, among 2 children, among 12 children, and "among" 1 child. The pictorial representations show that each sharing problem is basically a multiplicative situation.

Prepare for this page by using 18 counters and asking pupils to pretend that the counters are chocolates. If we are to share these 18 chocolates equally among 3 children, how many should each child get? Lay the counters out column by column:

Tom O O O
Sue O O etc.
Ed O O

If we wish to give 2 chocolates to each child, how many children can we give these 18 chocolates to? Lay the counters out row by row:

O O
O O
O O
etc.

It is helpful if pupils try to solve these problems by laying out the counters in a rectangular array. It is this approach to the problems that reveals their multiplicative nature.

Division Notation

You may wish to tell pupils that the notation:

$$3\overline{)\,12}$$

is read as 12 divided by 3. Also, you may wish to anticipate pupil page 98 by telling pupils that another way of writing this is:

$$12 \div 3$$

Drill

Here is a suggested drill sheet.

Add.

51 +29	69 +22	38 +47	18 +19	25 +17
68 +13	47 +25	36 +47	58 +29	29 +58
49 +11	59 +12	26 +57	29 +53	43 +39

Here are two stories for drill purposes:

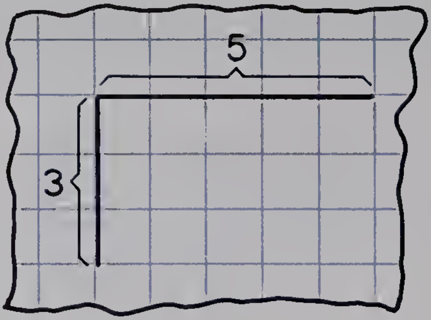
1) Nancy bought 2 yards of material to make a dress and 3 feet of material to make a blouse.
2) Tessie walked 3 blocks east and then 7 blocks west.

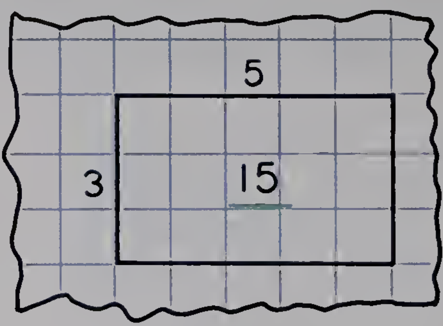
Headline for the Day: Write a number sentence that fits: Mary needs one dozen cookies if she and her 5 guests are to have ____ cookies each.

Problem

3 x 5 =

$\begin{array}{r} 5 \\ 3\overline{)} \end{array}$





Answer

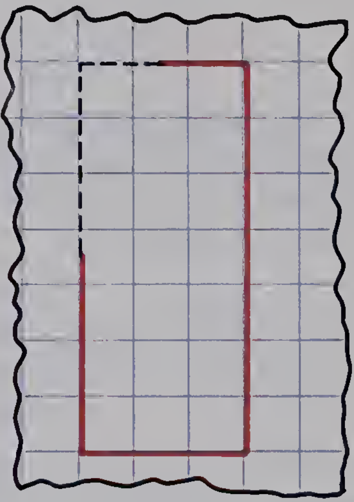
$\begin{array}{r} 5 \\ 3\overline{)15} \end{array}$

3 x 5 = 15

A.

7 x 3 = 21

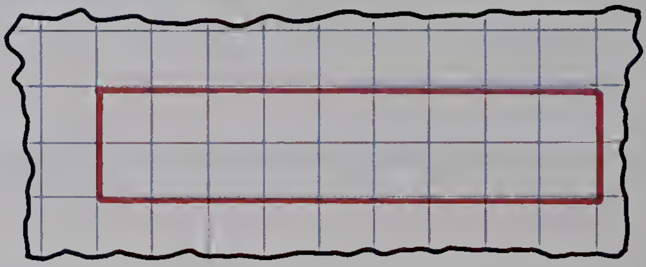
$\begin{array}{r} 3 \\ 7\overline{)21} \end{array}$



B.

2 x 9 = 18

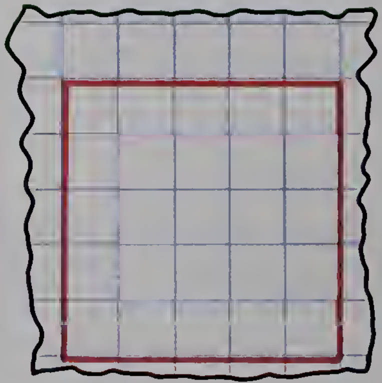
$\begin{array}{r} 9 \\ 2\overline{)18} \end{array}$



C.

5 x 5 = 25

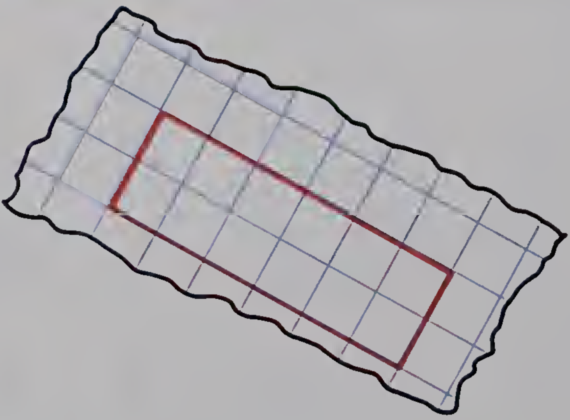
$\begin{array}{r} 5 \\ 5\overline{)25} \end{array}$



D.

2 x 6 = 12

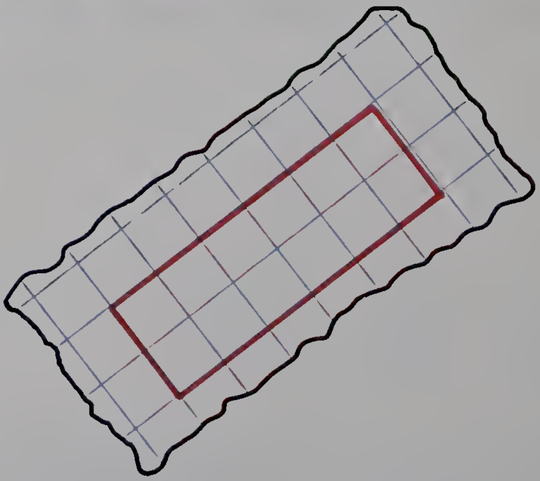
$\begin{array}{r} 6 \\ 2\overline{)12} \end{array}$



E.

6 x 2 = 12

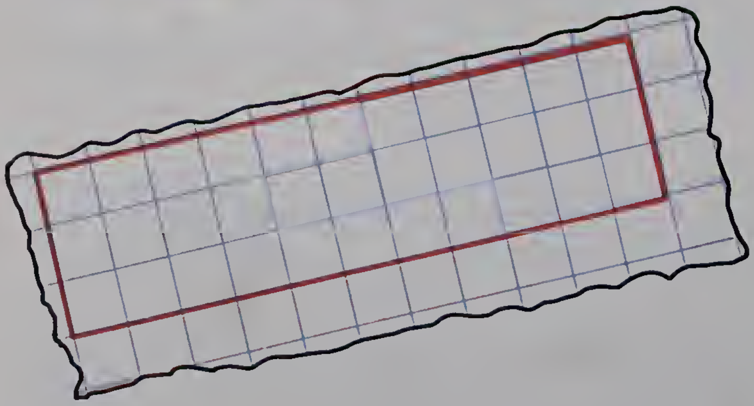
$\begin{array}{r} 6 \\ 2\overline{)12} \end{array}$



F.

11 x 3 = 33

$\begin{array}{r} 3 \\ 11\overline{)33} \end{array}$



G. $\begin{array}{r} 2 \\ 9\overline{)18} \end{array}$

H. $\begin{array}{r} 7 \\ 3\overline{)21} \end{array}$

I. $\begin{array}{r} 11 \\ 3\overline{)33} \end{array}$

Notes on Pupil Page 97

Tasks: *Pupils draw rectangles in exercises A-F to find multiplication and division facts. In G-I they use their awareness of the fact the multiplication is commutative to record multiplication and division facts (B→G, A→H, F→I).*

Purpose: To give pupils practice in using the area representation to find multiplication and division facts.

Unifying Ideas: Number and counting; multiplication and division.

Materials for Teacher: A penny, a nickel, two dimes, a quarter, and a half-dollar; two boxes.

Materials for Pupils: None.

The Lesson:

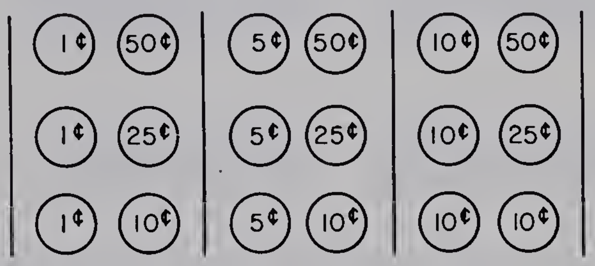
On this page pupils are asked to draw their own column-row representations of multiplication and division facts. Not only is this a check on understanding; it is also a reminder to the pupil that when he can't remember a fact, he can return to a model and work out the correct answer. If there is time, have pupils draw crossing points and dot arrays as representations of the facts.

Pairing Coins

Name	Box A	Box B	Total
Mary	1¢	10¢	11¢
John	5¢	25¢	30¢
Bill	10¢	50¢	60¢
Barbara	5¢	10¢	15¢
Alex	10¢	10¢	20¢
Jean	5¢	50¢	55¢
Bob	1¢	50¢	51¢
William	10¢	25¢	35¢
Helen	1¢	25¢	26¢

In one box put one penny, one nickel, and one dime. In another box put one dime, one quarter, and one half-dollar. Have individuals from the class close their eyes and pick one coin from each box. Make a record of the coins and total amount of money each pupil has picked as shown in the last column.

The above chart includes the nine combinations that are possible. After you have recorded the first nine draws, ask if there are any repetitions. If there are, eliminate them and continue as before. When the class has come up with six or seven unique combinations, ask: Can you think of any other possible combinations that we haven't had yet? Pupils may find it easier to find combinations if they are led to make a systematic record of the possibilities. For example:



Repeat this activity with fewer than three coins in one or both boxes. Keep reducing or changing the number of coins—2 and 3, 3 and 2, 2 and 2, 1 and 2, 2 and 1, 1 and 1, 0 and 3, 2 and 0.

When a record table is finished, erase the amounts in the columns for boxes A and B and ask: Can we tell which coins each pupil drew if we know only the total amount? (Of course!)

Drill

Here are two stories for drill purposes:

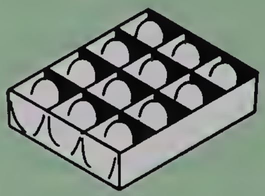
1) Fred brought 3 trucks to class. He took the 4 wheels off each truck.
2) Sue Ellen bought 3 ice-cream cones for 10¢ each.

Headline for the Day: Menu:

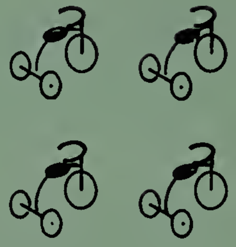
Sandwiches		*Drinks*	
Hamburger	25¢	Milk	10¢
Hot Dog	20¢	Pop	5¢
Cheese	15¢		

In how many ways can you order a sandwich and a drink? How much will each combination cost?

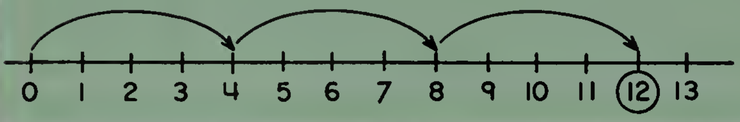
FACT TEAMS — MULTIPLICATION AND DIVISION

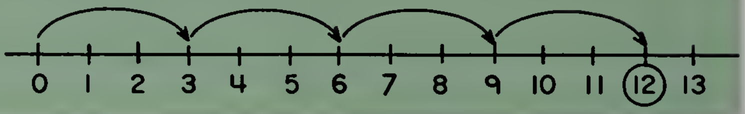


4 + 4 + 4 = 12



$$\begin{array}{r} 3 \\ 3 \\ 3 \\ +\ 3 \\ \hline 12 \end{array}$$


0 1 2 3 4 5 6 7 8 9 10 11 12 13


0 1 2 3 4 5 6 7 8 9 10 11 12 13

Each picture is about 3, 4, and 12. Here are some shorthand ways of writing what the pictures tell us.

a. 3 x 4 = 12

b. 4 x 3 = 12

c. $\begin{array}{r} 4 \\ \times 3 \\ \hline 12 \end{array}$

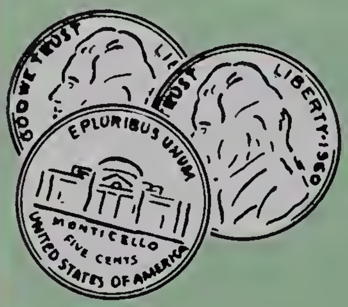
d. $\begin{array}{r} 3 \\ \times 4 \\ \hline 12 \end{array}$

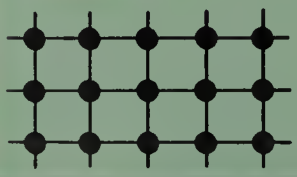
e. $\begin{array}{r} 4 \\ 3\overline{)12} \end{array}$

f. $\begin{array}{r} 3 \\ 4\overline{)12} \end{array}$

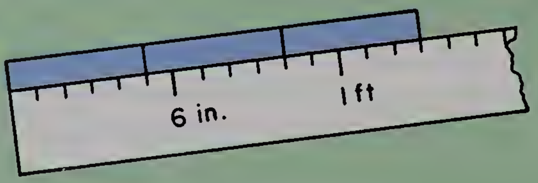
g. 12 ÷ 4 = 3

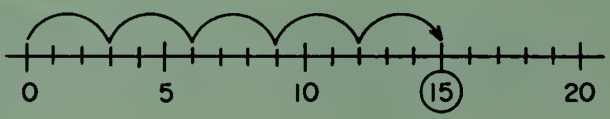
h. 12 ÷ 3 = 4


IN GOD WE TRUST
LIBERTY 1960
E PLURIBUS UNUM
MONTICELLO
FIVE CENTS
UNITED STATES OF AMERICA



$$\begin{array}{r} 3 \\ 3 \\ 3 \\ 3 \\ +\ 3 \\ \hline 15 \end{array}$$





3 + 3 + 3 + 3 + 3 = 15

5 + 5 + 5 = 15

$$\begin{array}{r} 5 \\ 5 \\ +\ 5 \\ \hline 15 \end{array}$$

Each of these pictures is about 3, 5, and 15. Fill the blanks in these shorthand ways of writing what the pictures tell you.

i. 3 x 5 = 15

j. 5 x 3 = 15

k. 15 ÷ 3 = 5

l. 15 ÷ 5 = 3

m. $\begin{array}{r} 5 \\ \times 3 \\ \hline 15 \end{array}$

n. $\begin{array}{r} 3 \\ \times 5 \\ \hline 15 \end{array}$

o. $\begin{array}{r} 3 \\ 5\overline{)15} \end{array}$

p. $\begin{array}{r} 5 \\ 3\overline{)15} \end{array}$

Notes on Pupil Page 98

Tasks: Pupils study the upper half of the page and then complete the various number sentences in the bottom half.

Purpose: (a) To emphasize that multiplication and division facts can have a variety of representations; (b) to introduce the division sign ÷ .

Unifying Ideas: Multiplication and division.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

How can we find the product of 4 by 5? There are many ways. Count the wheels on 5 wagons. Draw 4 horizontal lines, 5 vertical lines, and count the crossing points. Start at 0 and make 5 jumps of 4 on the number line, or make 4 jumps of 5. Write down five 4's in a column and add. Find out the value in pennies of 4 nickels, etc.

Once one knows what number 4 × 5 is, he is ready to state three more facts:

$$5 \times 4 = 20 \qquad 20 \div 5 = 4 \qquad 20 \div 4 = 5$$

To know how to multiply is to know how to divide. To know how to add is to know how to subtract.

Find the Pair

$$\square + \triangle = 7 \quad \text{and} \quad \square \times \triangle = 10$$

Find numbers for the □ and △ to make this compound sentence true. Remember that in any sentence with frames, you must use the same number for frames of the same shape and size.

Ask Bill to write down a pair of numbers, and tell the class what the sum is and what the product is. When Bill announces the sum and the product, write a compound sentence such as the one above. The class must discover Bill's numbers.

How Many Batteries?

A baseball team has 4 pitchers and 4 catchers. How many different batteries can they plan on using? (A battery is a pitcher-catcher combination.) Another team has 4 pitchers and 3 catchers. How many fewer batteries than the first team can the second team make? (4 × 4 = 16, 4 × 3 = 12, and 16 − 12 = 4; four less.) The chart below records the number of possible combinations which can be made if the number of pitchers is 0, 1, 2, 3, or 4 and the number of catchers is 0, 1, 2, 3, or 4. (In all cases we assume that no pitcher is a catcher.)

Pitchers

Catchers	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	6	8
3	0	3	6	9	12
4	0	4	8	12	16

How Many Combinations of Fingers?

In how many ways can you combine a finger of your right hand with a finger of your left hand? (25)

Draw the grid shown below. Ask a pupil to make a combination. Record it. Etc.

Right Hand

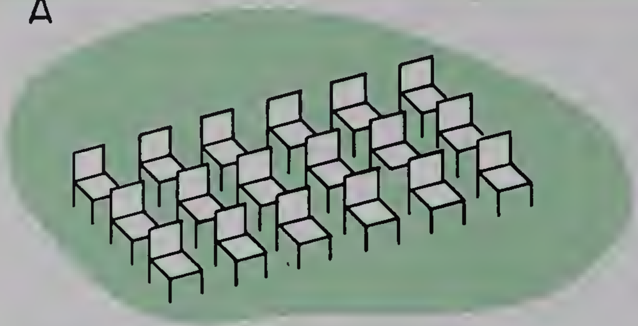
Left Hand	Thumb	Fore	Middle	Ring	Little
Thumb					
Fore					
Middle					
Ring					
Little					

What happens if we don't count thumbs? (Only 16 combinations.) Suppose that we don't count thumbs or little fingers? (Only 9 combinations.) Whenever a finger is eliminated from both hands, the number of possible combinations is still the square of the number of remaining fingers on each hand.

Headline for the Day: Write stories suggested by the number sentences on page 98.

Each picture suggests a fact team of four sentences. Complete the sentences.

A



Multiplication sentences

3 x 6 = 18

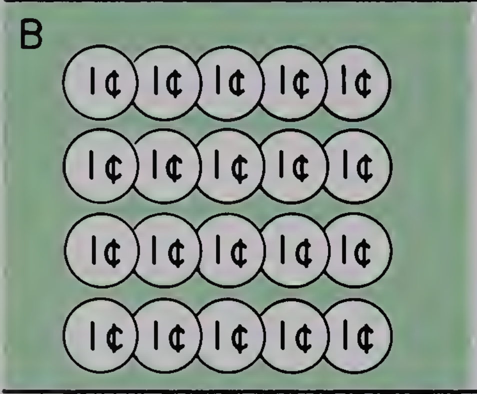
6 x 3 = 18

Division sentences

18 ÷ 6 = 3

18 ÷ 3 6

B



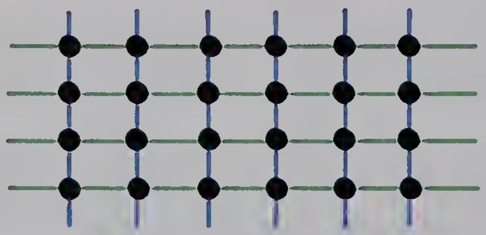
Multiplication sentences

$$\begin{array}{r} 4 \\ \times\ 5 \\ \hline 20 \end{array} \qquad \begin{array}{r} 5 \\ \times\ 4 \\ \hline 20 \end{array}$$

Division sentences

$$\begin{array}{r} 4 \\ 5\overline{)20} \end{array} \qquad \begin{array}{r} 5 \\ 4\overline{)20} \end{array}$$

C



Multiplication

4 x 6 = 24

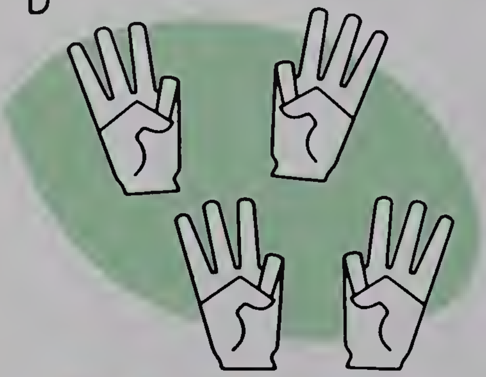
6 x 4 = 24

Division

24 ÷ 6 = 4

24 ÷ 4 = 6

D



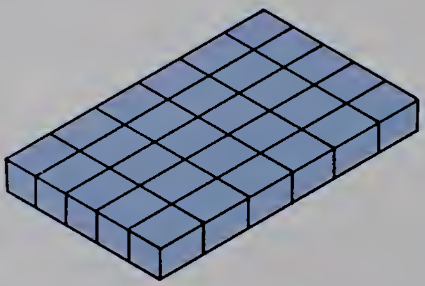
3 x 4 = 12

4 x 3 = 12

12 ÷ 4 = 3

12 ÷ 3 = 4

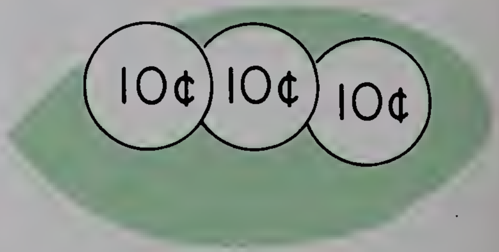
E



$$\begin{array}{r} 5 \\ \times\ 6 \\ \hline 30 \end{array} \qquad \begin{array}{r} 6 \\ \times\ 5 \\ \hline 30 \end{array}$$

$$\begin{array}{r} 5 \\ 6\overline{)30} \end{array} \qquad \begin{array}{r} 6 \\ 5\overline{)30} \end{array}$$

F



3 x 10 = 30

10 x 3 = 30

30 ÷ 3 = 10

30 ÷ 10 = 3

G

2 x 3 = 6 15 ÷ 3 = 5 2 x 4 = 8 3 x 3 = 9 4 x 4 = 16

Tasks: Pupils complete the various number sentences in the fact team suggested by each picture.

Purpose: To provide pupils with practice in interpreting representations of multiplication.

Unifying Ideas: Multiplication and division.

Materials for Teacher: None.

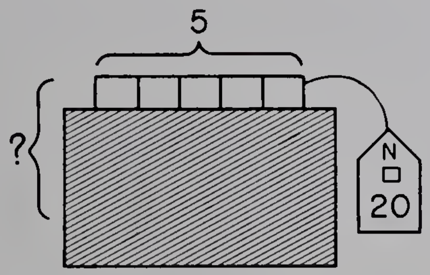
Materials for Pupils: None.

The Lesson:

Pupils may work on this page without specific preparation.

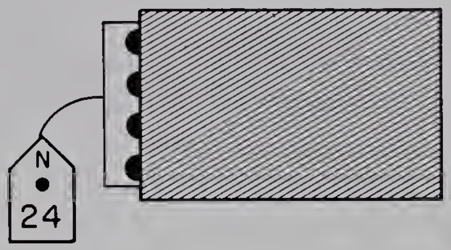
Multiplication and Division

Here is an activity which suggests the relationship between multiplication and division.



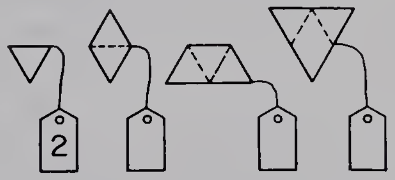
This picture represents 20 squares, 5 to a row and 4 to a column. The squares are covered. As the cover is lowered to show the first row of squares, ask: How many rows of little squares will there be? This question is much like the question: What is 20 ÷ 5? It is also like the question: If the product is 20 and one factor is 5, what is the other factor?

The following picture is similar to the one above:



There are 4 rows of dots and the tag tells us there are 24 dots in all. When we remove the cover, how many columns of dots will there be?

The If-Then Game



If we say that the size of the first shape suggests 2, as the label indicates, then how should we label the other shapes? The tags should read 2, 4, 6, and 8. Erase the 2, the 6, and the 8, leaving the 4. Ask pupils: Suppose that we were given the number 4 in

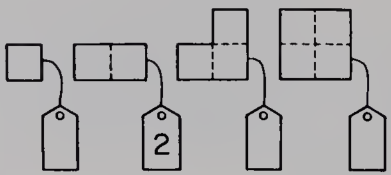
more

Headline for the Day: Write number stories suggested by the number sentences on page 99.

the second tag. How would we number the other tags? The same system of labeling would result: 2, 4, 6, and 8. Repeat the activity, starting with only the tag labeled 6, then with only the tag labeled 8.

Erase all labels and put a 6 in the second tag. If the tag on the second shape were marked with a 6 then how would we label the tags for the other shapes? Is there a shape twice as large as the one we have labeled with a 6? (Yes) What should its label be? (12) Now, how do we label the other tags? (3 and 9)

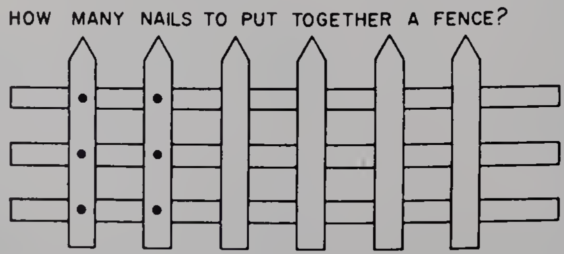
If some pupils are confused, draw another diagram on the board, such as:



Repeat the development as already outlined. There is a subtlety in this activity that may properly disturb some pupils. They are used to counting things —a dot is a dot, regardless of its size. Therefore, it must be made clear that the ground rules have been changed. Emphasize the fact that we are concerned here with size. Just as it is possible to arrange collections of things according to the number of objects in them, it is also possible to arrange objects according to their size.

How Many Nails?

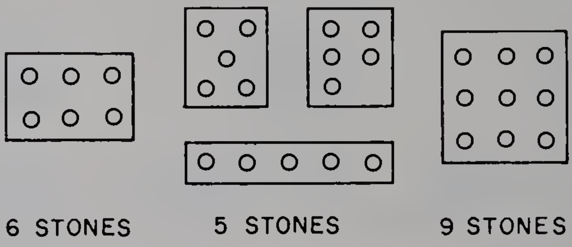
Ask the class to tell you how many nails it is going to take to put together a fence like this:



Cross Pieces	Uprights	Nails
3	2	6
3	3	9
3	4	12
3	5	15
3	6	18

Prime Numbers and Composite Numbers

Can 6 stones be arranged in rows and columns so that all rows are alike and all columns are alike? If we agree that there must be at least 2 stones in each row and column, can 5 stones be arranged in rows and columns so that all rows are alike and all columns are alike? (No) Can 9 stones be so arranged? (Yes)



2 stones—No	2 × 1	1 × 2
3 stones—No	3 × 1	1 × 3
4 stones—Yes	2 × 2	2 × 2
5 stones—No	5 × 1	1 × 5
6 stones—Yes	2 × 3	3 × 2
7 stones—No	7 × 1	1 × 7
8 stones—Yes	4 × 2	2 × 4
9 stones—Yes	3 × 3	3 × 3
10 stones—Yes	2 × 5	5 × 2
11 stones—No	1 × 11	11 × 1
12 stones—Yes	3 × 4 or 2 × 6	4 × 3 or 6 × 2

The No answers indicate prime numbers. The Yes answers indicate nonprime or composite numbers. Have teams of pupils test numbers from 13 on.

As the pupils study pupil page 96, you can point out that 12 cookies could be divided equally among 1, 2, 3, 4, 6, and 12 persons. Which numbers of cookies could not be divided into groups of equal size if each person must get at least two cookies but not all of them? (Breaking cookies doesn't count.)

Using Thought Starters

Many children are helped into a creative writing activity by thought starters. These thought starters can be a few key words—fog, vacation, secret, etc.; or phrases—buying a new dress, a trip to Mars, the talking frog, etc. Mathematical sentences can also be used as thought starters:

$2 + 3 = 5$ $17 - 16 = 1$

$24 = 3 \times 8$ $57 \div 3 = 19$

Incomplete mathematical sentences are, for some, even more effective in initiating a chain reaction that may result in a story or poem:

$12 = \square + 3$ $\qquad 3 \times \square = 21$

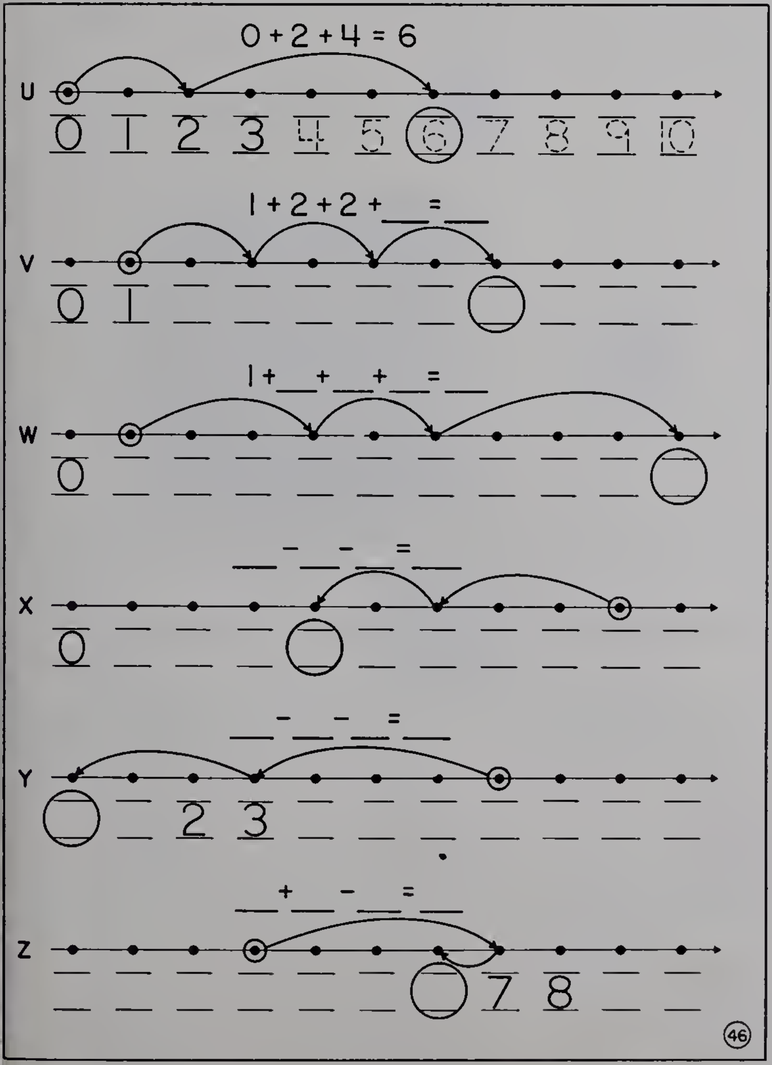
$18 - \square = 9$ $\qquad 24 \div 4 = 3 \times \square$

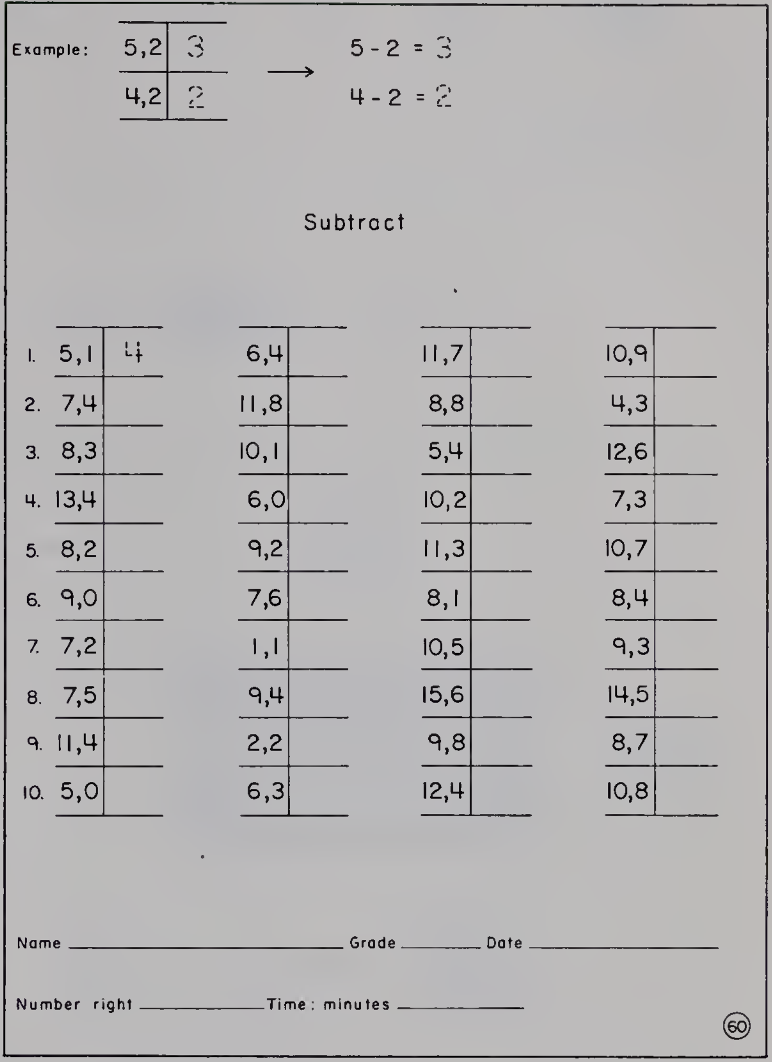
Using mathematical sentences in a language arts context helps pupils to understand how numbers and the operations of arithmetic can be used to suggest real-life situations. Other thought starters which contain ideas from arithmetic might be:

There were 4 sleds and twice as many boys.
14 men set out on a trip to the North Pole.
The score was tied.
I told you a thousand times.
The stream was 10 feet wide.
One too many

Drill

Consider using as drill sheets pages 46 and 60 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:





Example:

5,2	3
4,2	2

→ 5 - 2 = 3
4 - 2 = 2

Subtract

1.	5,1	4	6,4		11,7		10,9	
2.	7,4		11,8		8,8		4,3	
3.	8,3		10,1		5,4		12,6	
4.	13,4		6,0		10,2		7,3	
5.	8,2		9,2		11,3		10,7	
6.	9,0		7,6		8,1		8,4	
7.	7,2		1,1		10,5		9,3	
8.	7,5		9,4		15,6		14,5	
9.	11,4		2,2		9,8		8,7	
10.	5,0		6,3		12,4		10,8	

Name ______________ Grade ________ Date ____________

Number right ________ Time: minutes ________

(60)

Another suggested drill sheet:

Add.

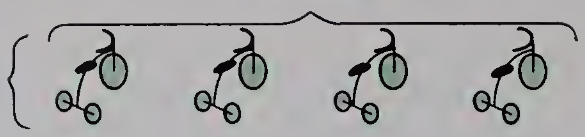
65 +19	57 +23	36 +48	39 +54	29 +59
56 +27	48 +25	67 +28	37 +29	35 +58
42 +49	28 +33	62 +24	34 +29	75 +16

Here are two stories for drill purposes:

1) Mother gave the clerk 5 one dollar bills and 2 quarters.
2) A toy tractor costs 40 cents. Ezra has 25 cents.

I

4 tricycles



3 wheels on each tricycle

a total of 12 wheels

III

7 days in a week

4 weeks

S	M	T	W	T	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28

There are 28 days in four weeks.

IV

5 days of school each week

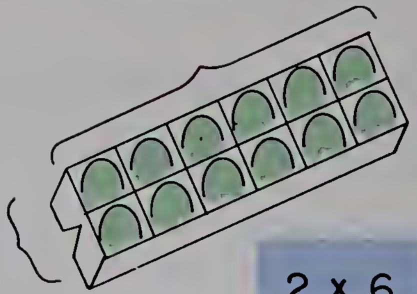
4 weeks

S	M	T	W	T	F	S
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28

There are 20 days of school in four weeks.

II

6 eggs in each row

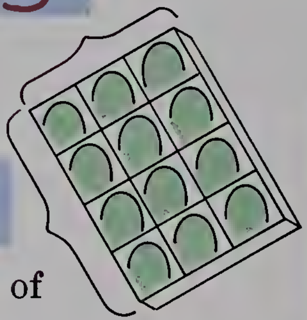


2 rows of eggs

2 x 6 = 12

6 x 2 = 12

3 eggs in each row



4 rows of eggs

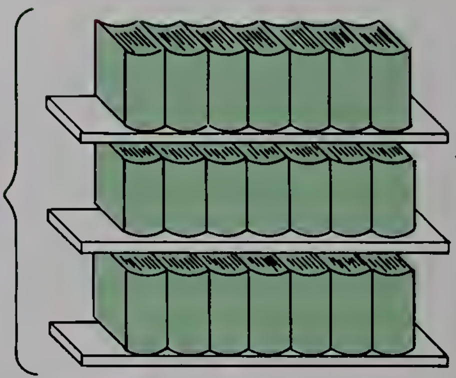
3 x 4 = 12

4 x 3 = 12

There are 12 eggs in a dozen.

V

7 books on each shelf



3 shelves

There are 21 books altogether.

VI

A $\begin{array}{r}6\\2\overline{)12}\end{array}$

B $\begin{array}{r}2\\6\overline{)12}\end{array}$

C $\begin{array}{r}7\\4\overline{)28}\end{array}$

D $\begin{array}{r}3\\4\overline{)12}\end{array}$

E $\begin{array}{r}7\\3\overline{)21}\end{array}$

F $\begin{array}{r}2\\1\overline{)2}\end{array}$

G $\begin{array}{r}10\\3\overline{)30}\end{array}$

H $\begin{array}{r}10\\4\overline{)40}\end{array}$

I $\begin{array}{r}4\\3\overline{)12}\end{array}$

J $\begin{array}{r}3\\3\overline{)9}\end{array}$

Notes on Pupil Page 100

Tasks: Pupils determine and record the missing factors and products from the clues supplied in the various pictures. In exercise VI they supply the missing products, hopefully from memory.

Purpose: (a) To provide practice in interpreting representations of multiplication; (b) to encourage pupils to memorize multiplication facts.

Unifying Ideas: Multiplication and division.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Pupils may proceed with this page without specific preparation.

Ten Little Indians

Have children sing the song that begins: One little, two little, three little Indians. Let's talk about an Indian boy. How many fingers on his right hand? (5) How many fingers on both hands?(10) How many tomahawks?(1) How many ears?(2) How many arms and legs?(4) How many fingers, not counting his thumbs?(8) How many toes, not counting his right big toe? (9) How many eyes, ears, noses, and mouths altogether?(6) He has 7 feathers in his headdress and 3 shiny teeth on his neckband. The same is true of all the other nine little Indian boys.

Make a record on the board to show how the numbers change with the addition of each Indian boy.

You may not have time to complete the record, but follow it through as far as you can. This humorous activity involves 121 multiplication facts, and many patterns are revealed in the chart. If the activity seems too lengthy, shorten it to suit your needs.

	Number of Indian Boys										
Number of:	0	1	2	3	4	5	6	7	8	9	10
Pockets (0)	0	0	0	0	0	0	0	0	0	0	0
Tomahawks (1)	0	1	2	3							
Ears (2)	0	2	4	6							
Teeth on neckband (3)	0	3	6	9							
Arms and legs (4)	0	4	8								
Fingers on right hand (5)	0	5	10								
Eyes, ears, nose and mouth (6)											
Feathers (7)											
Fingers on both hands, not counting thumbs (8)											
Toes, not counting right big toe (9)											
Fingers on both hands (10)											

Drill

Here are two stories for drill purposes:

1) The class needs 86 blocks to make a house. The children collected 57 blocks.
2) Fifteen children have to sit in 3 rows, with the same number of children in each row.

Headline for the Day: Make a number sentence to fit using < or > in the sentence: Which is more, 3 dozen eggs or 35 eggs?

	A		B	C	D	E
A wagon has	4	wheels.	4	4	4	4
So, on	2	wagons,	3	1	4	5
there is a total of	8	wheels.	12	4	16	20

	F		G	H	I	J
In a foot there are	12	inches.	12	12	12	12
So, in	3	feet,	2	0	1	4
there are	36	inches.	24	0	12	48

	K		L	M	N	O
A nickel is worth	5	¢.	5 ¢	5 ¢	5 ¢	5 ¢
Bill had	4	nickels.	3	2	5	7
Altogether Bill had	20	¢.	15 ¢	10 ¢	25 ¢	35 ¢

	P		Q	R	S	T
On a number line Maria took	2	jumps.	3	5	8	13
The size of each jump was	3	.	3	3	3	3
She started on 0 and landed on	6	.	9	15	24	39

	U		V	W	X
A drum costs	25	¢.	25 ¢	25 ¢	25 ¢
Mike bought	2	drums.	4	6	10
He paid	50	¢,	100 ¢	150 ¢	250 ¢
or $	.50	altogether.	$ 1.00	$ 1.50	$ 2.50

Tasks: Pupils determine and record the missing factors and products.

Purpose: To provide pupils with practice in multiplication and division in real-life situations but without pictures.

Unifying Ideas: Multiplication and division; measurement.

Materials for Teacher: Flash cards.

Materials for Pupils: None.

The Lesson:

All the exercises within each panel have a common factor. For example, 4 is a factor common to exercises A-E and 12 is common to exercises F-J. Thus, it is natural to expect the distributive principle for multiplication over addition to come into play. A pupil might add the products in B and C to obtain the product for D, and he might add the products for C and D (or for A and B) to obtain the product for E. Similarly, (L, M)→N, (M, N)→O, (K, L)→O, (P, Q)→R, (Q, R)→S, (R, S)→T, (U, V)→W, (V, W)→X.

Be sure that pupils understand the format of the exercises before they begin work. For example, in doing exercise B, they should think:

A wagon has 4 wheels.
So on 3 wagons,
there is a total of ? wheels.

Flash Cards

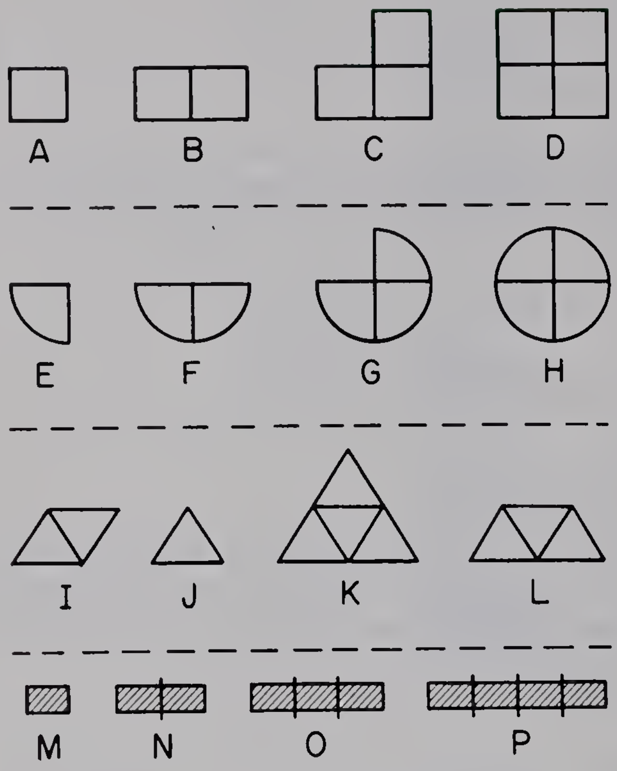
We suggest flash card drill for the multiplication facts up through 5 × 5 and the related division facts. See the Notes on Pupil Page 28 for suggestions on the use of flash cards.

If-Then

Here is an activity that gives pupils opportunities to practice the multiplication facts. Draw diagrams on the chalkboard like those on the next page.

more

Headline for the Day: In how many ways can 12 be divided into groups of the same size?



Then ask questions such as:

If A represents 5 pennies then what possible coins are represented by B, C, and D? B could represent 2 nickels, 1 dime, or 10 pennies.

If H represents a dozen eggs then what do E, F, and G represent?

Pairing Digits

Draw on the board a diagram like the one below.

Fill in the *Digit on the left* and *Digit on the right* columns. Have pupils give you the answers to put in the other columns.

Digit on the left	Digit on the right	Numbers	Total
1, 2	3	13, 23	2
2	0, 3	20, 23	2
1, 2	1, 2	11, 12, 21, 22	4
1, 2, 3	5	15, 25, 35	3
7, 8, 9	1, 2	71, 72, 81, 82, 91, 92	6
1, 2, 3	4, 5, 6	14, 15, 16, 24, 25, 26,	
		34, 35, 36	9

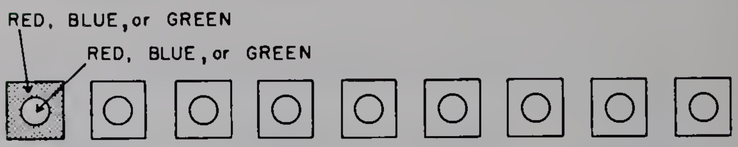
How Many Ways to Color?

How many ways are there to color the design in the next column, if the following conditions are met:

The region between the square and circle is to be red, blue, or green.

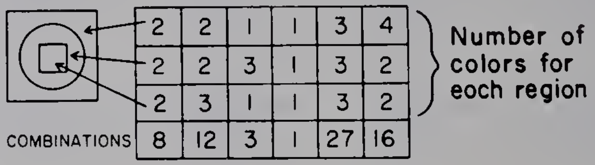
The region inside the circle is to be red, blue, or green.

Both regions must be colored.



There are 9 ways (because $3 \times 3 = 9$). Repeat the activity using 3 colors for the first region and 2 colors for the second region (result—6 combinations). To make the activity more difficult, increase the number of colors: 3 colors and 4 colors, for example, produce 12 possible combinations.

This activity can be extended to a design having three regions:

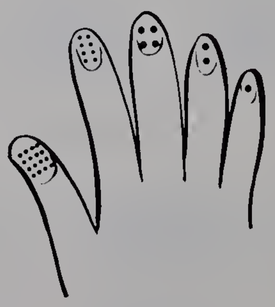


	2	2	1	1	3	4	Number of colors for eoch region
	2	2	3	1	3	2	
	2	3	1	1	3	2	
COMBINATIONS	8	12	3	1	27	16	

Fewest Fingers

In Quietland no one can talk because noise causes earthquakes. Consequently, the inhabitants have to count on their fingers. Of necessity they learned to count to 31 on the fingers of one hand—and to 1023 using both hands. How?

Well, they used to paint dots on their fingernails as shown below. (Of course, now they don't need these reminders!)



If the little finger is up alone, that shows 1. If the ring finger is up alone, that shows 2. If both are up, 2 + 1 or 3. If only the middle finger, that shows 4. What would show 5? (Middle finger and little finger.)

Here is how Quietlanders count to 31 on one hand. In the chart a 1 means that finger is up and a 0 means that finger is down. (You should draw the hand on the chalkboard.)

Number	Fingers				
	Thumb	Fore	Middle	Ring	Little
1	0	0	0	0	1
2	0	0	0	1	0
3	0	0	0	1	1
4	0	0	1	0	0
5	0	0	1	0	1
18	1	0	0	1	0
19	1	0	0	1	1
etc.					

(Compare with "Fewest Containers" in the Notes on Pupil Page 37.)

Drill

At the top of the next column is some developmental drill work.

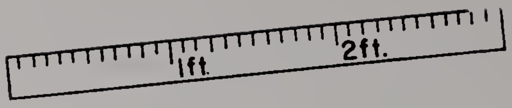
$$
\begin{aligned}
(80 + 4) - (20 + 6) &= (70 + \square) - (20 + 6) \\
&= (70 - 20) + (\square - 6) \\
&= \triangle \\
(60 + 2) - (30 + 9) &= (50 + \square) - (30 + 9) \\
&= (50 - 30) + (\square - 9) \\
&= \triangle \\
(70 + 2) - (40 + 8) &= (60 + \square) - (40 + 8) \\
&= (60 - 40) + (\square - 8) \\
&= \triangle
\end{aligned}
$$

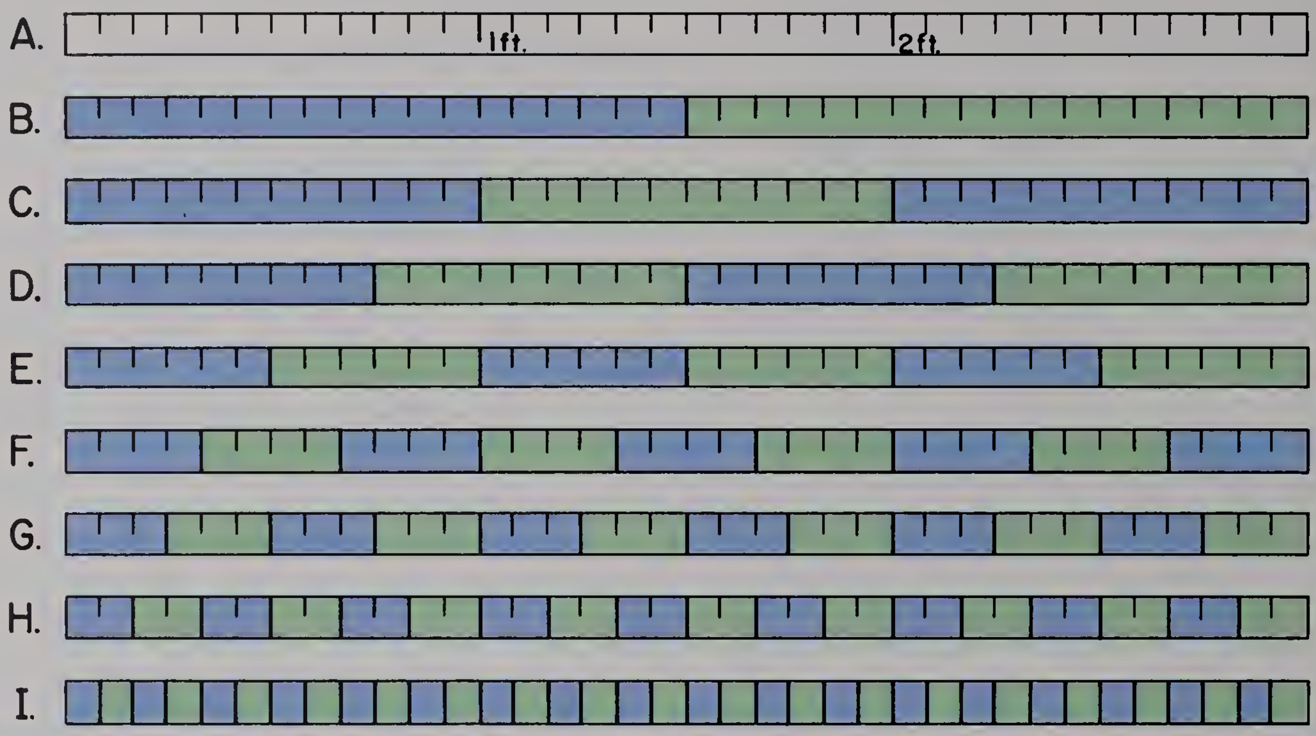
$$34 - 16 = \triangle \qquad 52 - 29 = \triangle \qquad 82 - 58 = \triangle$$

Here are two stories for drill purposes:

1) Monte went shopping with his father. He bought cookies for 32¢. His father bought grapes for 38¢ and milk for 27¢.
2) Milton ran up the staircase 2 steps at a time. The staircase had 12 steps.

Cutting a yardstick into pieces of the same length.




A.
1ft.
2ft.
B.
C.
D.
E.
F.
G.
H.
I.

B.
36 ÷ 2 = 18
or
36 ÷ 18 = 2

C.
36 ÷ 3 = 12
or
36 ÷ 12 = 3

D.
36 ÷ 4 = 9
or
36 ÷ 9 = 4

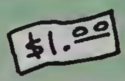
E.
36 ÷ 6 = 6
or
~~36 ÷ ___ = ___~~

F.
$\begin{array}{r} 4 \\ 9\overline{)36} \end{array}$ or $\begin{array}{r} 9 \\ 4\overline{)36} \end{array}$

G.
36 ÷ 12 = 3
or
36 ÷ 3 = 12

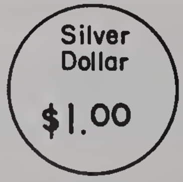
H.
$\begin{array}{r} 2 \\ 18\overline{)36} \end{array}$ or $\begin{array}{r} 18 \\ 2\overline{)36} \end{array}$

I.
36 ÷ 36 = 1
or
36 ÷ 1 = 36



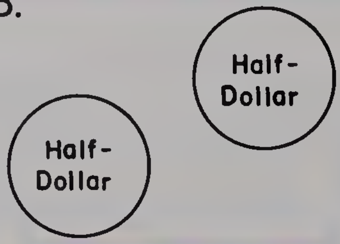
Exchange a dollar bill for coins of the same kind.

A.


Silver Dollar
$1.00

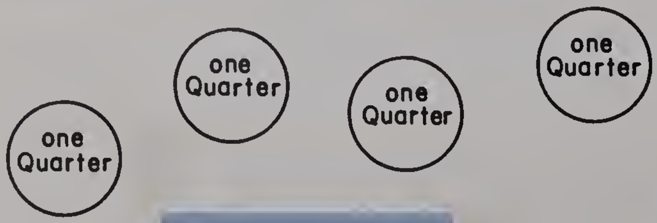
1 x 100 = 100

B.


Half-Dollar
Half-Dollar

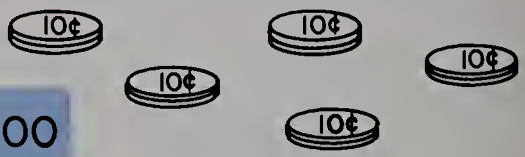
2 x 50 = 100

C.


one Quarter
one Quarter
one Quarter
one Quarter

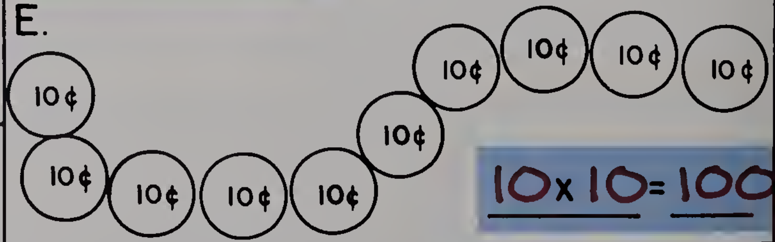
4 x 25 = 100

D. 5 stacks with 2 dimes in each stack.


10¢
10¢
10¢
10¢
10¢

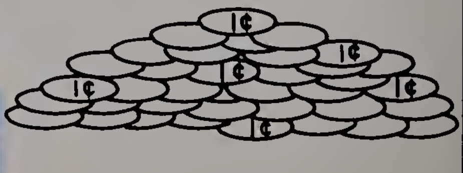
5 x 20 = 100

E.


10¢
10¢
10¢
10¢
10¢
10¢
10¢
10¢
10¢
10¢

10 x 10 = 100

F. A pile of pennies worth a dollar.



100 x 1 = 100

Notes on Pupil Page 102

Tasks: *Pupils determine the missing factors or products from the given clues.*

Purpose: To provide pupils with practice in using multiplication and division facts in concrete situations.

Unifying Ideas: Multiplication and division; measurement.

Materials for Teacher: Several demonstration yardsticks; legal-size Ditto masters.

Materials for Pupils: Dittoed sheets (legal-size); scissors.

The Lesson:

Prepare pupils for this page by distributing Dittoed sheets each showing six copies of a foot ruler marked in inches. Pupils should cut out the six copies of the ruler.

Cut a foot ruler into 2 pieces of the same length. How long is each piece? (6 inches) Cut a foot ruler into 6 pieces of the same length. How long is each piece? (2 inches)

Each of these two activities illustrates the fact team:

$$12 \div 2 = 6 \quad 12 \div 6 = 2 \quad 2 \times 6 = 12 \quad 6 \times 2 = 12$$

Cut a foot ruler into 3 pieces of the same length. How long is each piece? (4 inches)

If you cut a foot ruler into 4 pieces of the same length, how long would each piece be? Check your answer.

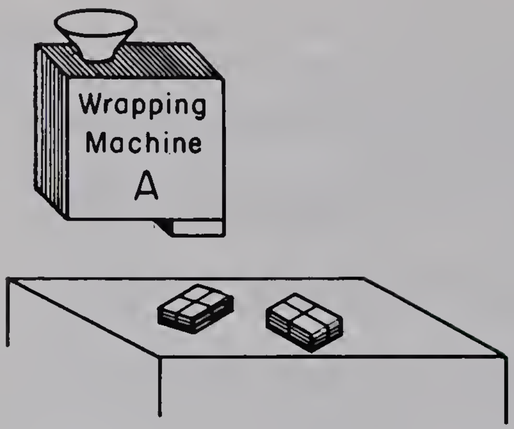
If you cut a foot ruler into 12 pieces of the same length, how long would each piece be? If you "cut" a foot ruler into 1 piece, how long is that piece?

Wrapping Machines

On the board, make a sketch such as the one on the next page, and label it Wrapping Machine A. Any sketch will do if it has a "place to put things in" (input) and a "delivery station" (output).

more

Headline for the Day: Which number doesn't belong with the others? 5, 18, 25, 15, 10, 35



Input	Output	
	Number of packages	Contents of each package
6 marbles	2	3 marbles
18 cookies		

We don't know what's inside this machine. We only know what we put in it and what comes out. If we put 6 marbles in, the machine gives us 2 packages, neatly wrapped. When we open them, we find that each contains 3 marbles. What kind of machine is this? Let's try it again. We put in 18 cookies. What do you suppose the machine delivers?

It turns out that this machine always tries to give us 2 packages with the same number of objects in each package. How many cookies do you suppose we'll find in each? 9, of course.

Now, let's see if we know what the machine will do in the following situations:

Input	Output	
	Number of packages	Contents of each package
6 marbles	2	3 marbles
18 cookies	2	cookies
30 pennies		
a dozen eggs		
10 trucks and		
4 cars		
a half-dozen		
apples		
2 dimes and 6 pennies		
3 nickels and 5 pennies		
		7 pencils
		25 stories 11 letters of the alphabet

The fourth item (a dozen eggs) may cause a feeling of doubt. Some pupils may think that the machine couldn't handle such an input any better than it could handle 1 marble, or 1 nickel. But most pupils will agree that the machine would give us 2 packages with 6 eggs in a package. The unit—a dozen eggs—can be regarded as 12 of a new unit—an egg.

With 10 trucks and 4 cars as the input, the machine might wear a puzzled look—if it had one to wear. The machine might put 2 cars and 5 trucks in each package. That would be consistent. Or it might put 7 vehicles in each package: one containing 4 trucks and 3 cars, the other containing 6 trucks and 1 car.

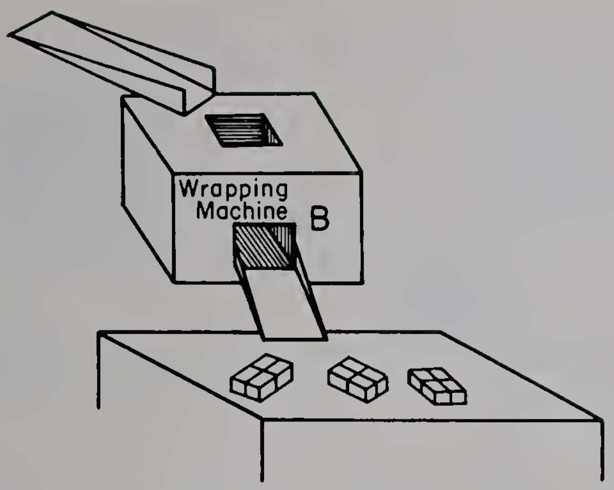
Next item—a half-dozen apples. This is easy. We think of this as 6 of a new unit. So each of the two neatly wrapped packages contains 3 apples or, if you prefer, a quarter-dozen apples.

Next item—2 dimes and 6 pennies. Does the machine recognize that a dime is worth 10 pennies? Well, in one package there were 4 pennies. What do you suppose was in the other? 2 dimes and 2 pennies. There were 4 coins in each package. Evidently the machine simply separates the objects into 2 groups with the same number in each group. The machine could have delivered one package containing 1 dime and 3 pennies and another package containing 1 dime and 3 pennies.

Suppose that we know the contents of one of the packages. Can we know anything about the input? If one package contains 7 pencils then the input must have been 14 objects, 7 of which were pencils. It is possible that there were 14 pencils, but it is also possible that there were 11 pencils and 3 erasers, etc.

What could we feed into the machine that would stump it completely? Certainly one quarter (25¢), one egg, three eggs, etc. This machine can't cut things in half.

Draw another machine—call it Wrapping Machine B—which is similar in operation, only it always delivers 3 packages. If you were to feed it 6 marbles, you would get them wrapped neatly in 3 packages, with 2 marbles in each package. Have pupils keep a record of output and input. How many of the examples of input we have considered could both machines handle? (6 marbles, 30 pennies, a dozen eggs; but not 10 trucks and 4 cars, nor 2 dimes and 6 pennies.)



Drill

Here are two stories for drill purposes:

1) It took Andy 29 minutes to walk to school. It took him only 19 minutes to walk home.

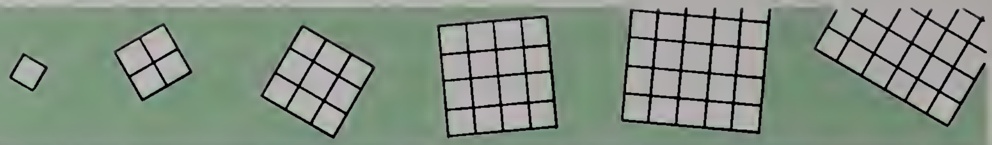
2) Merwin's father gave him a half-dollar for his birthday. He bought a toy car for 15 cents and a toy boat for 30 cents.

Here are some subtraction problems for drill:

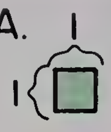
Subtract.

72 −45	53 −27	46 −29	50 −16	83 −26
91 −35	60 −28	43 − 8	85 −76	64 −19
52 −14	82 −29	90 −33	55 −28	73 −59

ALL ABOUT SQUARES.

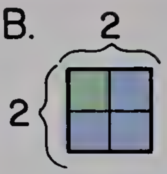


A.



1 x 1 = 1

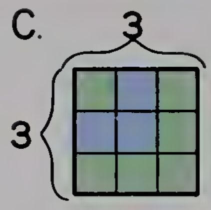
B.



2 x 2 = 4

1 + 3 = 4

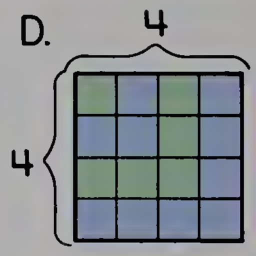
C.



3 x 3 = 9

1 + 3 + 5 = 9

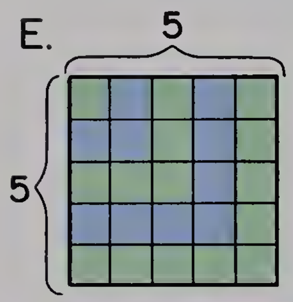
D.



4 x 4 = 16

1 + 3 + 5 + 7 = 16

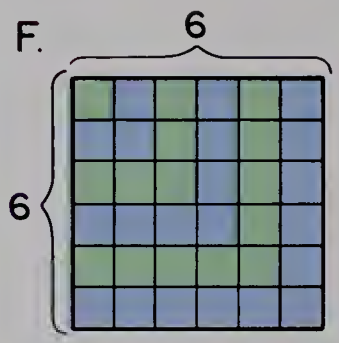
E.



5 x 5 = 25

1 + 3 + 5 + 7 + 9 = 25

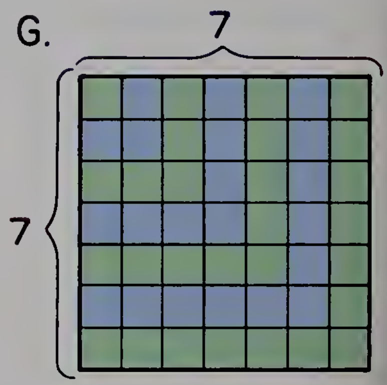
F.



6 x 6 = 36

1 + 3 + 5 + 7 + 9 + 11 = 36

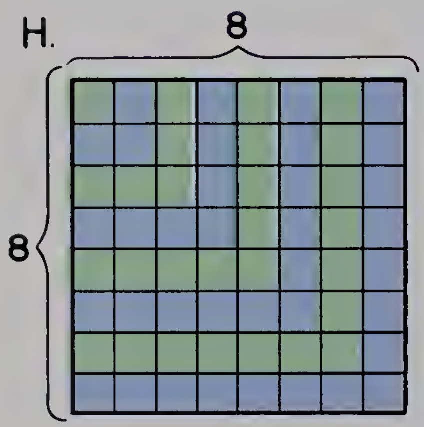
G.



7 x 7 = 49

1 + 3 + 5 + 7 + 9 + 11 + 13 = 49

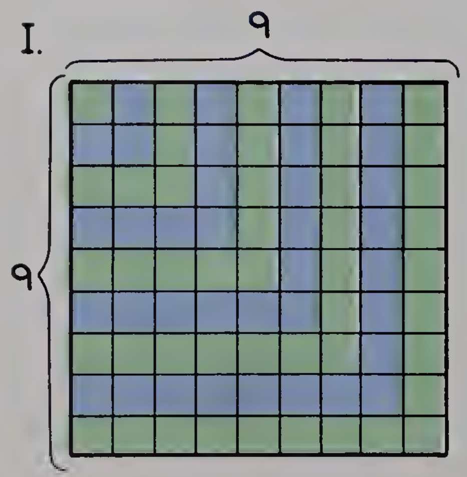
H.



8 x 8 = 64

1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 = 64

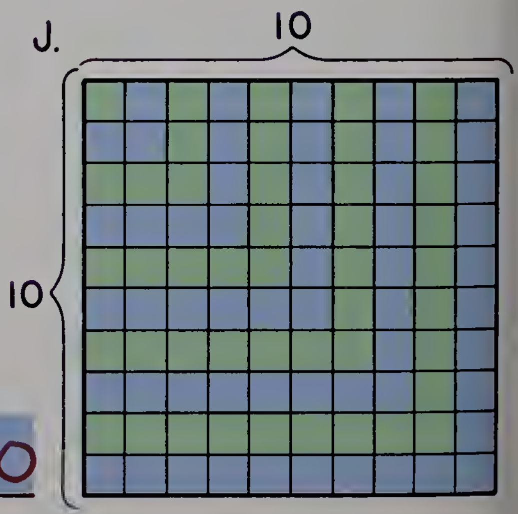
I.



9 x 9 = 81

1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 = 81

J.



10 x 10 = 100

1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 = 100

Notes on Pupil Page 103

Tasks: *Pupils determine the missing products (square numbers), and the sums of consecutive odd numbers.*

Purpose: This is an enrichment page to show pupils the connection between the sums of consecutive odd numbers and the square numbers.

Unifying Ideas: Multiplication and division; addition and subtraction; functions and relations.

Materials for Teacher: A square card with 1 dot on it; an L-shaped card with 3 dots on it so that it and the 1-dot card can be put together to form a square; L-shaped cards having 5 dots, 7 dots, 9 dots, 11 dots, 13 dots; flannel board. (See the Notes on Pupil Page 126.)

Materials for Pupils: Graph paper.

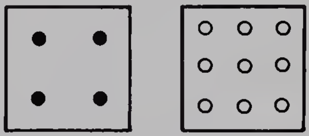
The Lesson:

The Notes on Pupil Page 126 present an approach to the topic of square numbers in which a Counting Board is used. You may wish to try that approach here.

Alternatively, you may wish to use the cards with dots in the activity described below. Pupils should follow the development at their seats, using graph paper.

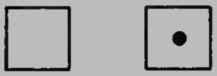
Square Numbers

Sometimes it is possible to arrange objects in a square array. For example:



In these pictures, 4 dots have been arranged in 2 columns and 2 rows; and 9 dots have been arranged in 3 columns and 3 rows.

The following pictures suggest that it is possible to arrange 0 things in 0 rows, and 1 thing in 1 row:

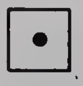


What numbers of things can be arranged in square arrays? 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, etc.

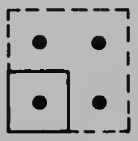
more

Headline for the Day: Which number doesn't belong with the others? 3, 4, 9, 16, 25

One way to suggest how this particular series of numbers grows is to begin with 1 row and 1 column:

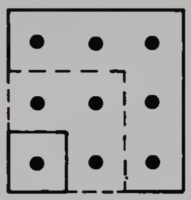


Then arrange 2 rows and 2 columns:



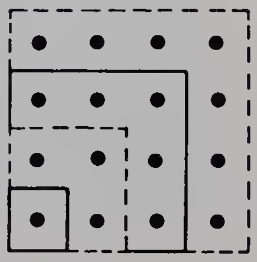
To a display of 1 dot we have annexed a display of 3 dots $(1+3=4)$.

Next make 3 rows and 3 columns of dots:



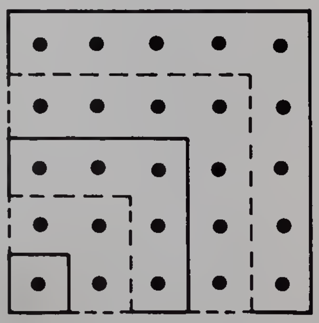
To a display of $1+3$ dots we have annexed a display of 5 dots $(1+3+5=9)$.

Continue by making 4 rows and 4 columns of dots:



To a display of $1+3+5$ dots we have annexed a display of 7 dots $(1+3+5+7=16)$.

Next:



To a display of $1+3+5+7$ dots we have annexed a display of 9 dots $(1+3+5+7+9=25)$.

The above series of examples results in a pattern that can be easily perceived:

$$
\begin{aligned}
1 &= 1\\
4 &= 1+3\\
9 &= 1+3+5\\
16 &= 1+3+5+7\\
25 &= 1+3+5+7+9
\end{aligned}
$$

Let's extend this pattern:

$$
\begin{aligned}
\_\_\_ &= 1+3+5+7+9+11\\
\_\_\_ &= 1+3+5+7+9+11+13\\
\_\_\_ &= 1+3+5+7+9+11+13+15
\end{aligned}
$$

How far can the pattern be extended? The sum of the first 9 odd numbers is 9×9; the sum of the first 10 odd numbers is 10×10, or 100; etc.

The numbers in the sequence 1, 4, 9, 16, etc., are called *square numbers*, for obvious reasons. In the listing of square numbers, it would seem logical to include 0, since 0 is 0×0.

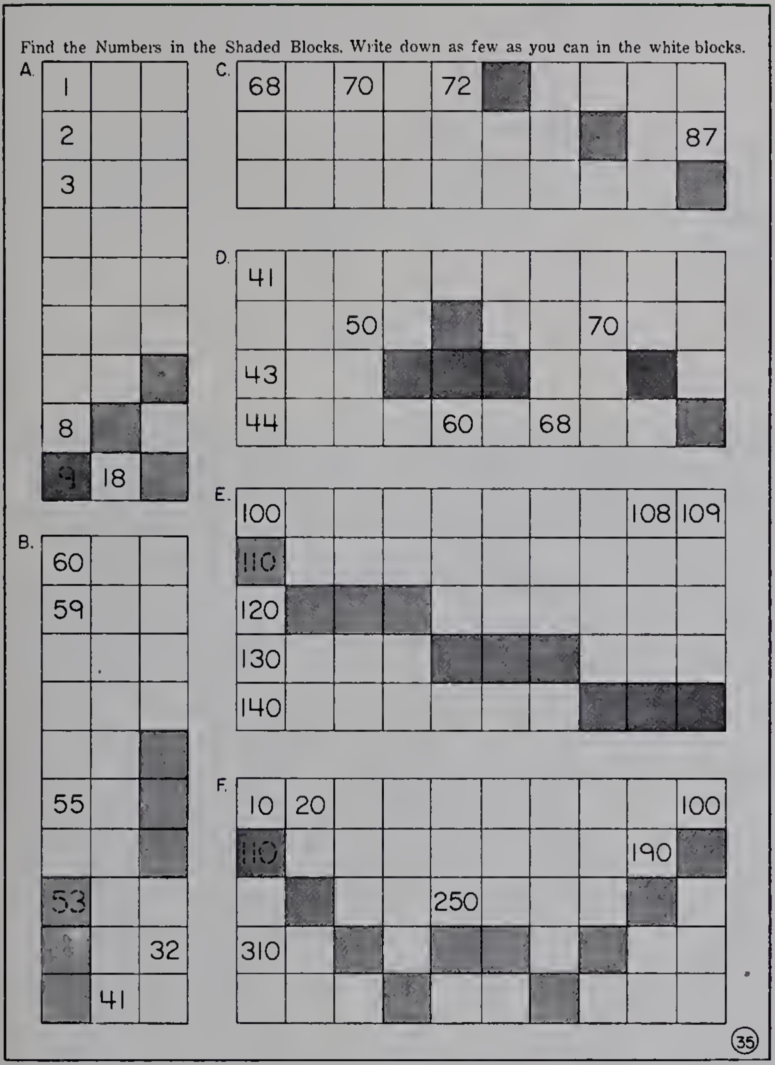
Columns	Rows	Totals	As sum of consecutive odd numbers
1	1	1	$1 = 1$
2	2	4	$4 = 1+3$
3	3	9	$9 = 1+3+5$
4	4	16	$16 = 1+3+5+7$
5	5	25	$25 = 1+3+5+7+9$
6	6	36	$36 = 1+3+5+7+9+11$
7	7	49	$49 = 1+3+5+7+9+11+13$
8	8	64	$64 = 1+3+5+7+9+11+13+15$

Drill

Here are two stories for drill purposes:

1) Father took Mary and Jack for a ride. They left at 7:10. They were gone for 30 minutes.
2) Terry's brother sent 5 airmail letters. An airmail stamp costs 8 cents.

For easy drill sheets consider pages 35 and 55 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B, as shown on the next page.



Find the Numbers in the Shaded Blocks. Write down as few as you can in the white blocks.

A.

1		
2		
3		
		▨
8	▨	
▨ 9	18	▨

B.

60		
59		
		▨
55		▨
		▨
▨ 53		
▨		32
▨	41	

C.

68		70		72	▨				
							▨		87
									▨

D.

41									
		50		▨			70		
43			▨	▨	▨			▨	
44				60		68			▨

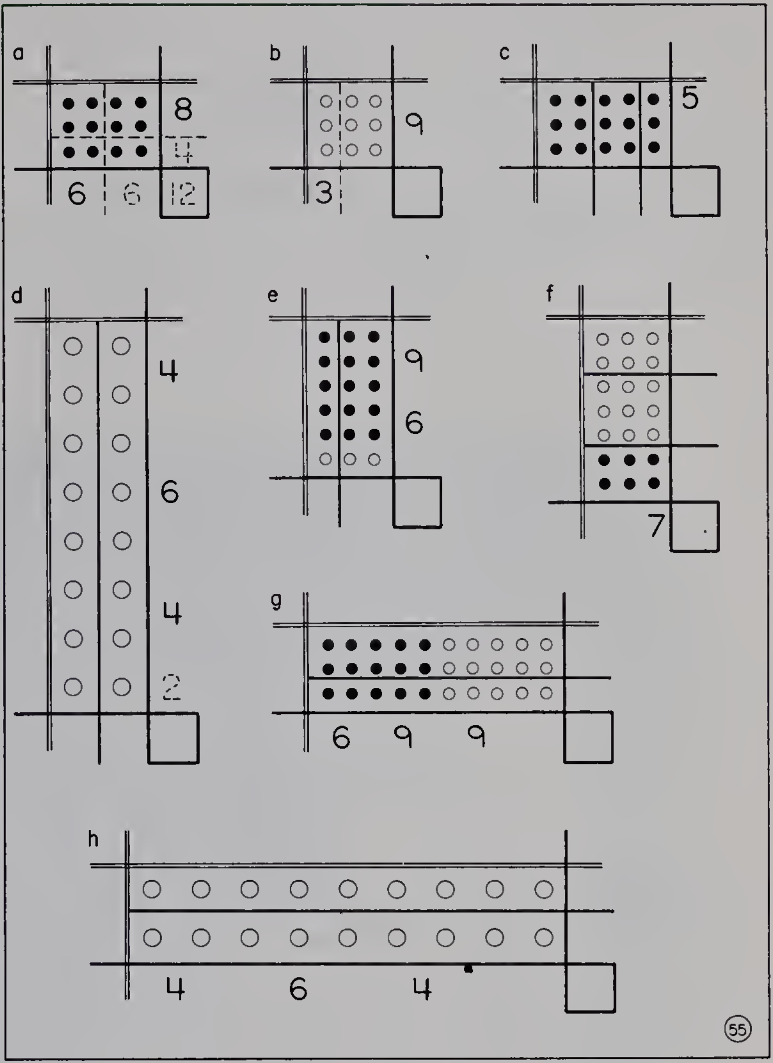
E.

100								108	109
▨ 110									
120	▨	▨	▨						
130				▨	▨	▨			
140							▨	▨	▨

F.

10	20								100
▨ 110								190	▨
	▨			250				▨	
310		▨		▨	▨		▨		
			▨			▨			

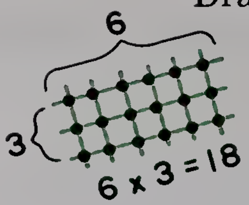
㉟

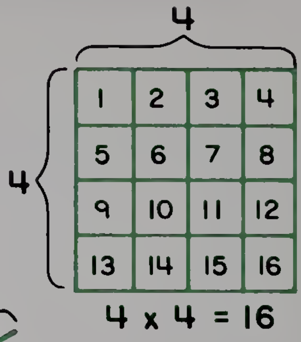

a
8
4
6
6
12
b
9
3
c
5
d
4
6
4
2
e
9
6
f
7
g
6 9 9
h
4 6 4
(55)

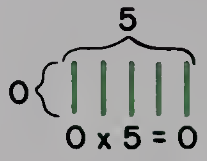
A USEFUL RECORD

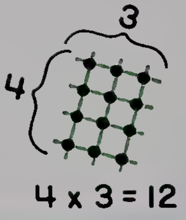
Draw sketches when in doubt.

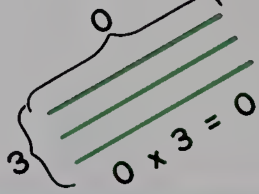
Fill in the blanks
(1) from memory,
or
(2) make a sketch and find out,
or
(3) check back on pages you have done before.











	0	1	2	3	4	5	6	7	8	9	10
x 0	0 x 0 0	1 x 0 0	2 x 0 0	3 x 0 0	4 x 0 0	5 x 0 0	6 x 0 0	7 x 0 0	8 x 0 0	9 x 0 0	10 x 0 0
x 1	0 x 1 0	1 x 1 1	2 x 1 2	3 x 1 3	4 x 1 4	5 x 1 5	6 x 1 6	7 x 1 7	8 x 1 8	9 x 1 9	10 x 1 10
x 2	0 x 2 0	1 x 2 2	2 x 2 4	3 x 2 6	4 x 2 8	5 x 2 10	6 x 2 12	7 x 2 14	8 x 2 16	9 x 2 18	10 x 2 20
x 3	0 x 3 0	1 x 3 3	2 x 3 6	3 x 3 9	4 x 3 12	5 x 3 15	6 x 3 18	7 x 3 21	8 x 3 24	9 x 3 27	10 x 3 30
x 4	0 x 4 0	1 x 4 4	2 x 4 8	3 x 4 12	4 x 4 16	5 x 4 20	6 x 4 24	7 x 4 28	8 x 4 32	9 x 4 36	10 x 4 40
x 5	0 x 5 0	1 x 5 5	2 x 5 10	3 x 5 15	4 x 5 20	5 x 5 25	6 x 5 30	7 x 5 35	8 x 5 40	9 x 5 45	10 x 5 50
x 6	0 x 6 0	1 x 6 6	2 x 6 12	3 x 6 18	4 x 6 24	5 x 6 30	6 x 6 36	7 x 6 42	8 x 6 48	9 x 6 54	10 x 6 60
x 7	0 x 7 0	1 x 7 7	2 x 7 14	3 x 7 21	4 x 7 28	5 x 7 35	6 x 7 42	7 x 7 49	8 x 7 56	9 x 7 63	10 x 7 70
x 8	0 x 8 0	1 x 8 8	2 x 8 16	3 x 8 24	4 x 8 32	5 x 8 40	6 x 8 48	7 x 8 56	8 x 8 64	9 x 8 72	10 x 8 80
x 9	0 x 9 0	1 x 9 9	2 x 9 18	3 x 9 27	4 x 9 36	5 x 9 45	6 x 9 54	7 x 9 63	8 x 9 72	9 x 9 81	10 x 9 90
x 10	0 x 10 0	1 x 10 10	2 x 10 20	3 x 10 30	4 x 10 40	5 x 10 50	6 x 10 60	7 x 10 70	8 x 10 80	9 x 10 90	10 x 10 100

Notes on Pupil Page 104

Tasks: Pupils complete the table by recording the products.

Purpose: (a) to review the multiplication facts pupils have already learned; (b) to acquaint pupils with new multiplication facts; (c) to stimulate pupils to search for patterns in finding products.

Unifying Ideas: Multiplication and division.

Materials for Teacher: None.

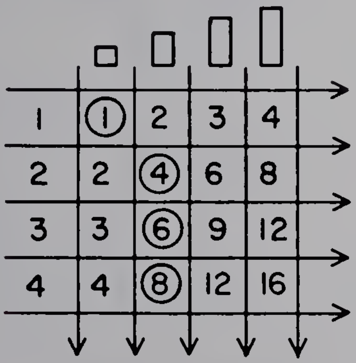
Materials for Pupils: Graph paper; L-shaped pieces of cardboard.

The Lesson:

Prepare pupils for this page by carrying out the following activity.

Visualizing the Multiplication Table

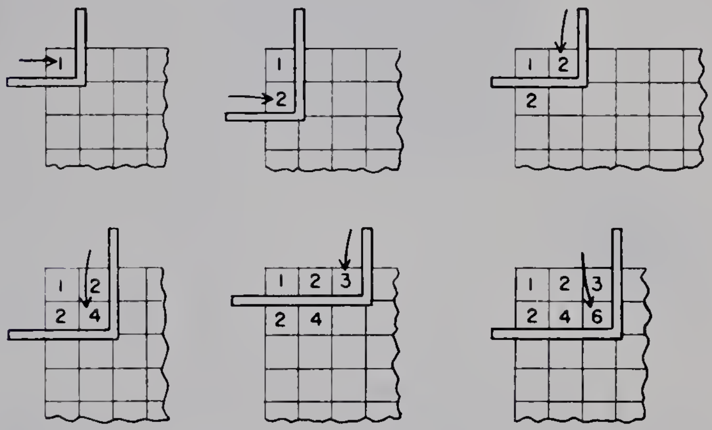
Make the following drawing on the board, but mark in only the numbers that have been circled. Encourage pupils to help you finish the chart.



1	①	2	3	4
2	2	④	6	8
3	3	⑥	9	12
4	4	⑧	12	16

It's a simple matter to extend the chart in both directions. Give pupils an opportunity to discover the familiar patterns.

Next, have each pupil use a piece of graph paper and a replica of a carpenter's framing square cut out of cardboard. Starting at the left-hand corner of the paper, pupils are to mark off square by square in succession, and to number each square as they proceed. For example:



In this activity pupils will be creating exactly the same chart as the one you drew on the board. Have pupils make the comparison. Ask: Are the patterns the same? Do the patterns become different as the numbers get bigger?

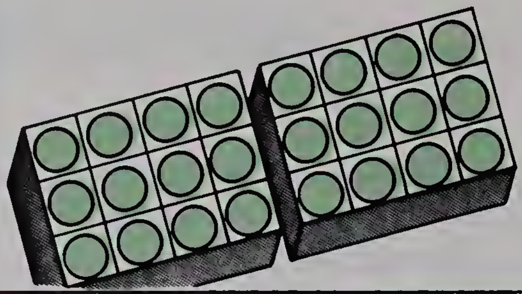
Drill

Here are two stories for drill purposes:

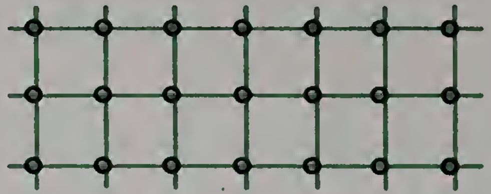
1) Mr. Smith drove 976 miles last month. He drove only 342 miles this month.
2) There are 28 pupils in Mrs. Solomon's room. Each pupil has 10 fingers.

Headline for the Day: Write a number sentence that fits: If Sarah has 15 choices of skirt and blouse outfits and has 3 blouses, how many skirts does she have?

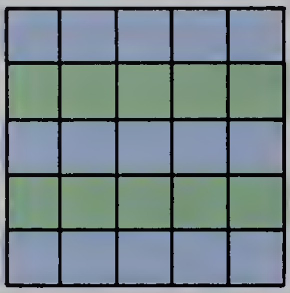
C



A



B



Complete each sentence. Then label the sketch which tells about the sentence.

A. 7 x 3 = 21

B. 25 ÷ 5 = 5

C. 3 x 8 = 24

D. 4 x 10 = 40

E. 18 ÷ 3 = 6

F. 20 ÷ 5 = 4

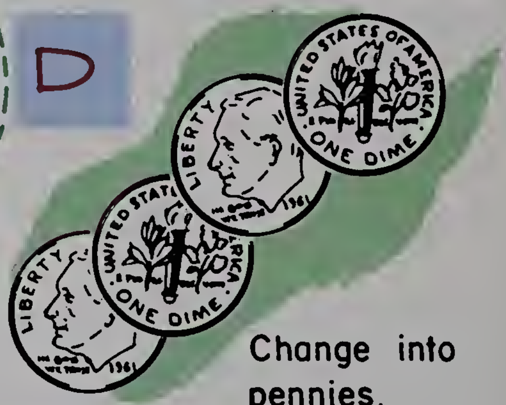
G. 8 x 4 = 32

H. 5 x 7 = 35

I. 28 ÷ 4 = 7

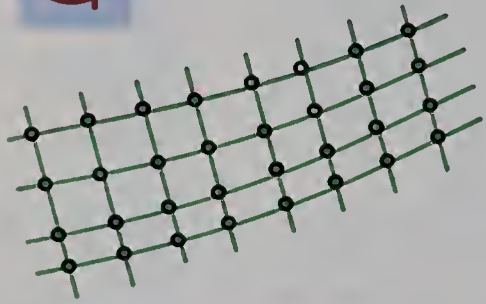
J. 7 x 10 = 70

D

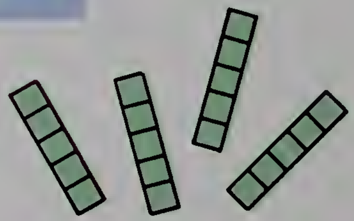

LIBERTY
UNITED STATES OF AMERICA
ONE DIME
1961

Change into pennies.

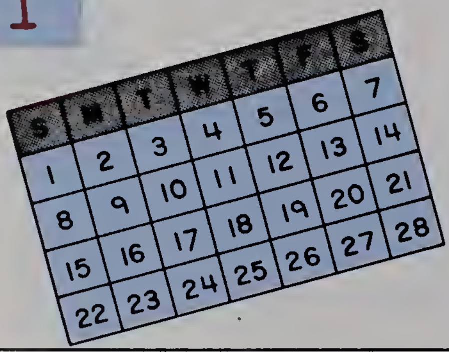
G



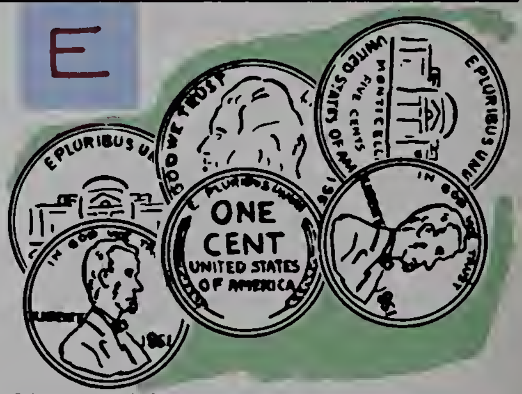
F



I

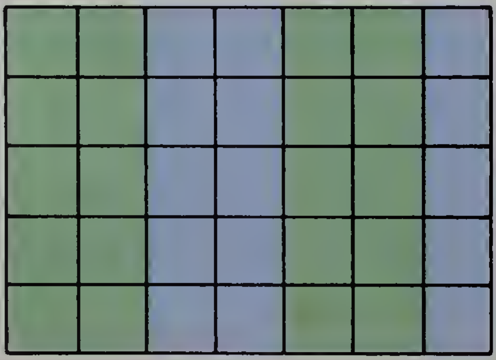

S M T W T F S
1 2 3 4 5 6 7
8 9 10 11 12 13 14
15 16 17 18 19 20 21
22 23 24 25 26 27 28

E

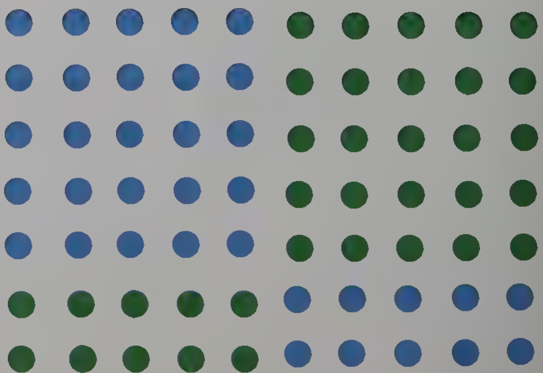

E PLURIBUS UNUM
IN GOD WE TRUST
LIBERTY
ONE CENT
UNITED STATES OF AMERICA
MONTICELLO
FIVE CENTS
1961

Share this equally among three children.

H



J



Notes on Pupil Page 105

Tasks: Pupils complete each number sentence and label the pictures that represent the sentences.

Purpose: To test pupils' ability to select representations for multiplication and division sentences.

Unifying Ideas: Structure; multiplication and division.

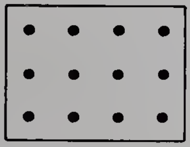
Materials for Teacher: The rectangular array flash cards described in the Notes on Pupil Pages 12 and 22.

Materials for Pupils: None.

The Lesson:

Suggested point distribution for the test: 1 point per exercise.

Display the flash cards described in the Notes on Pupil Pages 12 and 22. As you hold up a card ask pupils to state multiplication and division sentences and to tell related stories. Examples:



Sentences

$3\times4=12$ $4\times3=12$ $12\div3=4$ $12\div4=3$

Stories

3 wagons have 12 wheels.

Mary earned 3 cents a day for 4 days.

Miss Jones divided 12 chocolates equally among 3 children.

3 hippos have 12 legs.

You can cut 4 pieces, each 3 inches long, from a foot of string.

If 3 horizontal lines cross 4 vertical lines, there will be 12 crossing points.

4 tricycles have 12 wheels.

This kind of drill in which abstract sentences are interpreted in terms of physical situations will help your pupils substantially in developing problem solving ability and in applying their arithmetic outside the classroom.

The Special Numbers 0 and 1

Put on the board the following examples of multiplication and addition which make use of the two smallest whole numbers (0 and 1):

$0\times0=$__	$1\times0=$__	$0+0=$__	$1+0=$__
$0\times1=$__	$1\times1=$__	$0+1=$__	$1+1=$__
$0\times2=$__	$1\times2=$__	$0+2=$__	$1+2=$__
$0\times3=$__	$1\times3=$__	$0+3=$__	$1+3=$__
$0\times4=$__	$1\times4=$__	$0+4=$__	$1+4=$__

Reverse the order (which, of course, will not alter the results):

$0\times0=$__	$0\times1=$__	$0+0=$__	$0+1=$__
$1\times0=$__	$1\times1=$__	$1+0=$__	$1+1=$__
$2\times0=$__	$2\times1=$__	$2+0=$__	$2+1=$__

etc.

Point out that if either factor is 0, the product is 0. Verbalization is not too important in approaching new ideas, however. If a child says: Even if I multiply 999 by 0, all I get is 0, he has expressed the idea adequately.

Some interesting and significant patterns to be noted are:

$0\times1=\underline{0}$	$0+0=\underline{0}$
$1\times1=\underline{1}$	$1+0=\underline{1}$
$2\times1=\underline{2}$	$0+2=\underline{2}$
$1\times3=\underline{3}$	$3+0=\underline{3}$

If a factor is 1, the other factor and the product are equal (i.e., identical). If an addend is 0, the other addend and the sum are equal. The number 0 is sometimes called *the identity element for addition.*

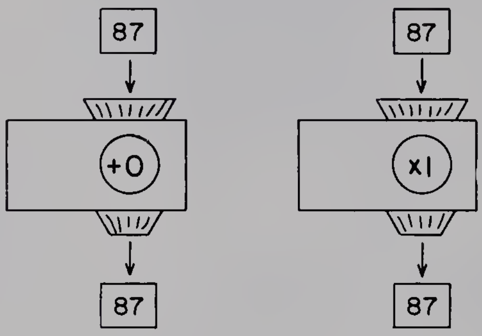
more

Headline for the Day: Make up stories that fit the number sentences on page 105.

Also, the number 1 is *the identity element for multiplication*. Clearly, this terminology is not for your pupils. All we need to be sure of at this time is that pupils can answer the following questions:

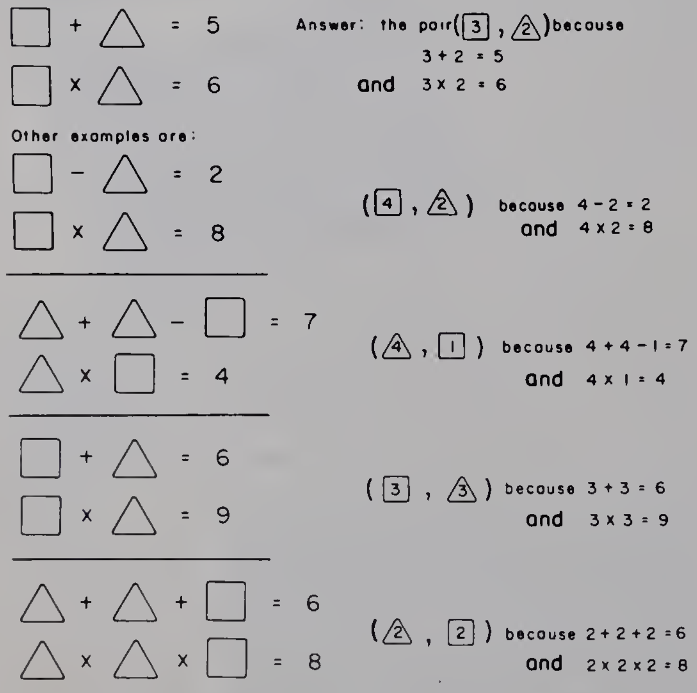
1) What do you get when you add 0 to a number? (The number.)
2) What do you get when you multiply a number by 1? (The number.)

Here are pictures of the "do nothing" machines at work:



Find the Pair

The game *Find the Pair* is based on an appreciation for the difference between the operations of multiplication and addition. The object of the game is for pupils to find a pair of whole numbers (alike or different) that satisfy both of two mathematical sentences, such as the following:



The last two examples are legitimate because the components of a pair of numbers could be alike. Usually, after pupils have played the game several times, it becomes perfectly clear to them that a number that appears in place of a certain frame in the first sentence of a pair of sentences must appear in place of the frame of that size and shape in the second sentence of the pair. In algebra, the same basic idea is part of the rules of substitution.

Grocery Store Problems

In grocery stores, you will find cans, cartons, bottles, and packages marked this way:

3/15 2/11 4/25 5/20 5/22 5/13

What do you think those marks mean? (3 for 15¢; 2 for 11¢; 4 for 25¢; etc.)

If a can is marked 3/15, how much do you think you would pay for 1 can? Someone may argue that if 3 cans cost 15¢ then 1 can would cost 5¢ because 15 ÷ 3 = 5. Someone else may contend that such is not the case, because then the cans would be marked 1/5. If the price is meant to indicate a bargain for purchasing 3 cans, then the price for 1 can must be at least 6¢. By purchasing 3 cans for 15¢, the customer saves at least 1¢ per can.

On the board draw a chart for each case:

No. of Cans	1	2	3	4	5	6	7	8	9	10
Price in cents	*5*	*10*	15			30				

No. of Cans	1	2	3	4	5	6	7	8	9	10
Price in cents	*6*	*12*	15	*21*	*27*	30				

We agree that the price is 3 cans for 15¢ in either case. How much would 6 cans cost? How much would 9 cans cost? The same in both records?

Now, how about 1 can? We have agreed that 1 can might cost 5¢ or 6¢. In each case, how much would 2 cans cost? (10¢; 12¢)

But someone might argue that if 1 can costs 6¢ and 3 cans cost 15¢, then you probably would get a small bargain for buying 2 cans at a time. So, while 1 can costs 6¢, 2 cans may cost only 11¢. The chart would look like this:

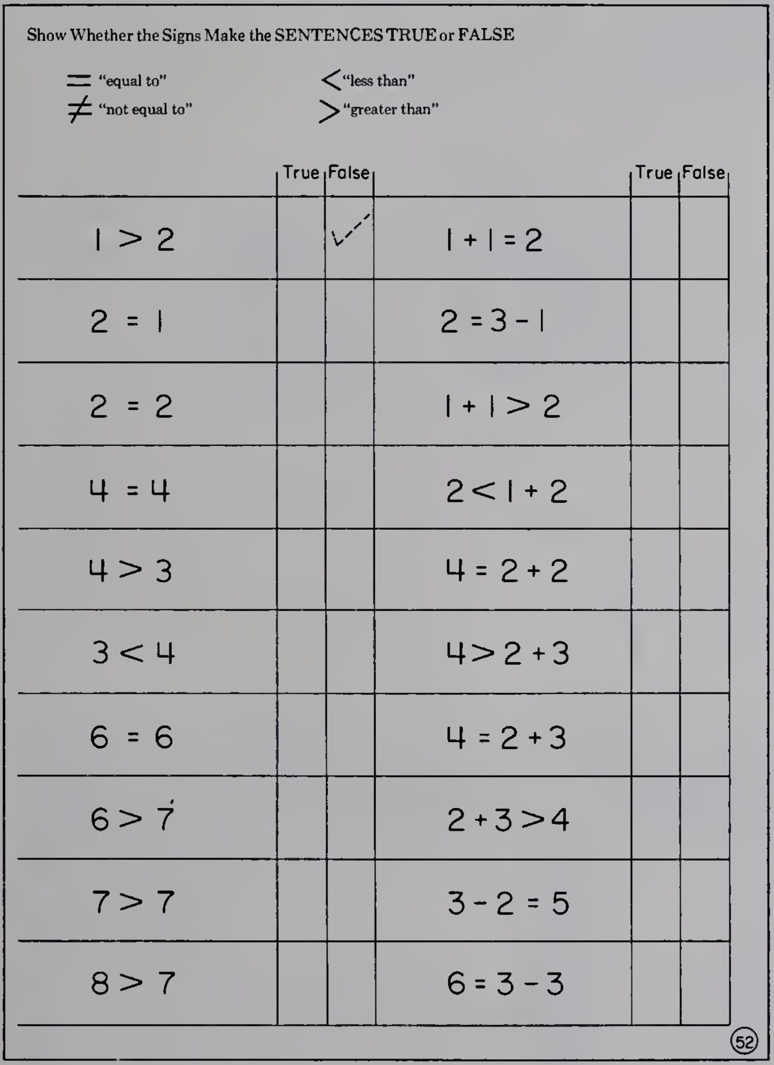
		Savings
1 can	6¢	0
2 cans	11¢	1¢
3 cans	15¢	3¢

So, we need another chart, such as the one on the following page.

No. of Cans	1	2	3	4	5	6	7
Price in cents	*6*	*11*	15			30	

Drill

Consider using as a drill sheet on the equality and inequality relations page 52 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



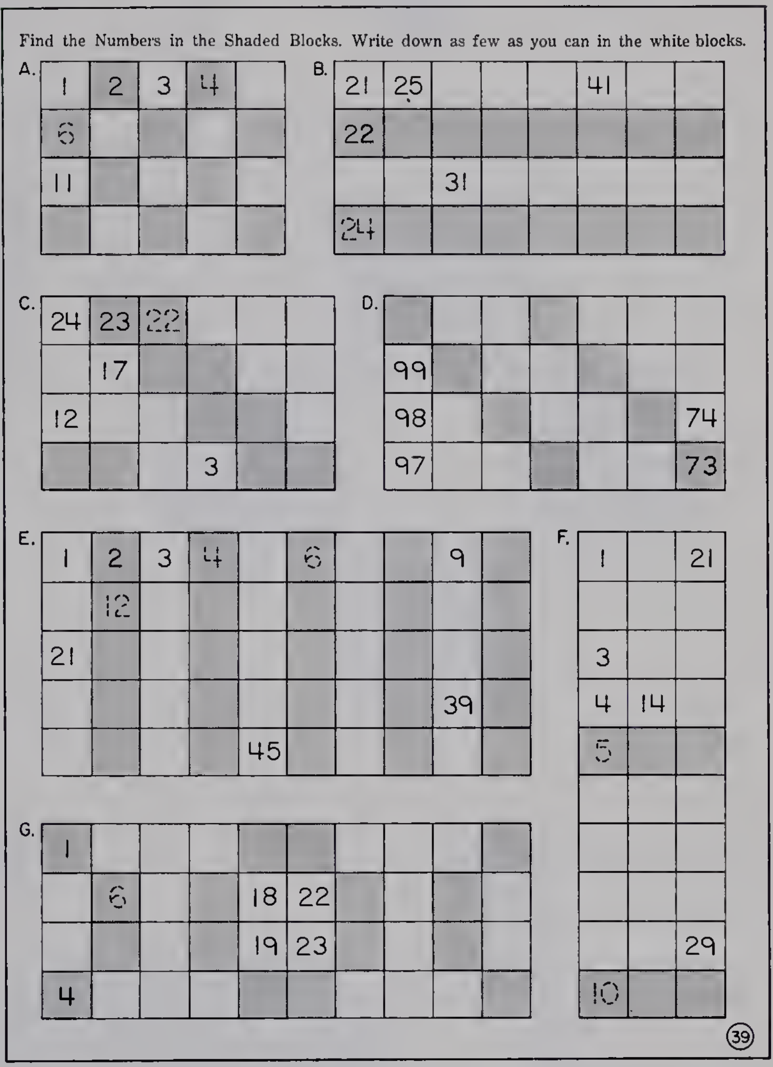
Show Whether the Signs Make the SENTENCES TRUE or FALSE

= "equal to"
≠ "not equal to"
< "less than"
> "greater than"

	True	False		True	False
1 > 2		✓	1 + 1 = 2		
2 = 1			2 = 3 - 1		
2 = 2			1 + 1 > 2		
4 = 4			2 < 1 + 2		
4 > 3			4 = 2 + 2		
3 < 4			4 > 2 + 3		
6 = 6			4 = 2 + 3		
6 > 7			2 + 3 > 4		
7 > 7			3 - 2 = 5		
8 > 7			6 = 3 - 3		

(52)

Also consider using as a drill sheet page 39 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Find the Numbers in the Shaded Blocks. Write down as few as you can in the white blocks.

A.

1	2	3	4	
6		▨		▨
11	▨		▨	
▨		▨		▨

B.

21	25				41		
22	▨	▨	▨	▨	▨	▨	▨
		31					
24	▨	▨	▨	▨	▨	▨	▨

C.

24	23	22			
	17	▨	▨		
12			▨	▨	
▨	▨		3	▨	▨

D.

▨			▨			
99	▨			▨		
98		▨			▨	74
97			▨			73

E.

1	2	3	4		6		▨	9	▨
	12		▨		▨		▨		▨
21	▨		▨		▨		▨		▨
	▨		▨		▨		▨	39	▨
	▨		▨	45	▨		▨		▨

F.

1		21
3		
4	14	
5	▨	▨
		29
10	▨	▨

G.

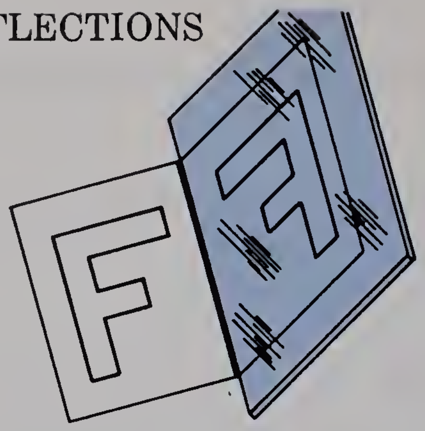
1				▨	▨				▨
	6		▨	18	22	▨		▨	
	▨		▨	19	23	▨		▨	
4				▨	▨				▨

㊴

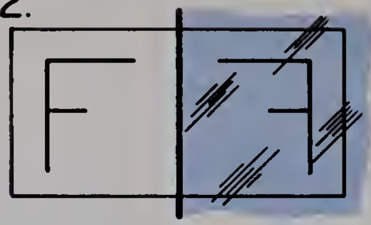
Here are two stories for drill purposes:

1) Dora read three books. The first book had 253 pages. The second book had 423 pages. The last book had 321 pages.
2) Pete bought 6 ice milk bars at 6 cents each.

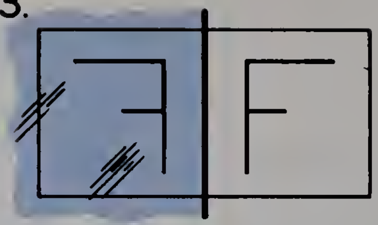
1.



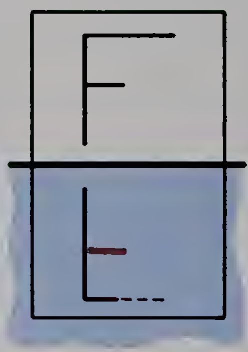
2.



3.



A.



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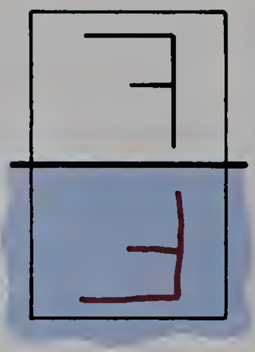
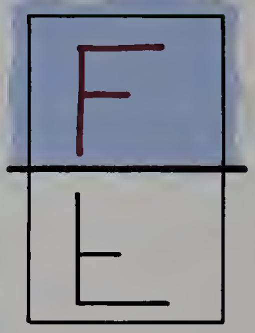
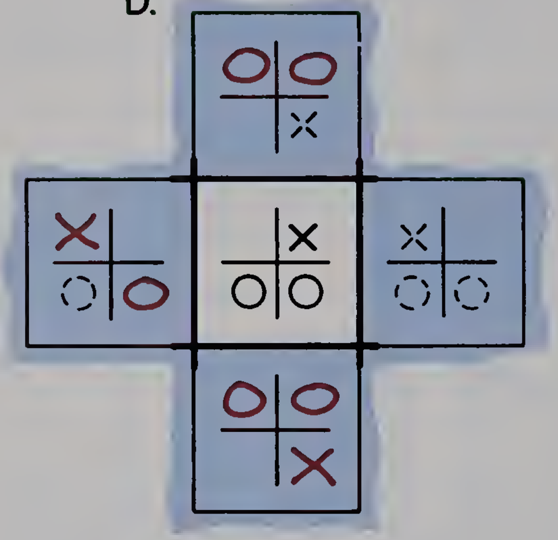


C.



D.



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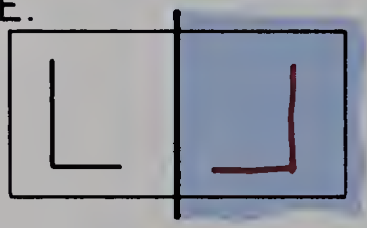
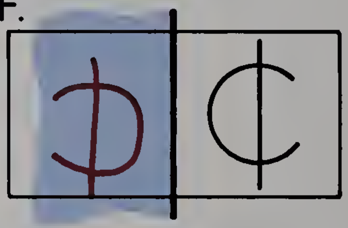
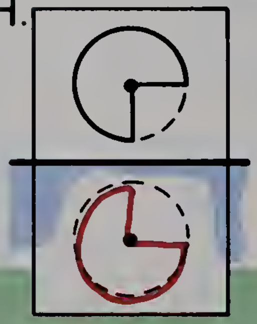


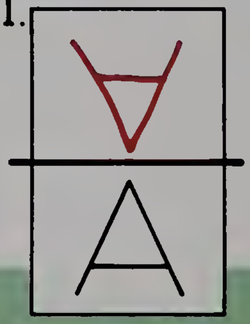
F.



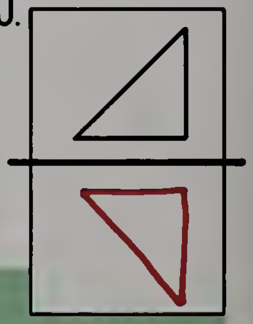
H.



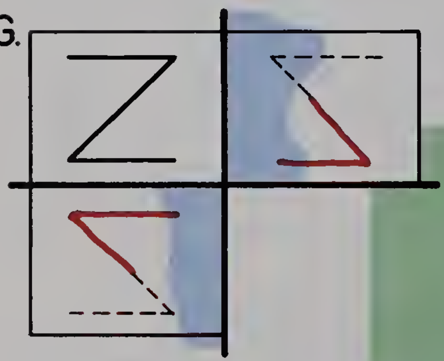
I.



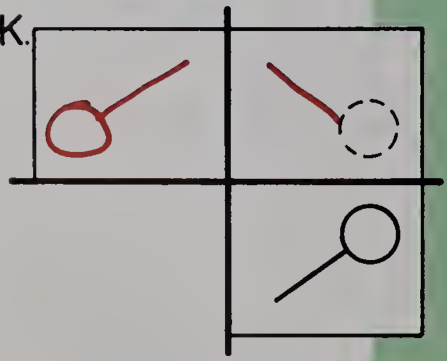
J.



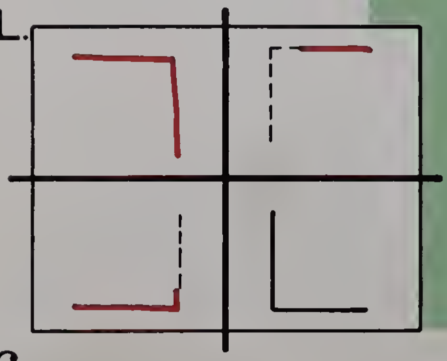
G.



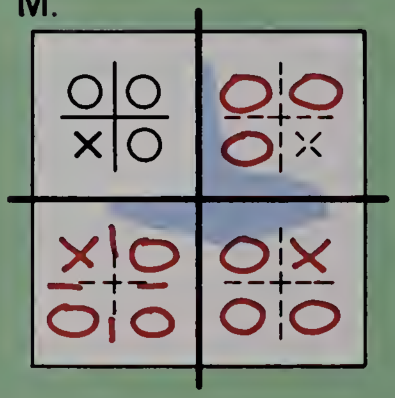
K.



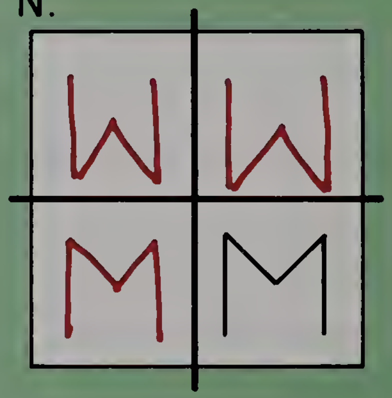
L.



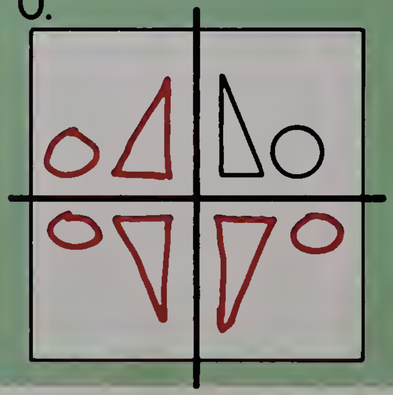
M.



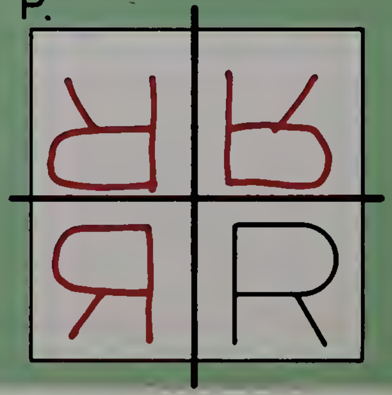
N.



O.



P.



Notes on Pupil Page 106

Tasks: *Pupils sketch the reflections of the given figures as suggested in samples 1-3 at the top of the page.*

Purpose: To introduce pupils to the geometry of reflections.

Unifying Ideas: Geometry.

Materials for Teacher: Ditto masters.

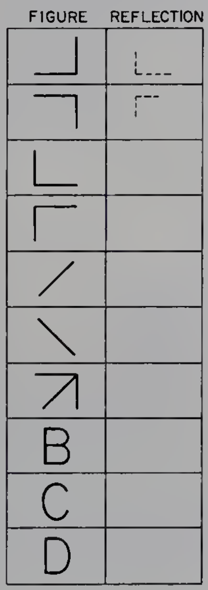
Materials for Pupils: Cuisenaire rods; pocket mirrors; Dittoed sheets.

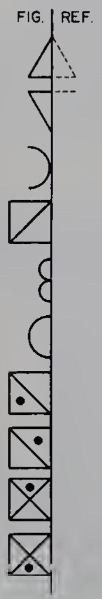
The Lesson:

Page 36 (item 32) of the booklet *Games and Enrichment Activities* suggests activities with Cuisenaire rods which help develop the idea of reflections in a line.

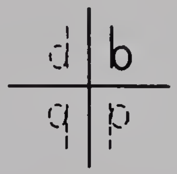
Let pupils use a pocket mirror to judge what the reflected image should look like before they make the drawing. Prepare a Dittoed sheet with pictures like those on pupil page 106 so that pupils can practice before they work in the text.

Here are some possible exercises for this preparatory work:





An interesting problem is to let them explore the reflections for the letter *b*:



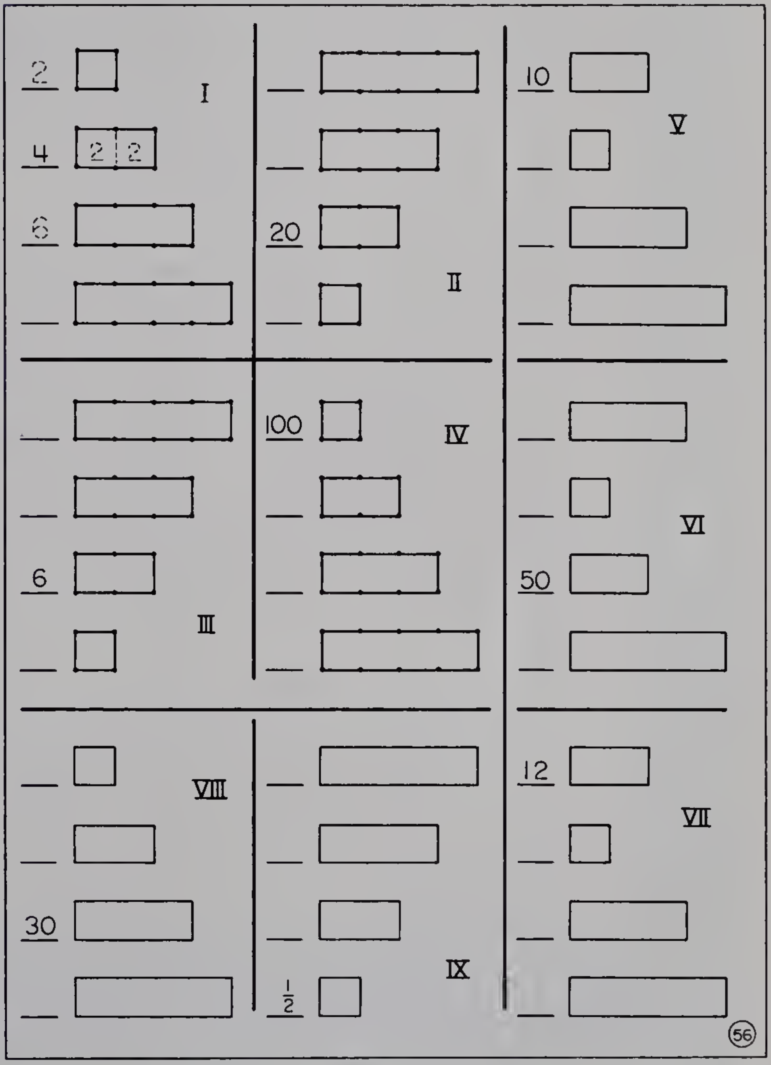
Drill

Here are two stories for drill purposes:

1) Frankie and Johnny saved bottle caps. Frankie collected 343 and Johnny 329.

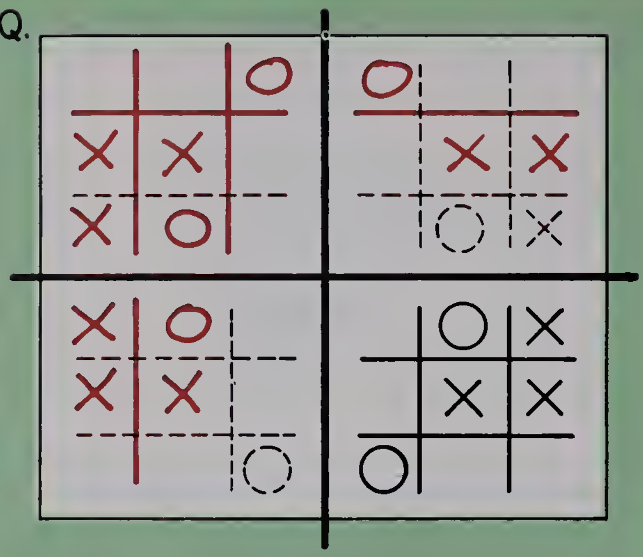
2) Florence planted 6 rows of tulips, 5 to a row.

Consider using for drill page 56 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:

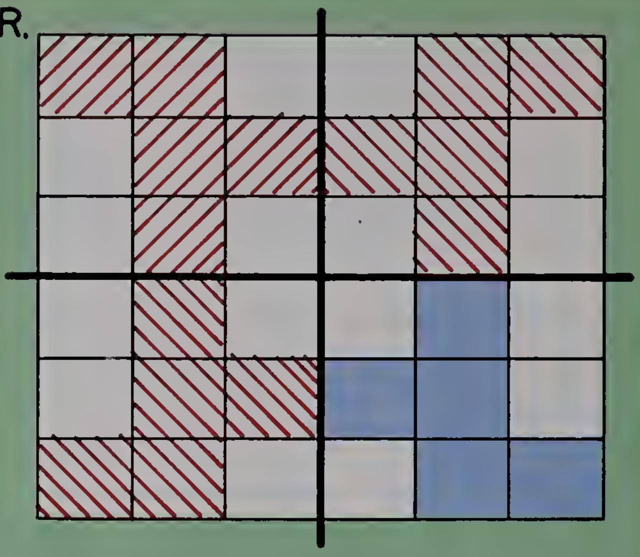


Headline for the Day: Write a story that fits: $(39 + 24) - 24 = \square$

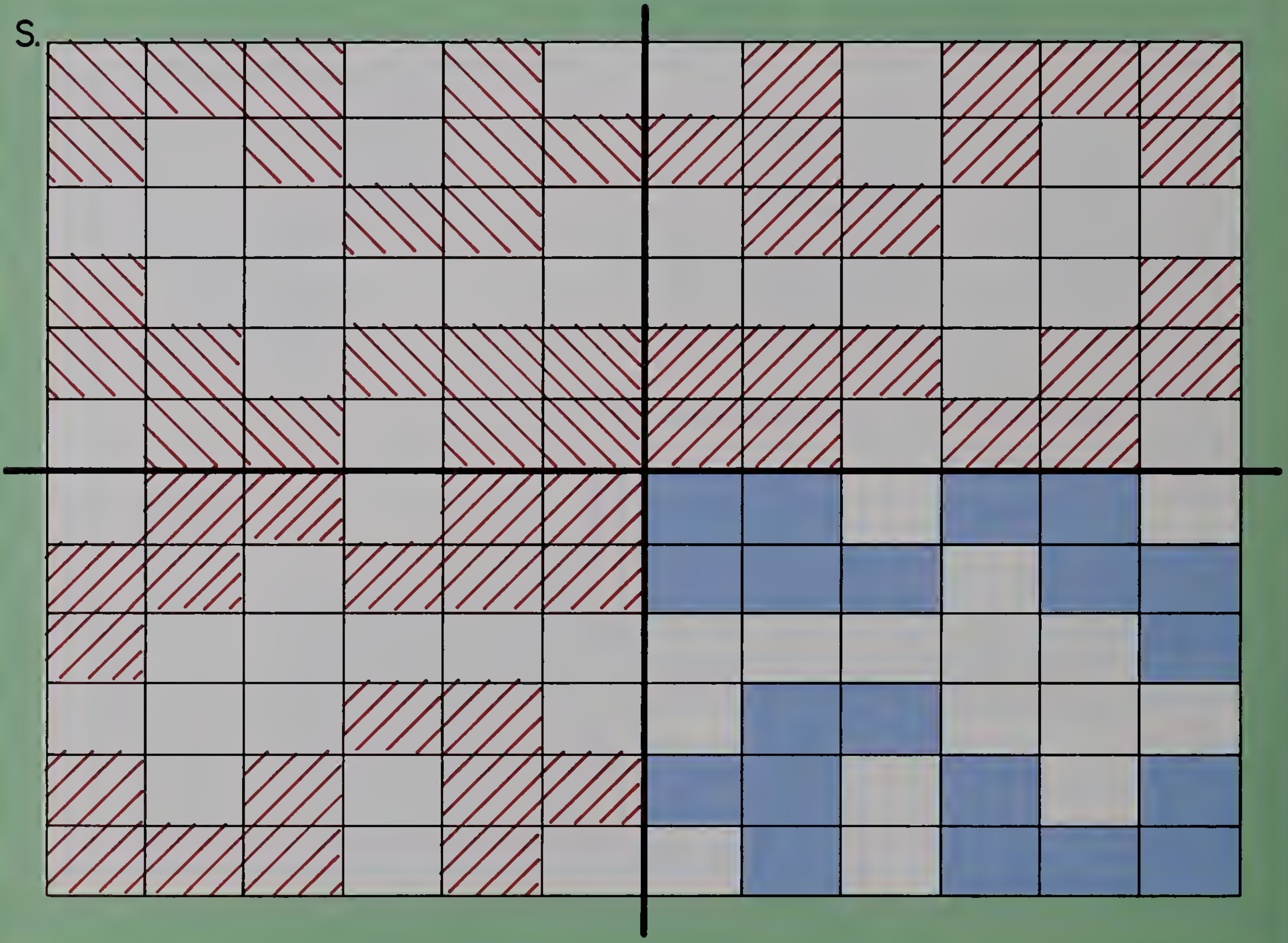
Q.



R.



S.



T.

$$\begin{array}{r}30\\+40\\\hline 70\end{array}\quad\begin{array}{r}5\\+3\\\hline 8\end{array}\quad\begin{array}{r}35\\+43\\\hline 78\end{array}$$

$$\begin{array}{r}50\\+40\\\hline 90\end{array}\quad\begin{array}{r}8\\+7\\\hline 15\end{array}\quad\begin{array}{r}58\\+47\\\hline 105\end{array}$$

$$\begin{array}{r}60\\-40\\\hline 20\end{array}\quad\begin{array}{r}9\\-5\\\hline 4\end{array}\quad\begin{array}{r}69\\-45\\\hline 24\end{array}$$

$35 + 29 = \boxed{64}$

$84 + \textcircled{42} = 126$

$34 - \triangle 9 = 25$

Tasks: *Pupils draw in the upper right quadrant the reflection of the figure in the lower right quadrant; then they draw in the upper left quadrant the reflection of the figure in the upper right quadrant; etc. In exercise T they do computation.*

Purpose: (a) To extend pupils' skill in the gometry of reflections; (b) to provide pupils with developmental practice in addition and subtraction.

Unifying Ideas: Geometry; addition and subtraction.

Materials for Teacher: Demonstration graph board.

Materials for Pupils: Stiff graph paper with 1″ by 1″ squares; scissors; masking tape.

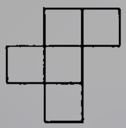
The Lesson:

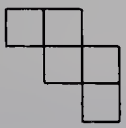
Use the demonstration graph board to help pupils see how to draw the reflections. In exercises R and S the work can be done numerically. Try it.

Pentominoes and Open Boxes

In July 1965 one of the authors observed Professor Marion Walters working with a second grade class on reflections. One brilliant sequence of lessons dealt with the problem of drawing a pentomino that could be folded to form an open box.

A pentomino is a plane figure consisting of five squares such that each square shares an edge with another square. Each of the four given figures in exercise S is a pentomino but only two of them can be folded to form open boxes (with masking tape):

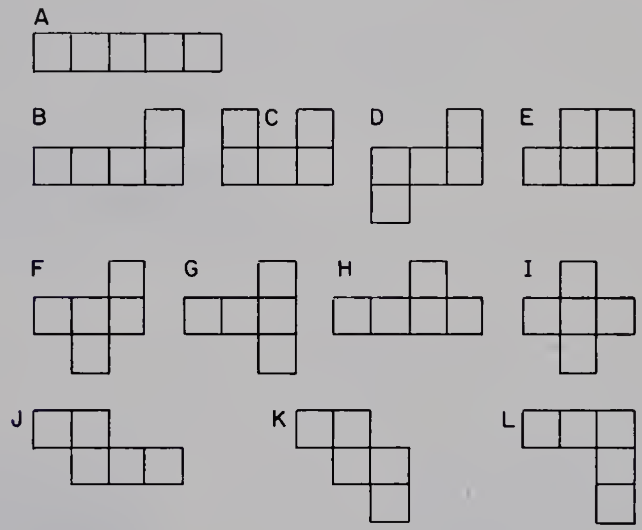




Professor Walters' class grappled with the problem of determining the total number of possible pentominoes. To solve this problem one needs to decide what is meant by saying that two pentominoes are different. Of course, one test is to cut out both figures and try to superpose one on the other. If you can make the figures fit, the pentominoes are congruent; otherwise, they are different.

There is another test for congruence which utilizes the idea of reflection. Two pentominoes drawn on graph paper are congruent if one can be obtained from the other by one or by two reflections. The drawings made by the pupils in exercise S produce pentominoes congruent to the given ones. You can't get a new pentomino by reflection.

With this in mind one can now embark on a search for all possible pentomino types. It turns out that there are precisely 12 types. Here they are:

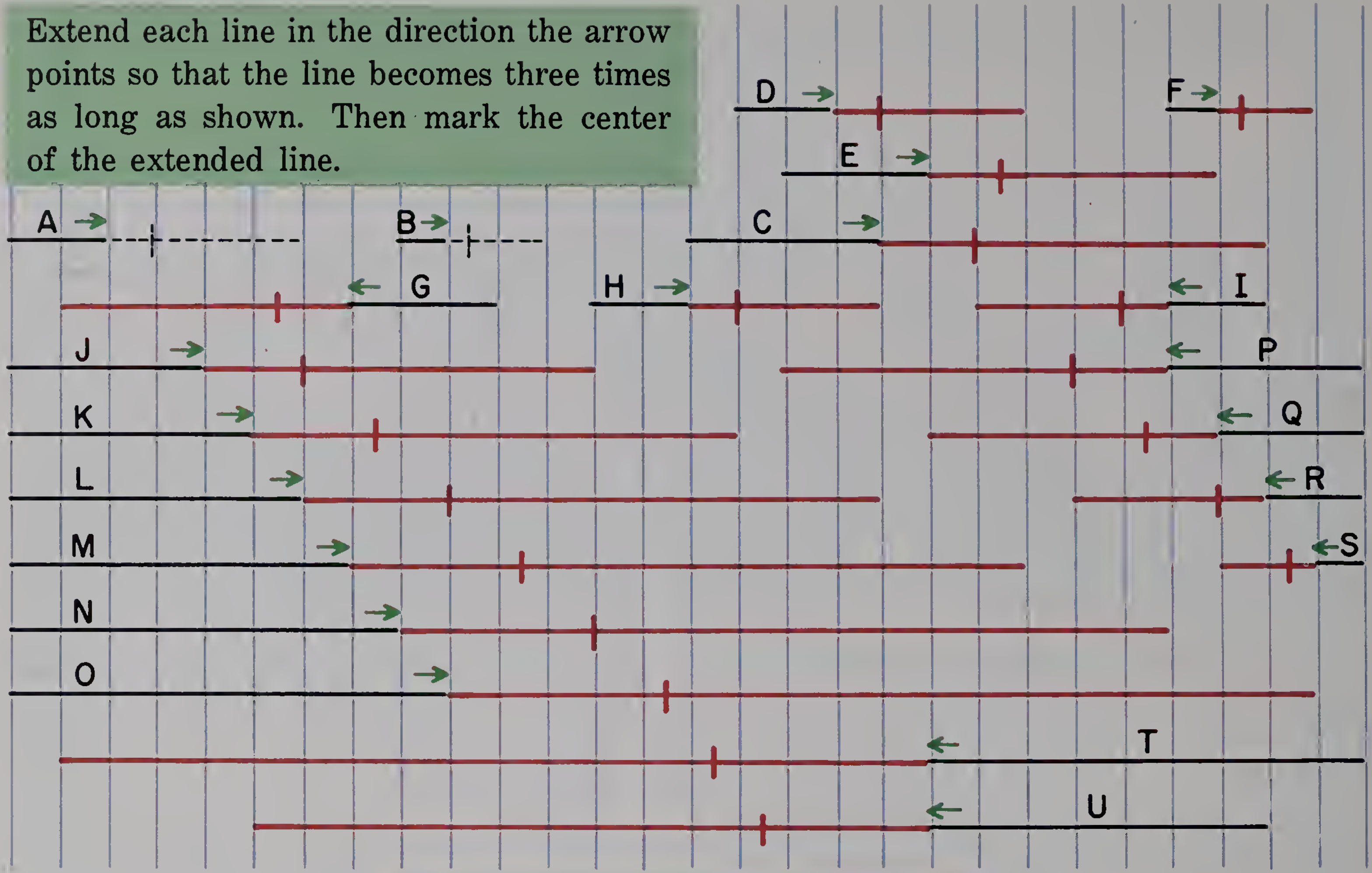


Notice that the pentominoes given on pupil page 107 are of types C, E, F, and K. Which of the 12 types can be folded into open boxes?

An interesting property of the 12 pentomino types is that, given one of each type to be used as a jigsaw puzzle piece, the 12 pieces fit together to form a rectangle. We challenge you to do this. (For more on pentominoes and related matters, see Solomon W. Golomb's book *Polyominoes*, published by Charles Scribner's Sons in 1965.)

Headline for the Day: Write stories suggested by the number sentences at the bottom of page 107.

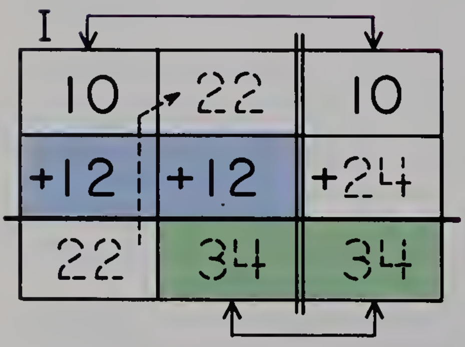
Extend each line in the direction the arrow points so that the line becomes three times as long as shown. Then mark the center of the extended line.



A game called CHAIN ADDITION

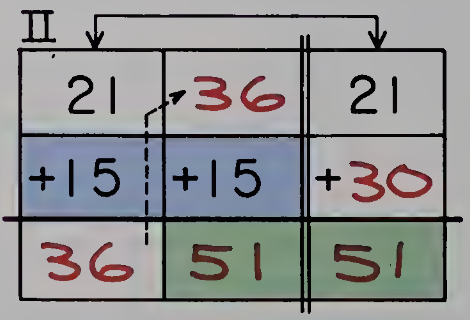
What are the rules?

I



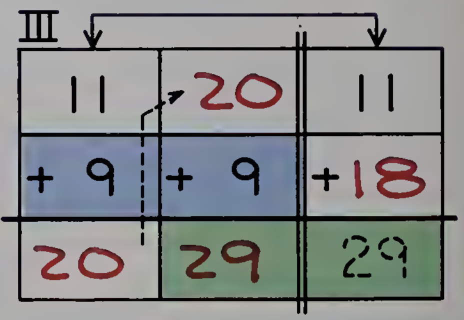
10	22	10
+12	+12	+24
22	34	34

II



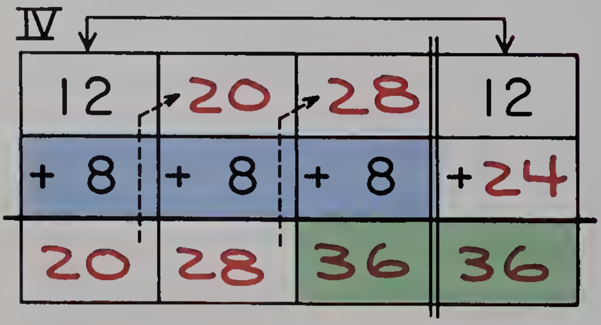
21	36	21
+15	+15	+30
36	51	51

III



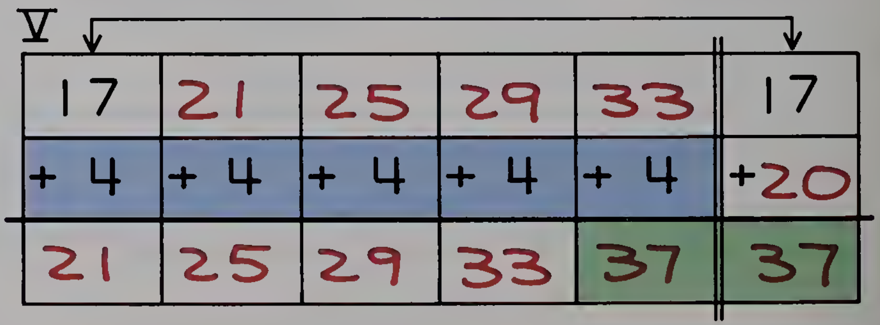
11	20	11
+ 9	+ 9	+18
20	29	29

IV



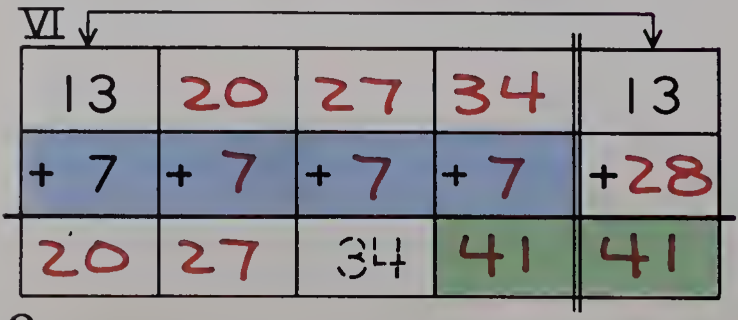
12	20	28	12
+ 8	+ 8	+ 8	+24
20	28	36	36

V



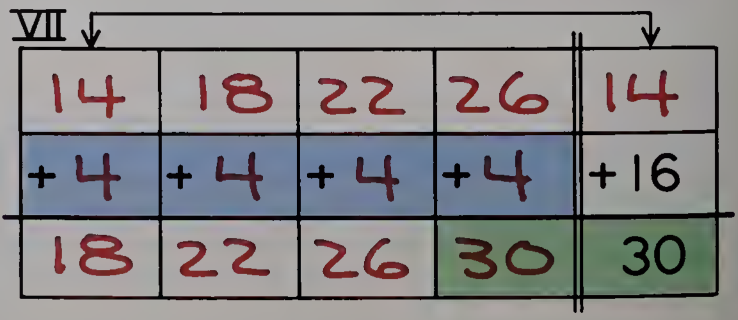
17	21	25	29	33	17
+ 4	+ 4	+ 4	+ 4	+ 4	+20
21	25	29	33	37	37

VI



13	20	27	34	13
+ 7	+ 7	+ 7	+ 7	+28
20	27	34	41	41

VII



14	18	22	26	14
+4	+4	+4	+4	+16
18	22	26	30	30

Notes on Pupil Page 108

Tasks: *Pupils extend each of the given line segments and mark midpoints in accordance with the instructions at the top of the page. They study exercise I to determine the rules for playing* Chain Addition, *and apply their discoveries in doing exercises II-VII.*

Purpose: (a) To review and extend pupils' understanding of linear measurement; (b) to provide pupils with practice in addition and multiplication.

Unifying Ideas: Geometry; measurement; addition and subtraction; multiplication and division; structure.

Materials for Teacher: None.

Materials for Pupils: Straightedges.

The Lesson:

In doing exercises A-U we hope that pupils will be on the alert for shortcuts. For example, pupils might use multiplication by 3 to find the lengths of some of the new segments. Or they might use multiplication by 2 to find the lengths of some of the extensions. In doing exercise T they should capitalize on what they did in exercise O and extend the line segment to the next to the last vertical line.

Exercises J-O make a nice sequence (as do exercises P-S). Since segment K is 1 unit longer than segment J, the new segment for K should be 3 units longer than the new segment for J. Similarly, the midpoint for the new K should be 1½ units to the right of the midpoint for the new J.

The *Chain Addition* game illustrates certain properties of addition. For example, in exercise I the fact that the green boxes carry the same number is a consequence of the fact that addition is an associative operation.

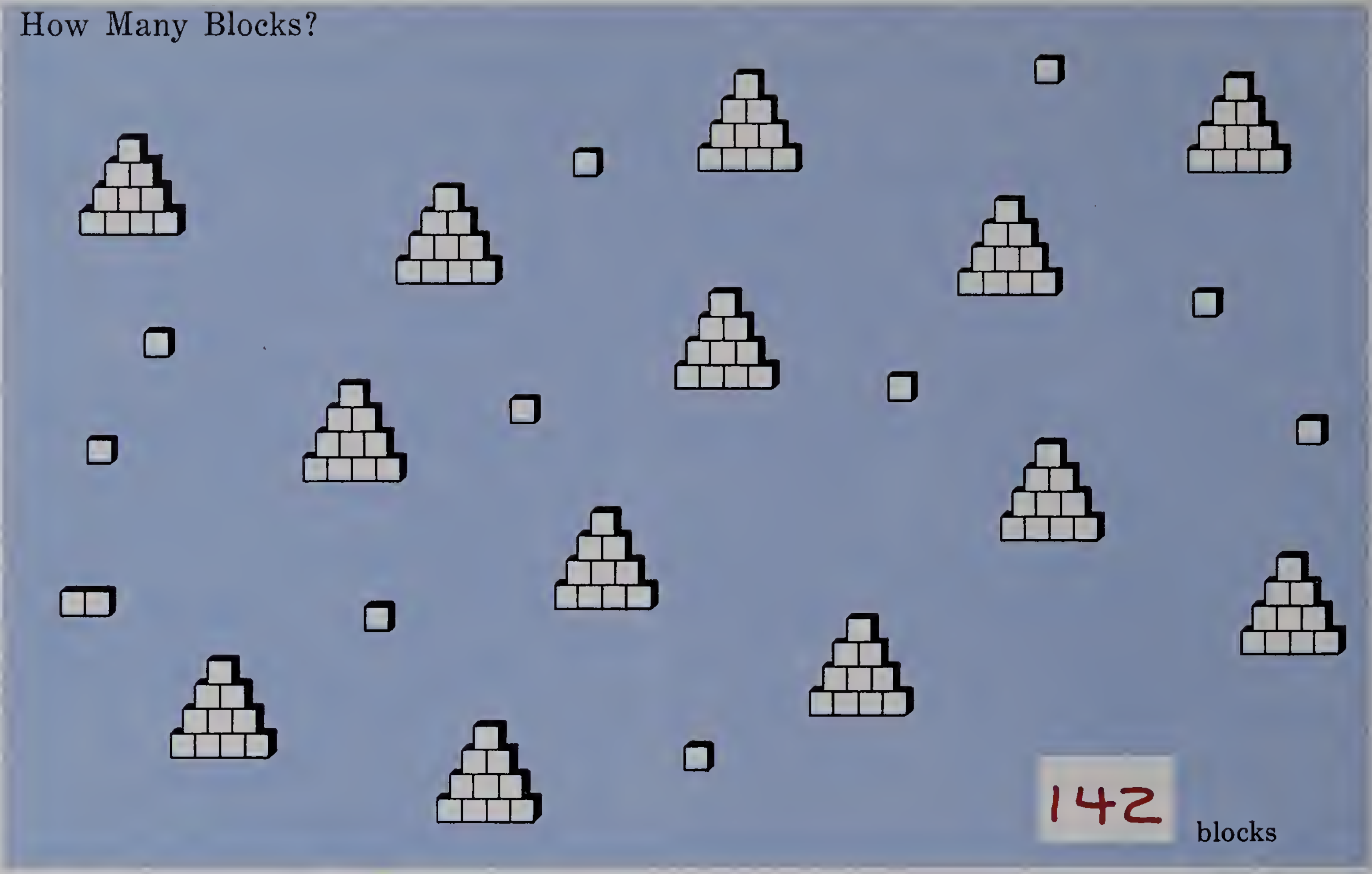
$$\underbrace{\underbrace{(10+12)}_{22}+12}_{34} = \underbrace{10+\underbrace{(12+12)}_{24}}_{34}$$

Chain Addition games involving more than three columns involve repeated applications of the associative principle for addition. For example, consider exercise VII:

$$\begin{aligned}(((14+4)+4)+4)+4 &= ((14+4)+4)+(4+4) && \text{[Associativity]}\\ &= (14+4)+(4+(4+4)) && \text{[Associativity]}\\ &= 14+(4+(4+(4+4))) && \text{[Associativity]}\\ &= 14+(4\times 4) && \text{[Multiplication is repeated addition.]}\end{aligned}$$

Headline for the Day: Dad has 10 choices of outfits. If he has __ pants, he has . . . shirts. Make a table.

How Many Blocks?

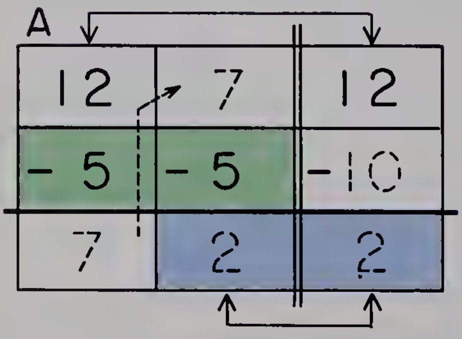


142 blocks

A game called CHAIN SUBTRACTION

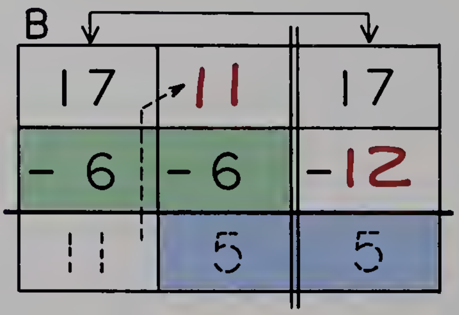
What are the rules?

A



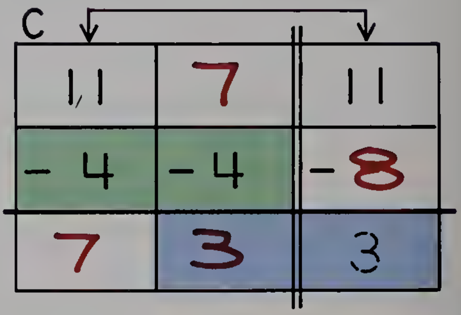
12	7	12
- 5	- 5	-10
7	2	2

B



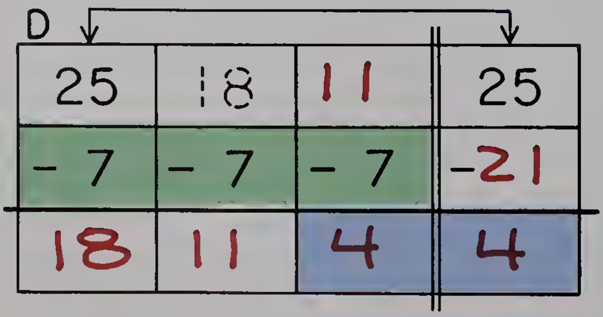
17	11	17
- 6	- 6	-12
11	5	5

C



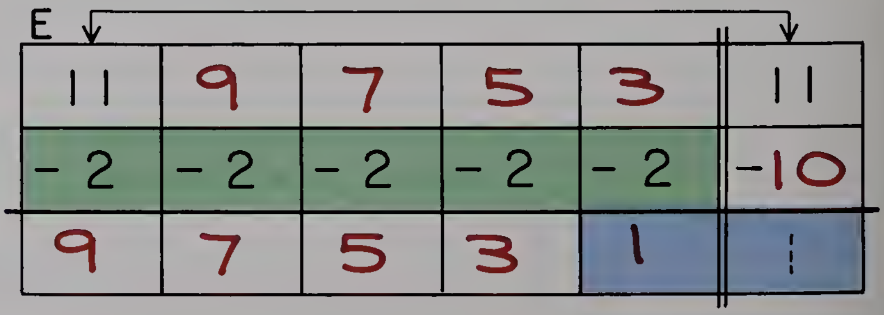
11	7	11
- 4	- 4	- 8
7	3	3

D



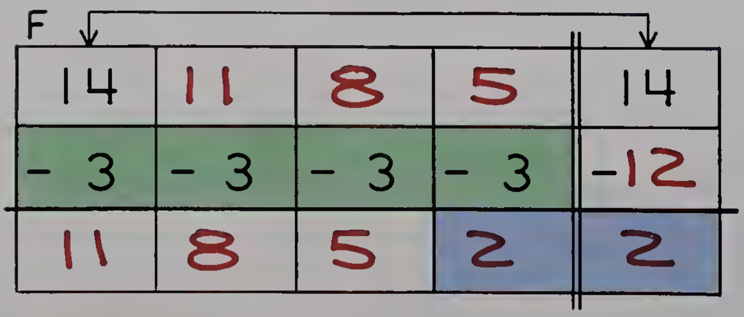
25	18	11	25
- 7	- 7	- 7	-21
18	11	4	4

E



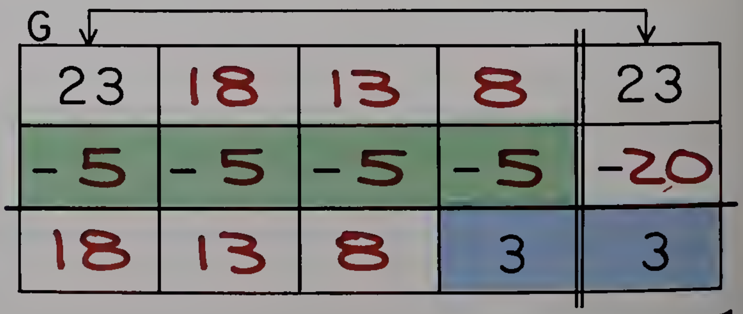
11	9	7	5	3	11
- 2	- 2	- 2	- 2	- 2	-10
9	7	5	3	1	1

F



14	11	8	5	14
- 3	- 3	- 3	- 3	-12
11	8	5	2	2

G



23	18	13	8	23
- 5	- 5	- 5	- 5	-20
18	13	8	3	3

Notes on Pupil Page 109

Tasks: *Pupils determine the total number of blocks in the top section of the page. They study exercise A to determine the rules for playing* Chain Subtraction, *and apply their discoveries in doing exercises B-G.*

Purpose: (a) To remind pupils of the connection between decimal partitioning and reporting on the numerosity of a display; (b) to provide pupils with practice in subtraction and multiplication; (c) to prepare pupils for the division-with-remainder process.

Unifying Ideas: Number and counting; numeration; addition and subtraction; multiplication and division; structure.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Pupils should be able to do the counting exercise without prior explanation. You may wish to follow up on the exercise by asking pupils to describe the picture if there were 183 blocks, 279 blocks, 305 blocks, etc. Some typical answers—18 stacks and 3 loose blocks, 16 stacks and 23 loose blocks.

The *Chain Subtraction* game asks questions like those asked in carrying out division-with-remainder problems. For example, in exercise F we are asking: What remains of a 14-group when we have removed as many 3-groups as possible? Each *Chain Subtraction* game leads to a number sentence which is of the form:

dividend = (divisor × quotient) + remainder

For example, exercise D leads to the sentence:

$$25 = (7 \times 3) + 4$$

The quotient is the maximum number of 7-groups which can be removed from a 25-group.

Drill

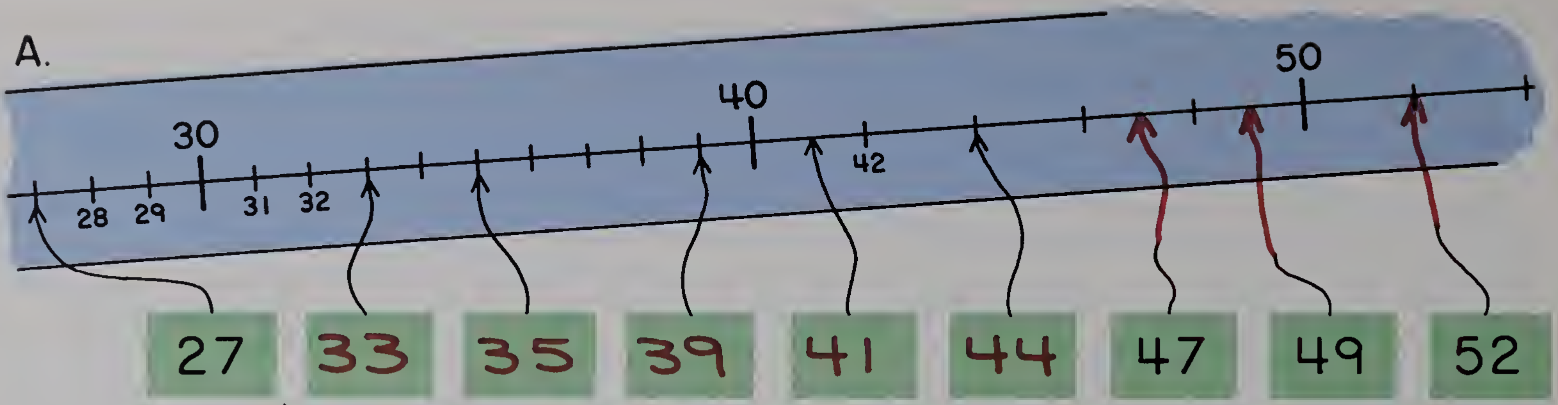
Here are some suggestions for a drill sheet:

Subtract.

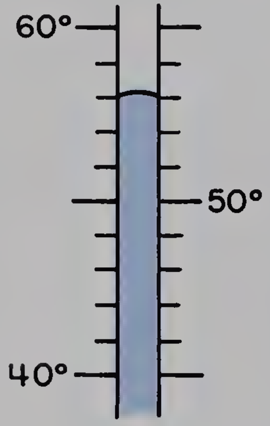
74 −28	80 −43	48 −29	97 −38	61 −16
87 −39	83 −15	95 −37	80 −25	42 −19
64 −25	72 −34	93 −46	66 −29	73 −29

Headline for the Day: Make up stories that fit the chain subtraction problems on page 109.

A.

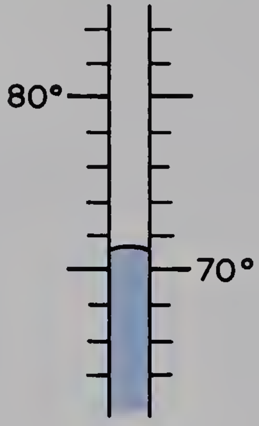

28
29
30
31
32
40
42
50
27
33
35
39
41
44
47
49
52

B.



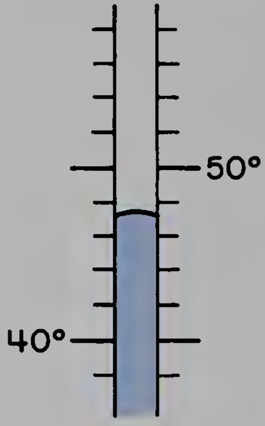
56° above zero

C.



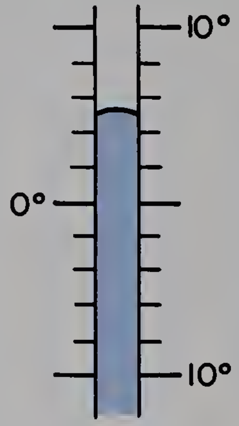
71° above zero

D.



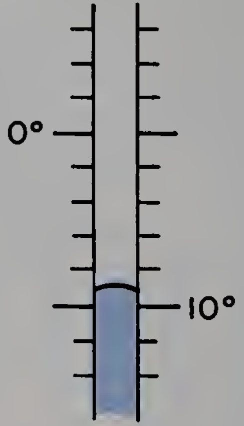
47° above zero

E.



5° above zero

F.



9° below zero

CHAIN ADDITION

①

36	52	68	84	36
+16	+16	+16	+16	+64
52	68	84	100	100

②

19	26	33	40	19
+ 7	+ 7	+ 7	+ 7	+28
26	33	40	47	47

③

12	17	22	27	12
+ 5	+ 5	+ 5	+ 5	+20
17	22	27	32	32

④

8	31	54	77	8
+23	+23	+23	+23	+92
31	54	77	100	100

⑤

17	23	29	35	17
+ 6	+ 6	+ 6	+ 6	+24
23	29	35	41	41

⑥

10	18	26	34	10
+ 8	+ 8	+ 8	+ 8	+32
18	26	34	42	42

Notes on Pupil Page 110

Tasks: Pupils assign labeled number cards to appropriate points on the number line, record the indicated temperature readings, and complete each Chain Addition *game.*

Purpose: (a) To review and extend pupils' understanding of numeration systems; (b) to provide pupils with practice in applying this understanding in reading a thermometer; (c) to provide pupils with practice in addition and subtraction.

Unifying Ideas: Numeration; addition and subtraction; measurement; structure.

Materials for Teacher: Thermometers; containers for holding water.

Materials for Pupils: None.

The Lesson:

Mix quantities of hot and cold water and let pupils determine the temperatures of the mixtures.

Pupils may be put to work on the *Chain Addition* games without special preparation.

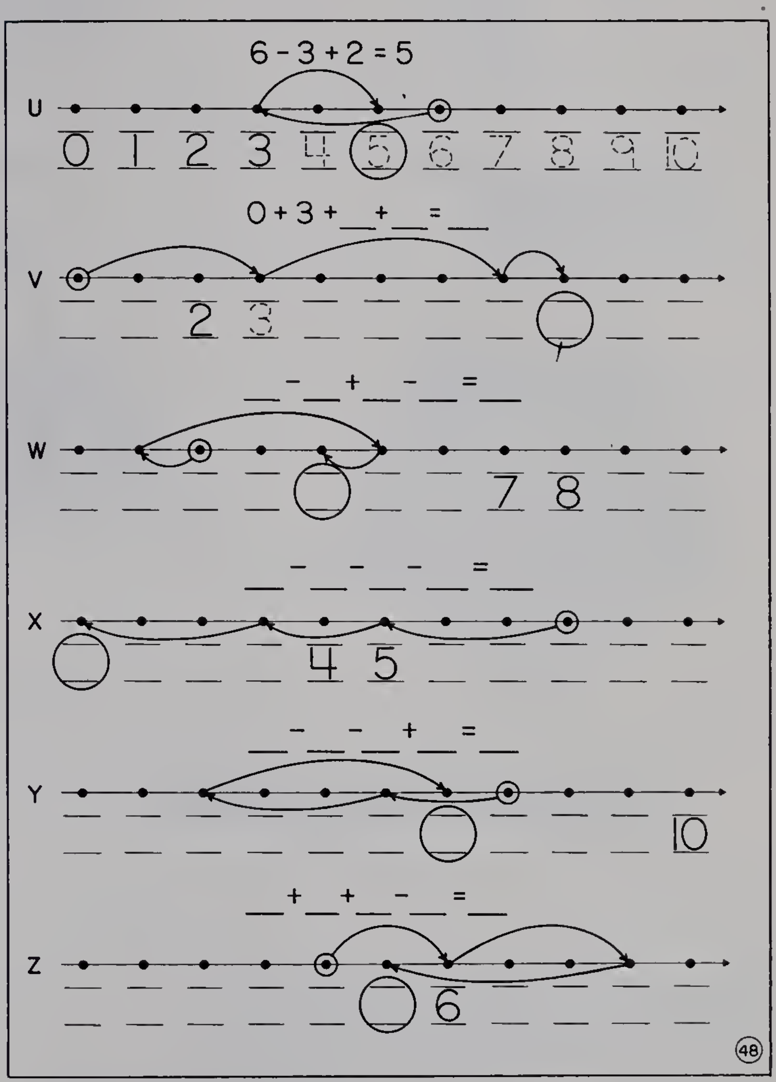
Drill

Here is a suggested drill sheet:

Watch the Signs!

46 −39	55 +26	37 −21	42 −21	38 +27
59 −46	22 −13	92 −38	27 +45	53 −19
46 −29	46 +29	31 −21	45 +55	55 −45

Also consider using as a drill sheet page 48 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:

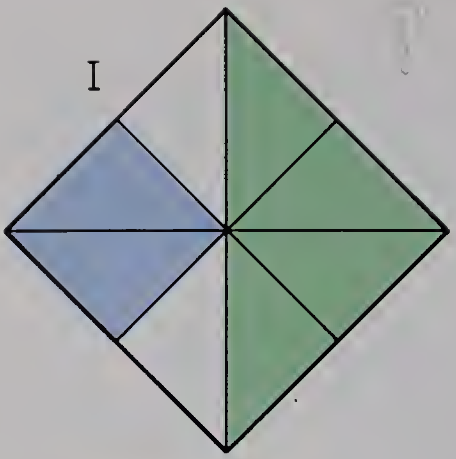


Here are some stories for drill purposes:

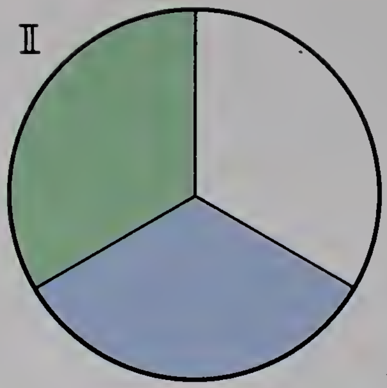
1) Sue colored 3 pictures in each of 5 coloring books. Phil colored 5 pictures in each of 3 coloring books.
2) Amos had 15 cents. He spent 3 cents, 2 cents, and 4 cents.

Headline for the Day: Make up stories that fit the six chain addition problems on page 110.

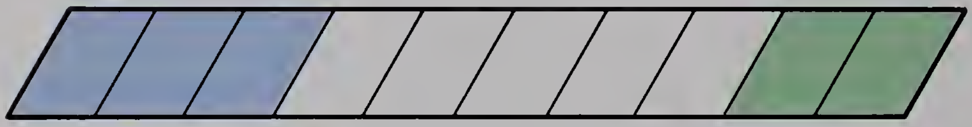
I



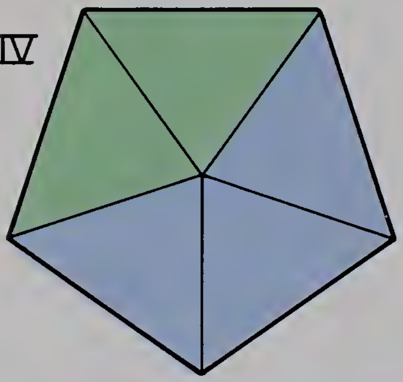
II



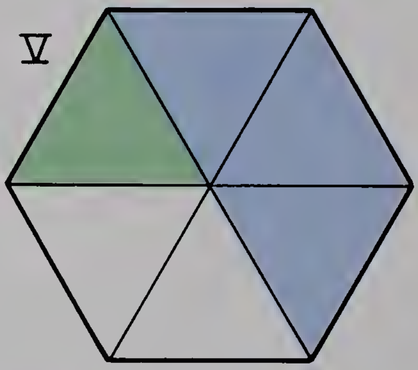
III



IV



V



	Part shaded green	Part shaded blue	Part not shaded
I	$\frac{4}{8}$ or $\frac{1}{2}$	$\frac{2}{8}$ or $\frac{1}{4}$	$\frac{2}{8}$ or $\frac{1}{4}$
II	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
III	$\frac{2}{10}$ or $\frac{1}{5}$	$\frac{3}{10}$	$\frac{5}{10}$ or $\frac{1}{2}$
IV	$\frac{2}{5}$	$\frac{3}{5}$	NONE
V	$\frac{1}{6}$	$\frac{3}{6}$ or $\frac{1}{2}$	$\frac{2}{6}$ or $\frac{1}{3}$

CHAIN SUBTRACTION

a

29	25	21	17	29
− 4	− 4	− 4	− 4	−16
25	21	17	13	13

b

50	38	26	14	50
−12	−12	−12	−12	−48
38	26	14	2	2

c

43	34	25	16	43
− 9	− 9	− 9	− 9	−36
34	25	16	7	7

d

32	25	18	11	32
− 7	− 7	− 7	− 7	−28
25	18	11	4	4

e

37	29	21	13	37
− 8	− 8	− 8	− 8	−32
29	21	13	5	5

f

31	26	21	16	31
− 5	− 5	− 5	− 5	−20
26	21	16	11	11

Notes on Pupil Page 111

Tasks: *Pupils record in the table the fraction of each figure shaded blue, green, or unshaded, and complete each* Chain Subtraction *game.*

Purpose: (a) To review and extend pupils' understandings of fractions; (b) to provide further opportunities for pupils to perfect their skills in working with the addition and subtraction algorithms.

Unifying Ideas: Number and counting; numeration; addition and subtraction; geometry; structure.

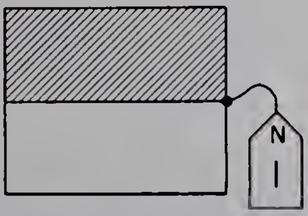
Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Pupils should be able to complete this page without prior explanation. You may wish to follow up with the following activities with fractions.

If-Then



Shaded	Unshaded	Tag
½	½	1
		2
		½

Suppose that we change the label or tag. Let's write 2 in the tag. Now, what do the shaded and unshaded regions suggest? (1 and 1.)

Suppose that we change the tag to read ½. Now, what do the shaded and unshaded regions suggest? (¼ and ¼) Without discussion, make the comment: It seems that a half of one half is one fourth. In money this would mean that half of a half-dollar is the same as a quarter of a dollar.

Fractions and the Real World

Pupils have by now discovered that whole numbers taken singly may not be adequate in describing all real-life situations. List a few situations that demonstrate this and ask pupils to add to this list.

half of an object
half of a dollar or dime
half past two
half price

$\boxed{\frac{1}{2}}$

each child got a third
one third of the way around a circle
one third off
one third of the objects
one out of every three

$\boxed{\frac{1}{3}}$

a quarter of a dollar
a quarter of an inch
quarter after four
a quarter of a gallon (one quart)
a fourth of a pie or an apple

$\boxed{\frac{1}{4}}$

three fourths of the way around
three out of four times at bat
three quarters of an hour
three quarters as large

$\boxed{\frac{3}{4}}$

two thirds of a pie
two thirds of the way
two out of every three
two thirds of the apples are ripe

$\boxed{\frac{2}{3}}$

two half-dollars
two halves of a sandwich
two out of two

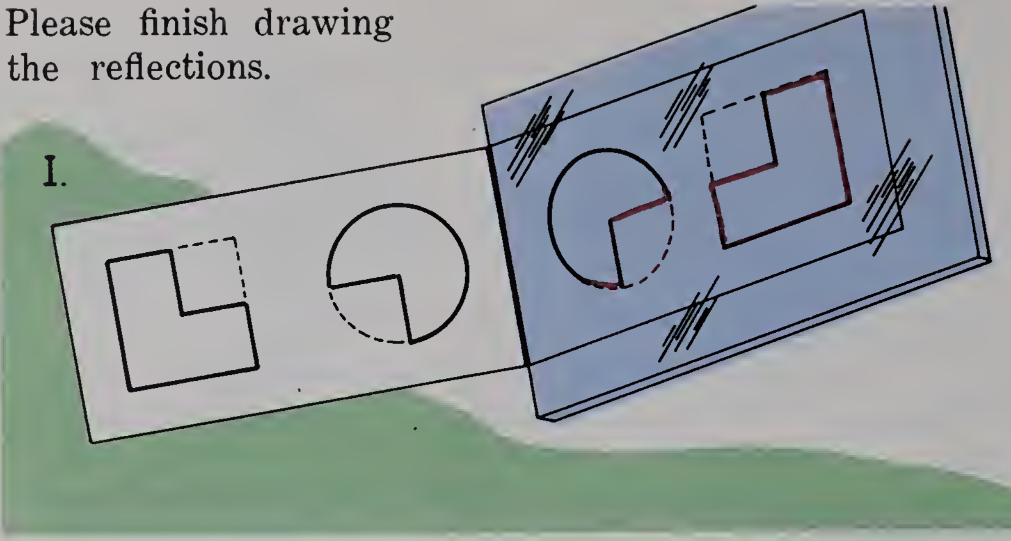
$\boxed{\frac{2}{2}}$

It is not necessary to discuss the mathematical symbols. The only purpose in this discussion is to be aware of the ideas behind these common expressions.

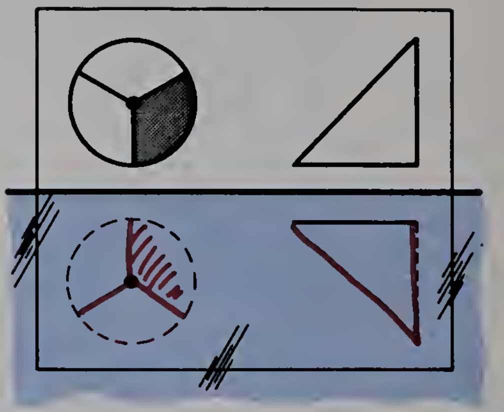
Headline for the Day: Tom gave the clerk $1.00 to pay for purchases of 32¢ and 19¢. He received his change in the fewest coins possible. What were the coins?

Please finish drawing
the reflections.

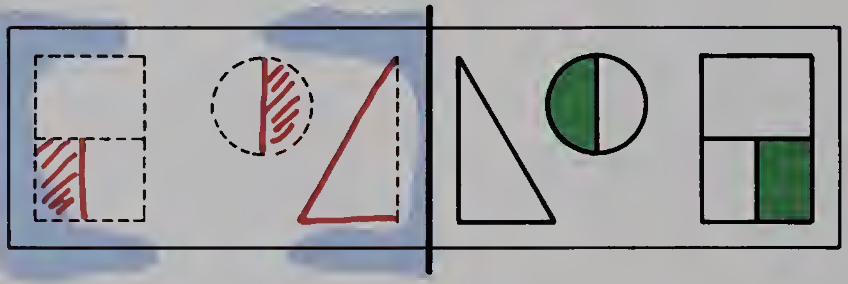
I.



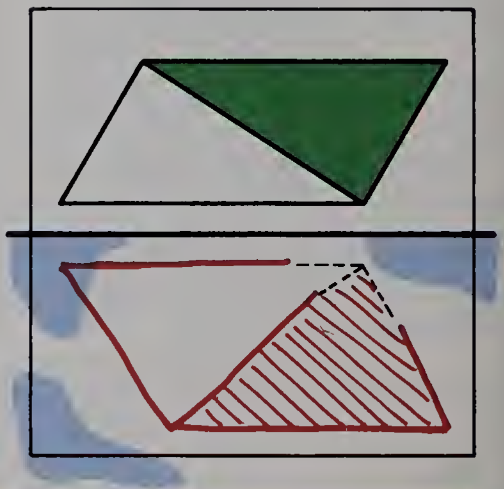
II.



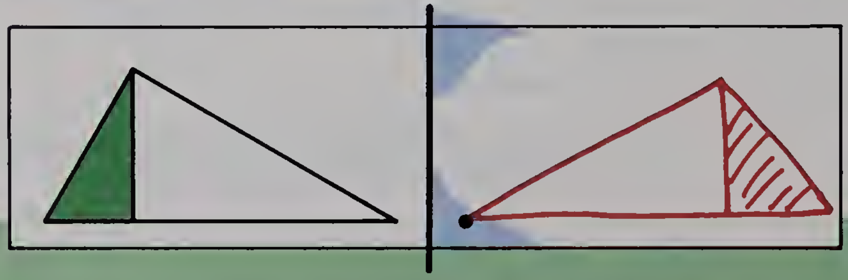
III.



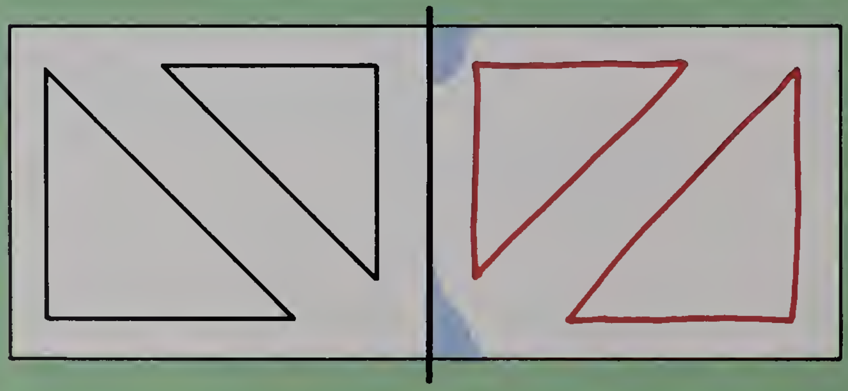
IV.



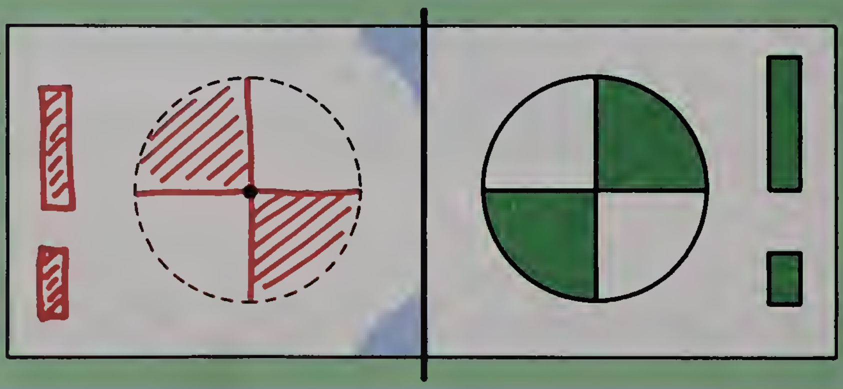
V.



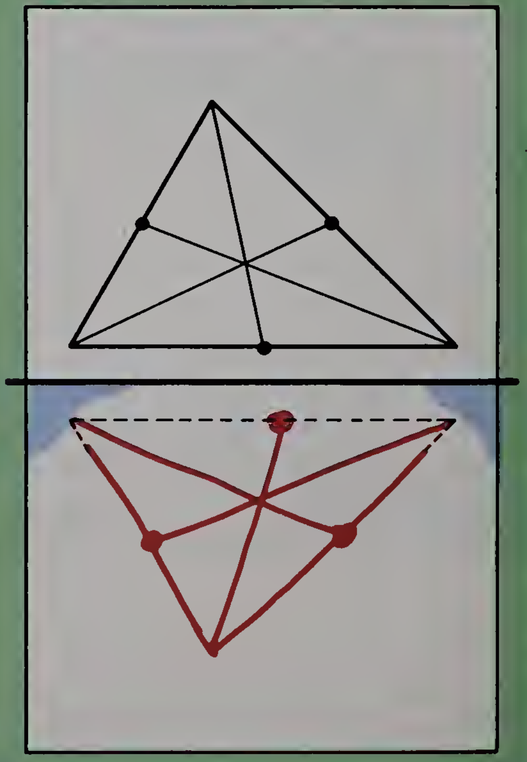
VI.



VII.



VIII.



IX. What's
My Rule? 3 , 6 , 9 , 12 , 15 , 18 , 21 , 24

Notes on Pupil Page 112

Tasks: Pupils sketch the reflections and record the missing entries in the What's My Rule? *game.*

Purpose: To further extend pupils' understanding of the geometry of reflections.

Unifying Ideas: Geometry; functions and relations.

Materials for Teacher: None.

Materials for Pupils: Pocket mirrors.

The Lesson:

See the Notes on Pupil Page 106 for suggestions for handling pupil page 112.

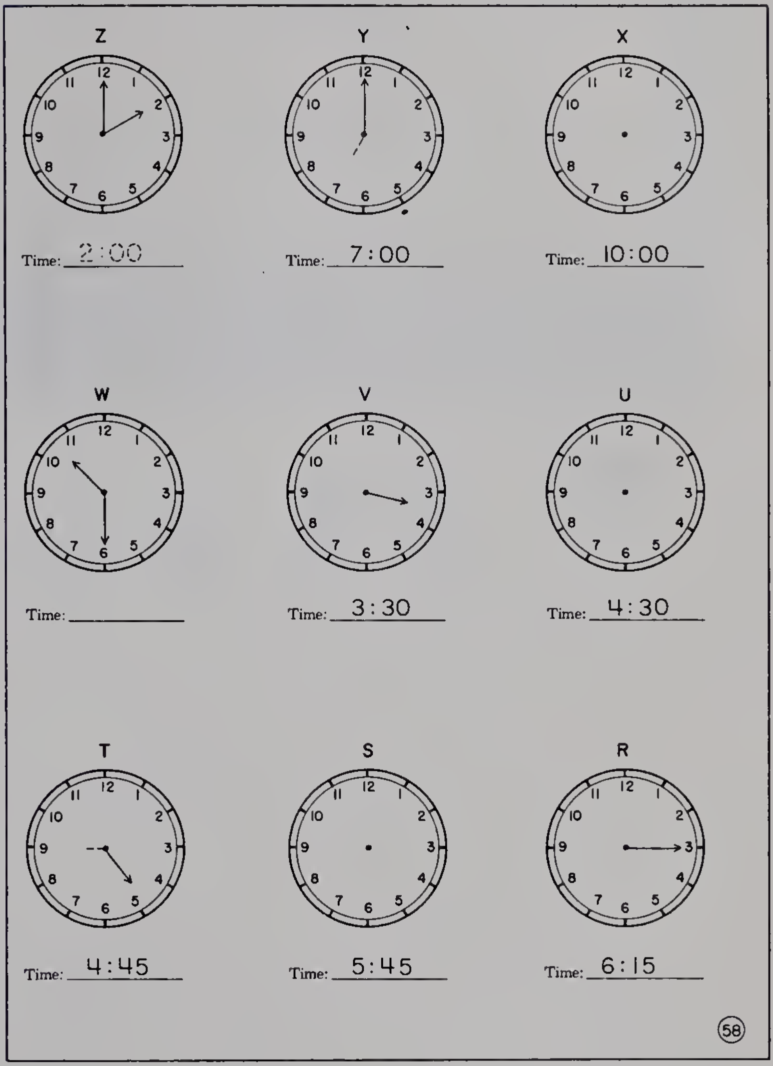
Drill

Consider using the following drill sheet:

Watch the Signs!

82	93	65	75	51
−54	−74	−18	+17	−13
47	76	81	78	36
+37	−37	−58	−49	+45
71	95	34	92	60
−24	−39	+58	−47	−25

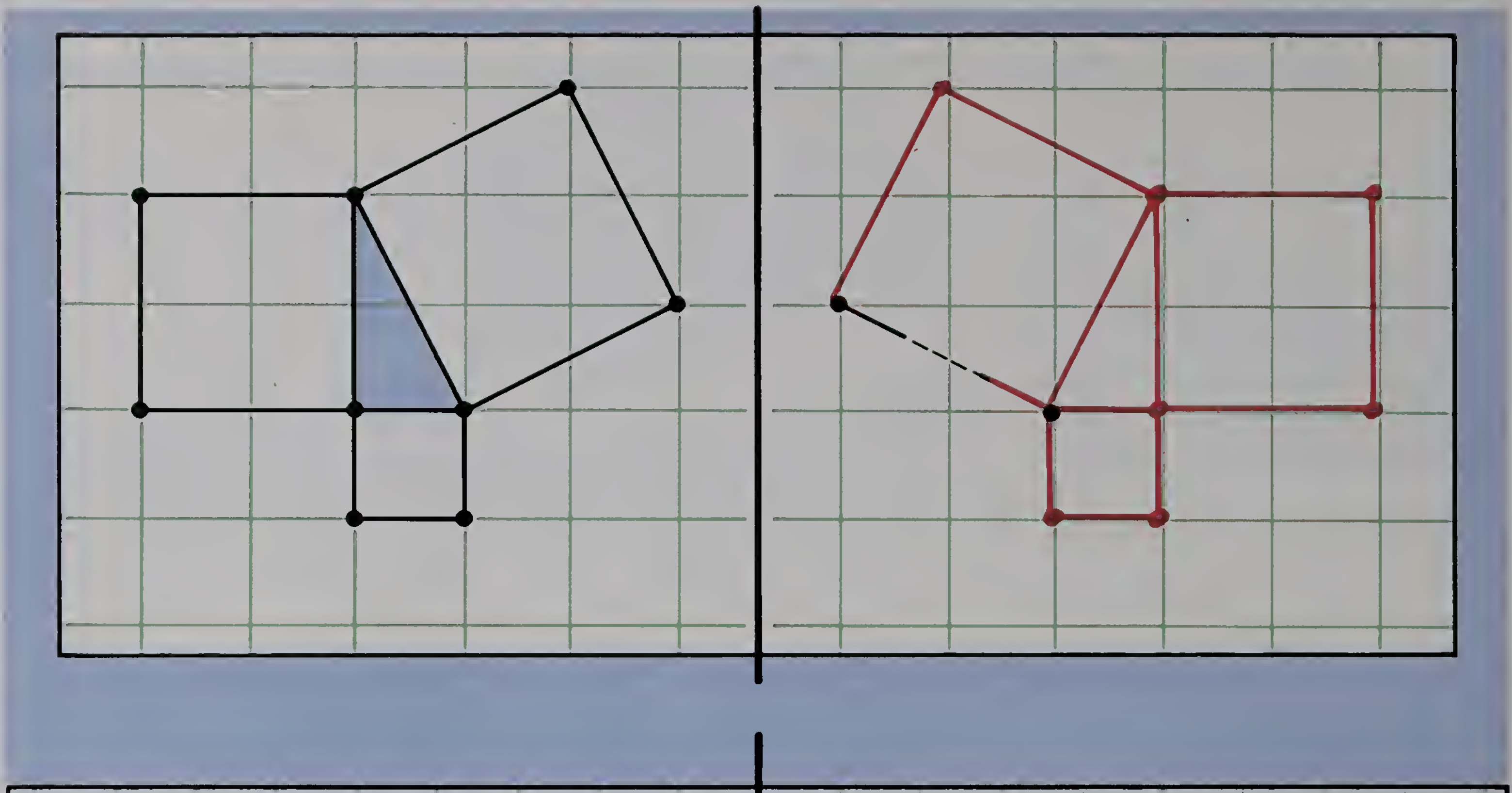
Also, consider using as a drill sheet page 58 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:

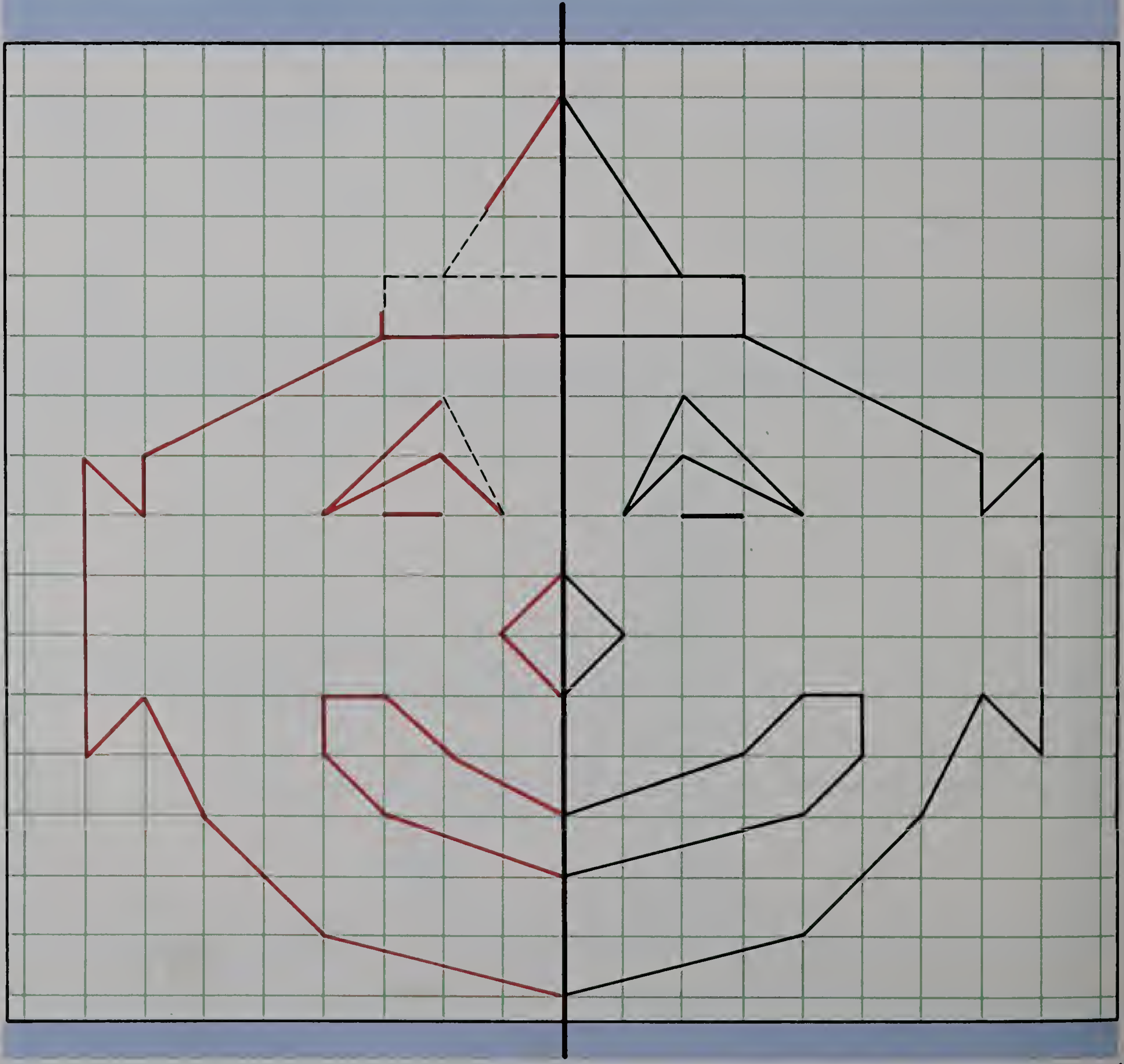


Time: 2:00
Time: 7:00
Time: 10:00
Time:
Time: 3:30
Time: 4:30
Time: 4:45
Time: 5:45
Time: 6:15

(58)

Headline for the Day: Write number sentences that fit: The length of 5 posts laid end to end in a straight line, if each post is 5 feet long; if each post is 10 feet long.





Notes on Pupil Page 113

Tasks: Pupils sketch reflections.

Purpose: To further extend pupils' understanding of the geometry of reflections.

Unifying Ideas: Geometry.

Materials for Teacher: Bottle of ink.

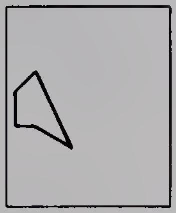
Materials for Pupils: Sheets of blank paper; pencils with very soft lead.

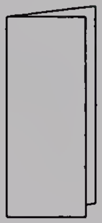
The Lesson:

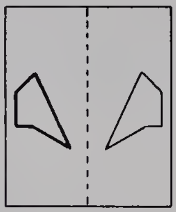
Pupil page 113 is a continuation of pupil page 112.

Ink Blot Figures

Draw a figure on a sheet of paper, using soft lead and pressing hard. Fold the paper lengthwise and rub hard. Open the fold and notice that the transferred image is a reflection (the crease represents the mirror).

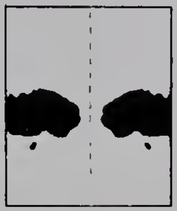


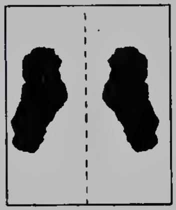


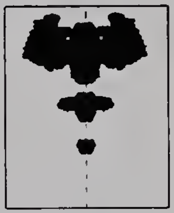


Let pupils try this for several figures. Some of the figures should be drawn so that the crease crosses the figure.

Rather spectacular results can be obtained by folding a sheet of blank paper, unfolding it, spattering a few drops of ink on the paper, and refolding. Here are some samples:







Headline for the Day: If 2 balls cost 18 cents, how much do 3 balls cost?

ALL ABOUT COINS:
pennies, nickels, dimes and quarters.

A

5¢ can be made with two different combinations of coins. A nickel is worth 5¢. Five pennies are worth 5¢. We can record this in this way.

(1) 0 p, 1 n.
(2) 5 p, 0 n.

B

10¢ can be made in four ways.

(1) 10 p, 0 n, 0 d.
(2) 0 p, 0 n, 1 d.
(3) 5 p, 1 n, 0 d.
(4) 0 p, 2 n, 0 d.

C

8¢ can be made in two ways.

(1) 8 p, 0 n.
(2) 3 p, 1 n.

D

12¢ can be made in four ways.

(1) 12 p, 0 n, 0 d.
(2) 7 p, 1 n, 0 d.
(3) 2 p, 2 n, 0 d.
(4) 2 p, 0 n, 1 d.

E

15¢

(Six ways)

15 p, 0 n, 0 d.
10 p, 1 n, 0 d.
5 p, 2 n, 0 d.
5 p, 0 n, 1 d.
0 p, 3 n, 0 d.
0 p, 1 n, 1 d.

F

18¢

(Six ways)

18 p, 0 n, 0 d.
13 p, 1 n, 0 d.
8 p, 2 n, 0 d.
8 p, 0 n, 1 d.
3 p, 3 n, 0 d.
3 p, 1 n, 1 d.

G

20¢

(Nine ways)

20 p, 0 n, 0 d.
15 p, 1 n, 0 d.
10 p, 2 n, 0 d.
10 p, 0 n, 1 d.
5 p, 3 n, 0 d.
5 p, 1 n, 1 d.
0 p, 4 n, 0 d.
0 p, 2 n, 1 d.
0 p, 0 n, 2 d.

Tasks: Pupils record the various possible combinations of coins for each amount of money specified on pupil pages 114 and 115.

Purpose: To review and extend pupils' understanding of numeration systems and measurement.

Unifying Ideas: Numeration; measurement.

Materials for Teacher: Coins for activities described below.

Materials for Pupils: None.

The Lesson:

Prepare pupils for these pages by doing the following. Put some coins in your hand and close your fingers over them. Say: In my hand, I have 9¢ in coins. Can you be sure of what coins I am holding? (No; you could have 9 pennies or you could have a nickel and 4 pennies.) No, you cannot be sure what coins I am holding. Can you be sure of anything about the coins I am holding?

If pupils have not played this kind of game before, you might ask some of the following questions:

Can you say for sure whether or not I have any dimes? (Yes) Can you say for sure that I have at least 5 coins? (Yes) Can you say for sure that I do not have more than 9 coins? (Yes) Can you say for sure that I have at least 1 nickel? (No) Can you say for sure that I have at least 4 pennies? (Yes)

In this game of *Can you say for sure?* precise expression is important. Can you say for sure that I have either 5 coins or 9 coins? (Yes)

Change the coins in your hand. Now the coins I am holding are worth 13¢ altogether. What can you say for sure about these coins? Some acceptable answers are:

You have at least 3 pennies. You have at least 4 coins. You do not have more than 13 coins. If you have 5 coins, they are 2 nickels and 3 pennies. If you have more than 9 coins, then you have 13 pennies. If you have 4 pennies, then you have at least 8 pennies.

Without telling the children what coins you actually have in your hand, change the question to: Tell me some combinations of coins I could have. After a few have been suggested ask: How many different combinations of coins are there that are worth 13¢? Make a chart:

Dimes	Nickels	Pennies	Total No. of Coins
		13	13
	1	8	9
	2	3	5
1		3	4

There are 4 possible combinations.

A new game: Now I have coins worth 15¢ in my hand. What can you say for sure about these coins? Name some of the possible combinations. How many combinations are there?

Headline for the Day: Write a number sentence that fits: 5 posts are driven into the ground 5 feet apart in a straight line. What is the distance between the first and last post?

25¢

(Thirteen ways)

H

25 p, 0 n, 0 d, 0 q.
20 p, 1 n, 0 d, 0 q.
15 p, 2 n, 0 d, 0 q.
15 p, 0 n, 1 d, 0 q.
10 p, 3 n, 0 d, 0 q.
10 p, 1 n, 1 d, 0 q.
5 p, 4 n, 0 d, 0 q.
5 p, 2 n, 1 d, 0 q.
5 p, 0 n, 2 d, 0 q.
0 p, 5 n, 0 d, 0 q.
0 p, 3 n, 1 d, 0 q.
0 p, 1 n, 2 d, 0 q.
0 p, 0 n, 0 d, 1 q.

Did you use the same number of coins in any two combinations? No

30¢

J

30 p, 0 n, 0 d, 0 q.
25 p, 1 n, 0 d, 0 q.
20 p, 2 n, 0 d, 0 q.
20 p, 0 n, 1 d, 0 q.
15 p, 3 n, 0 d, 0 q.
15 p, 1 n, 1 d, 0 q.
10 p, 4 n, 0 d, 0 q.
10 p, 2 n, 1 d, 0 q.
10 p, 0 n, 2 d, 0 q.
5 p, 5 n, 0 d, 0 q.
5 p, 3 n, 1 d, 0 q.
5 p, 1 n, 2 d, 0 q.
5 p, 0 n, 0 d, 1 q.
0 p, 6 n, 0 d, 0 q.
0 p, 4 n, 1 d, 0 q.
0 p, 2 n, 2 d, 0 q.
0 p, 0 n, 3 d, 0 q.
0 p, 1 n, 0 d, 1 q.

Did you use the same number of coins in any two combinations? YES

28¢

I

28 p, 0 n, 0 d, 0 q.
23 p, 1 n, 0 d, 0 q.
18 p, 2 n, 0 d, 0 q.
18 p, 0 n, 1 d, 0 q.
13 p, 3 n, 0 d, 0 q.
13 p, 1 n, 1 d, 0 q.
8 p, 4 n, 0 d, 0 q.
8 p, 2 n, 1 d, 0 q.
8 p, 0 n, 2 d, 0 q.
3 p, 5 n, 0 d, 0 q.
3 p, 3 n, 1 d, 0 q.
3 p, 1 n, 2 d, 0 q.
3 p, 0 n, 0 d, 1 q.

Tasks: Pupils continue the work started on pupil page 114.

Purpose: To review and extend pupils' understanding of numeration systems and measurement.

Unifying Ideas: Numeration; measurement.

Materials for Teacher: None.

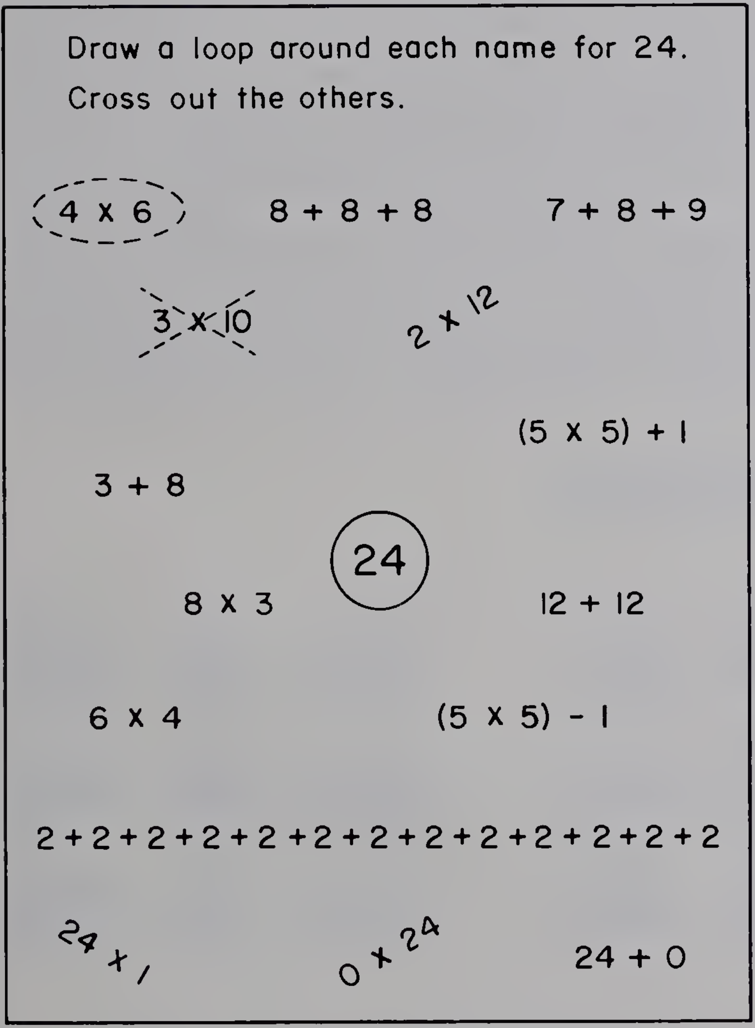
Materials for Pupils: None.

The Lesson:

See the remarks in the Notes on Pupil Page 114.

Drill

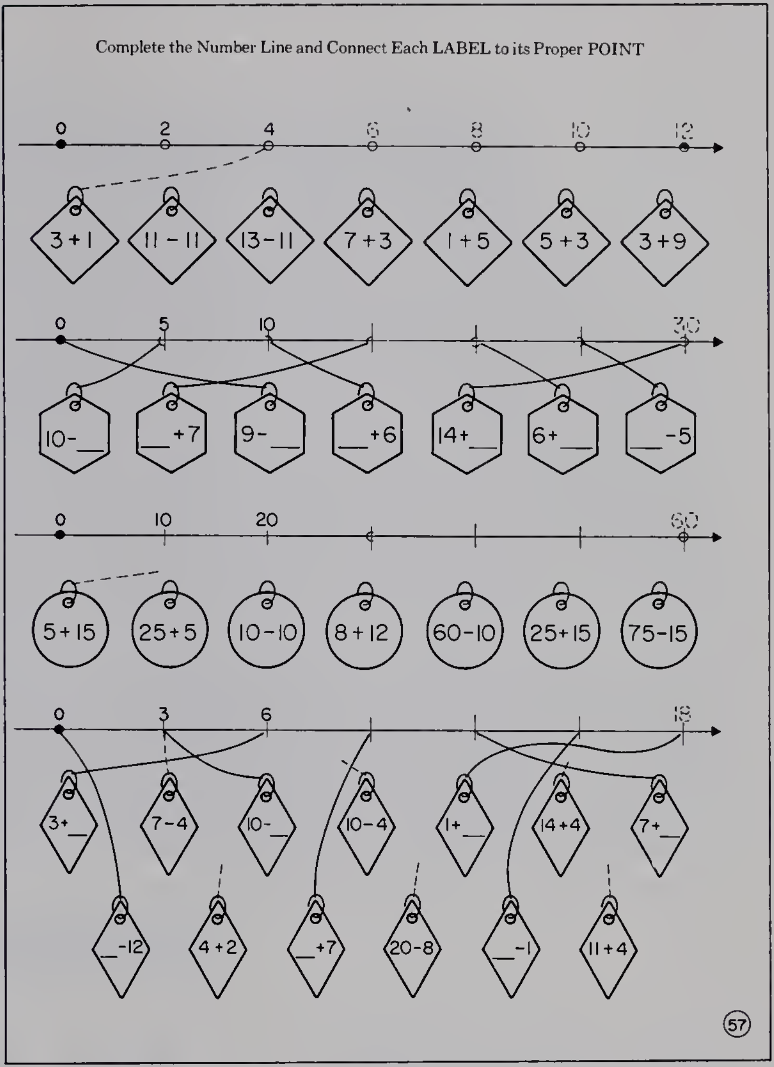
Here is a suggested drill sheet:



Draw a loop around each name for 24.
Cross out the others.

4 x 6
8 + 8 + 8
7 + 8 + 9
~~3 x 10~~
2 x 12
(5 x 5) + 1
3 + 8
24
8 x 3
12 + 12
6 x 4
(5 x 5) - 1
2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2 + 2
24 x 1
0 x 24
24 + 0

Also, consider using as a drill sheet page 57 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



Complete the Number Line and Connect Each LABEL to its Proper POINT

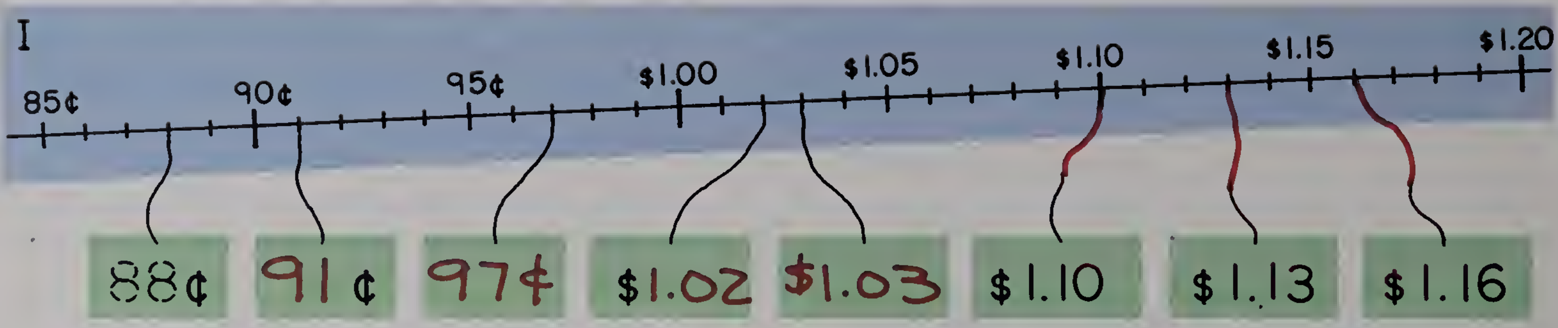
0
2
4
6
8
10
12
3 + 1
11 - 11
13 - 11
7 + 3
1 + 5
5 + 3
3 + 9
0
5
10
30
10-__
__+7
9-__
__+6
14+__
6+__
__-5
0
10
20
60
5 + 15
25 + 5
10 - 10
8 + 12
60 - 10
25 + 15
75 - 15
0
3
6
18
3+__
7 - 4
10-__
10 - 4
1+__
14 + 4
7+__
__-12
4 + 2
__+7
20 - 8
__-1
11 + 4
57

Here are some stories for drill purposes:

1) Charles bought a game for $1.59 and a toy for $.79. He gave the clerk a $5 bill.
2) Bill's little brother used 72 blocks to build a tower. The bottom of the tower was 6 blocks wide and 4 blocks deep.
3) Each shelf of the bookcase had 10 books on it. The bookcase had 4 shelves and Phil took 3 books from each shelf.

Headline for the Day: Write a number story that fits: If hamburgers are 18¢ each and bottles of pop are 12¢ each, how much will 3 hamburgers and 3 bottles of pop cost?

I


85¢
90¢
95¢
$1.00
$1.05
$1.10
$1.15
$1.20
88¢
91¢
97¢
$1.02
$1.03
$1.10
$1.13
$1.16

II

Find the sum	13¢	21¢	33¢	54¢	87¢	$1.41
9¢	22¢	30¢	42¢	63¢	96¢	$1.50
29¢	42¢	50¢	62¢	83¢	$1.16	$1.70
38¢	51¢	59¢	71¢	92¢	$1.25	$1.79
62¢	75¢	83¢	95¢	$1.16	$1.49	$2.03
$1.00	$1.13	$1.21	$1.33	$1.54	$1.87	$2.41
$1.62	$1.75	$1.83	$1.95	$2.16	$2.49	$3.03

WATCH THE SIGNS

A

50 + 13 = 63
− − −
20 + 5 = 25
= = =
30 + 8 = 38

B

80 − 20 = 60
− − −
30 − 7 = 23
= = =
50 − 13 = 37

C

40 − 10 = 30
+ + +
16 − 9 = 7
= = =
56 − 19 = 37

D

41 − 5 = 36
− + −
15 + 4 = 19
= = =
26 − 9 = 17

E

14 + 60 = 74
− − −
9 + 30 = 39
= = =
5 + 30 = 35

F

40 + 12 = 52
− − −
20 + 8 = 28
= = =
20 + 4 = 24

Tasks: *In exercise I pupils label the number cards and draw connecting lines. In exercise II they compute the missing numbers. Pupils are to find the missing numbers in the* Watch the Signs! *games.*

Purpose: (a) To review and extend pupils' understanding of the decimal system; (b) to provide pupils with practice in adding and subtracting quantities of money and in using money notation; (c) to let pupils see the "inner working" of the subtraction algorithm in the *Watch the Signs!* format.

Unifying Ideas: Numeration; addition and subtraction; measurement; structure.

Materials for Teacher: The chart described in the Notes on Pupil Page 27.

Materials for Pupils: None.

The Lesson:

Pupils may work on this page without special preparation.

Note that the *Watch the Signs!* exercises A, E, and F show the partitioning used in the subtraction algorithm for 63—25, 74—39, and 52—28, respectively.

Drill

Here is a suggested drill sheet on multiplication and division:

$4 \times \square = 8$
$6 \times 1 = \square$
$\square \times 5 = 25$
$\square \times 2 = 10$
$\square \times 10 = 80$
$2 \times \square = 6$

$3 \times \square = 15$
$2 \times \square = 4$
$\square \times 3 = 3$
$10 \times 7 = \square$
$\square \times 5 = 50$
$3 \times \square = 6$

Headline for the Day: Make up stories suggested by some of the addition and subtraction facts on page 116.

1

Label the longest amount of time with an A in the blue box; next longest with a B; next with a C; and the shortest with a D.

Four weeks C

Twenty-five days .... D

The month of April .... B

Thirty-five days .... A

Find SUMS of amounts indicated:

A and B: 65 da. C and D: 53 da.

B and C: 58 da. B and D: 55 da.

A and C: 63 da. A and D: 60 da.

2

Pens .... 10¢ each
Pencils .... 7¢ each

Alice gave the clerk 50¢ to pay for a purchase of items listed above.

Indicate what she might have bought and the cost. Also show the amount of change.

	Number of Pens @ 10¢	Number of Pencils @ 7¢	Total Cost	Amount of change from 50¢
A	1	1	17¢	33¢
B	2	2	34¢	16¢
C	1	4	38¢	12¢

$$\begin{array}{r} 25 \\ -\ 5 \\ \hline \end{array}$$

(A) 19 (B) 20 (C) 25 (D) 30

Mark the correct answer.

Mark the furthest from the correct answer.

A ○ B ● C ○ D ⊗ 10 Difference between the answers you marked

3

Which is largest?

(A) 43 (C) 25

(B) 52 (D) 34

A ○ B ● C ⊗ D ○ 27

4

What number comes next in this series?
14, 20, 26, 32, 38,

(A) 50 (C) 44

(B) 48 (D) 40

A ⊗ B ○ C ● D ○ 6

5

Which is largest?

(A) One hundred twenty-one

(B) Fifty plus fifty

(C) Ninety

(D) One hundred nineteen

A ● B ○ C ⊗ D ○ 31

6

Which is longest?

(A) Thirty-one inches

(B) One yard

(C) One and one-half feet

(D) Nineteen inches

A ○ B ● C ⊗ D ○ 18 in.

7

$$\begin{array}{r} 12 \\ 23 \\ 9 \\ +18 \\ \hline \end{array}$$

(A) 45

(B) 51

(C) 62

(D) 73

A ⊗ B ○ C ● D ○ 17

8

Nancy gave the clerk one dollar for a sixty-five cent purchase.
How much change should she get?

(A) 65¢ (C) 45¢

(B) 44¢ (D) 35¢

A ⊗ B ○ C ○ D ● 30¢

Notes on Pupil Page 117

Tasks: *Pupils follow the directions given for each test item.*

Purpose: To test pupils' achievement.

Unifying Ideas: Numeration; addition and subtraction; multiplication and division; measurement.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Suggested point distribution:

1—10 points
2— 6 points
3— 3 points
4— 3 points
5—3 points
6—3 points
7—3 points
8—3 points

If pupils have not yet been taught a jingle about the number of days in each month, now is the time to teach it to them. Exercise I requires pupils to know the number of days in April.

Drill

Here's a suggested drill sheet on "looking for tens" in adding:

$4 + 3 + 6 + 7 = \boxed{20}$

$7 + 2 + 3 = \square$ $\quad$ $8 + 6 + 2 = \square$

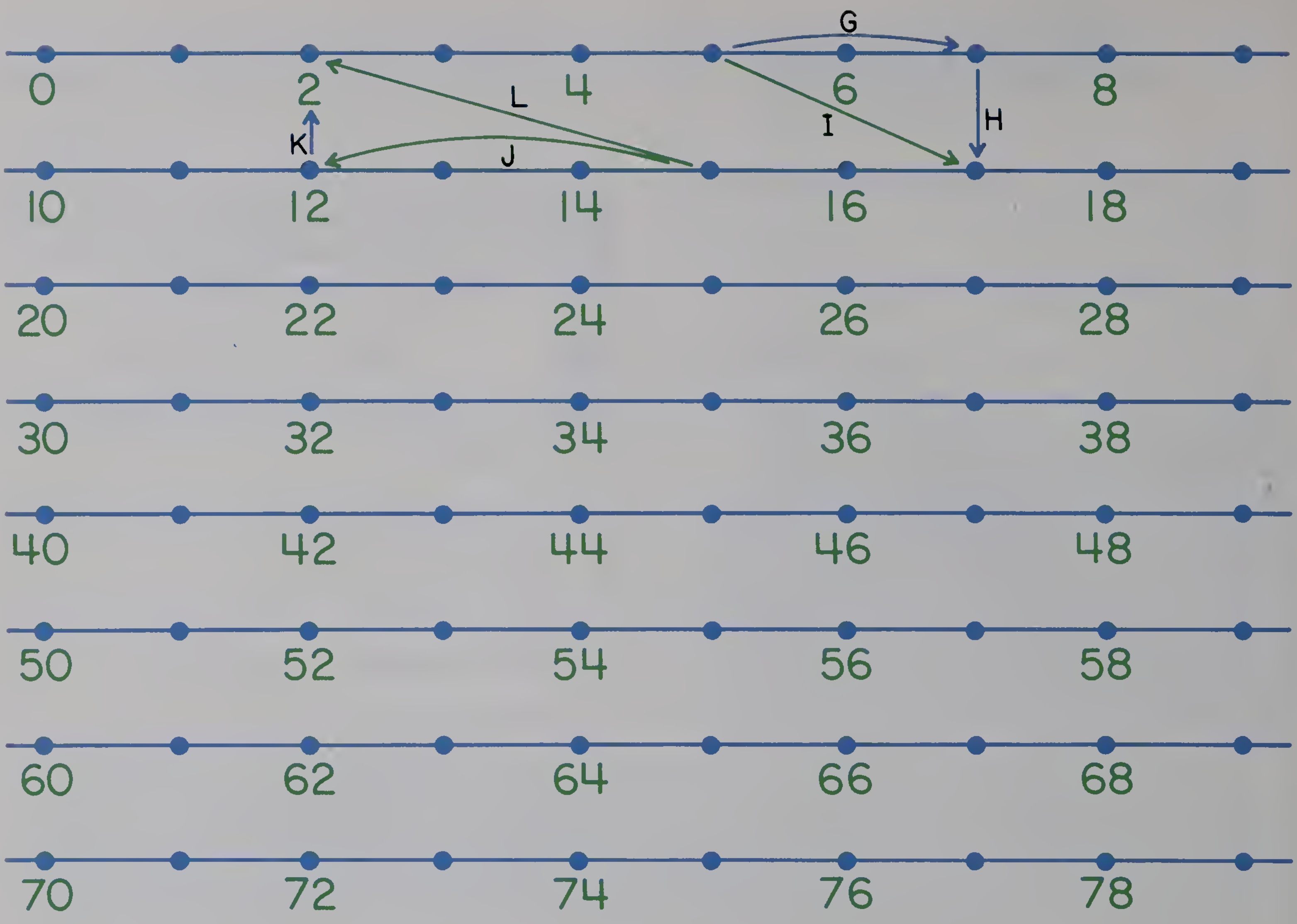
$2 + 5 + 3 + 8 + 7 + 5 + 4 = \square$

$7 + 9 + 13 + 1 + 10 = \square$

$4 + 7 + 16 + \square = 30$

$9 + 9 + 9 + 11 + 11 + 11 = \square$

Headline for the Day: If candy bars are 3 for a quarter, how much would 1 cost?


G
H
I
J
K
L
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
40
42
44
46
48
50
52
54
56
58
60
62
64
66
68
70
72
74
76
78

G

5 → 7
17 → 19
23 → 25
49 → 51

H

7 → 17
19 → 29
25 → 35
51 → 61

I

5 → 17
17 → 29
23 → 35
49 → 61

J

15 → 12
34 → 31
40 → 37
31 → 28

K

12 → 2
31 → 21
37 → 27
28 → 18

L

15 → 2
34 → 21
40 → 27
31 → 18

Tasks: *Pupils use the clues at the top of the page to help them complete the six* What's My Rule? *games displayed beneath the number line chunks.*

Purpose: (a) To help pupils develop an awareness of the fact that the composition of two functions is a function; (b) to provide pupils with practice in addition and subtraction of 2, 10, 12, 3, and 13.

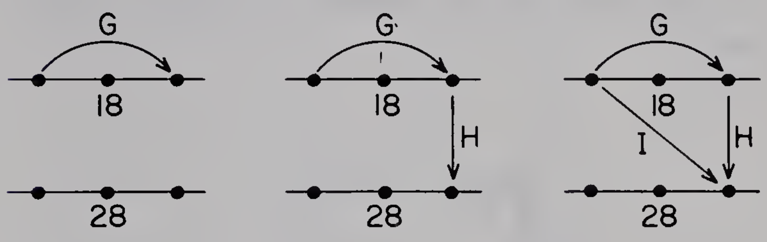
Unifying Ideas: Addition and subtraction; functions and relations.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

The arrow-triangle (top of pupil page 118) consisting of two green arrows and a blue arrow refers to the first lines in the tables for J, K, and L. The other arrow-triangle (two blue arrows and a green arrow) refers to the first lines in the tables for G, H, and I. One could draw similar arrow-triangles for the other lines in the table. For example, consider the second lines in the tables for G, H, and I. Here are the three steps to be used in drawing the triangle:



Composition of Functions

The ideas behind pupil page 118 are of great mathematical importance and much can be done with them.

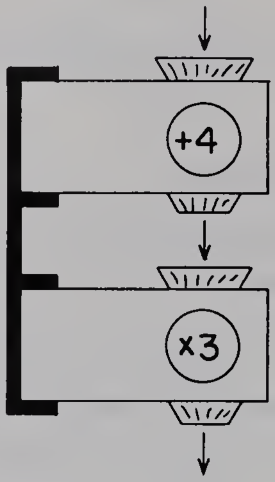
The basic idea is that of an operation on functions called *composition of functions*. This idea is a familiar one to your pupils although they don't

more

Headline for the Day: Write a story that fits: $0 = (75 + 27) - \square$

know (and need not be told) the technical name for it. For example, adding 3 to a number and then adding 7 to the result amounts to adding 10 to the given number. The function addition-of-10 is said to be the result of composing the function addition-of-7 with the function addition-of-3.

When you hook up a pair of machines (functions)—



—the action of the hook-up on numbers put into the first machine is identical in terms of results to the action of a single machine. Here is a *What's My Rule?* table for this pair of machines:

2	5	0	11	16	96	
18	27	12	45	60	300	etc.

Anyone studying this table would be inclined to look for a single rule that could be applied to any number in the top row to produce the corresponding number in the bottom row. Although the function at work here does not have a short name like addition-of- . . . or multiplication-by- . . . , it does have a name. We could call it addition-of-4-followed-by-multiplication-by-3. What is customarily done in mathematics is to assign a temporary name to the function, say, the letter F. Then we might write sentences like:

$2 \xrightarrow{F} 18 \quad 5 \xrightarrow{F} 27 \quad 0 \xrightarrow{F} 12$

On pupil page 118 we use the letter G to name the function addition-of-2. Also, H is addition-of-0, I is addition-of-12, J is subtraction-of-3, K is subtraction-of-10, and L is subtraction-of-13.

We could show the result of applying these functions by writing sentences like:

$5 \xrightarrow{G} 7 \quad 7 \xrightarrow{H} 17 \quad 5 \xrightarrow{I} 17$

$40 \xrightarrow{J} 37 \quad 31 \xrightarrow{K} 21 \quad 29 \xrightarrow{L} 16$

We could extend the shorthand a bit further and write sentences to show the result of applying compositions of these functions:

$5 \xrightarrow{G} \xrightarrow{H} 17$ or even shorter: $5 \xrightarrow{GH} 17$

$3 \xrightarrow{HJ} 10$ (because $3 \xrightarrow{H} 13$ and $13 \xrightarrow{J} 10$)

$12 \xrightarrow{HK} 12 \quad 98 \xrightarrow{HK} 98 \quad 19 \xrightarrow{IL} 18$

It is easy to see that G followed by H is I, and that J followed by K is L.

As you can readily see, there is much game-like activity that can be done with this idea and this notation. For example, pupils can be asked to complete the following sentences:

$9 \xrightarrow{GGGG} \boxed{17} \qquad 6 \xrightarrow{HHH} \boxed{36}$

$143 \xrightarrow{KKKKJ} \boxed{100} \qquad 143 \xrightarrow{KKJKK} \boxed{100}$

$17 \xrightarrow{GGJGJ} \boxed{17} \qquad 5 \xrightarrow{IILILL} \boxed{2}$

The computations involved can be carried out in terms of jumps on the number line. Example—to complete the sentence:

$17 \xrightarrow{GGJGJ} \boxed{}$

Start at 17, make two jumps of 2 to the right, a jump of 3 to the left, a jump of 2 to the right, and a jump of 3 to the left. This brings you to 17.

Pupils can be encouraged to speculate on whether it makes any difference in which order you apply the functions. In the case of addition functions it makes no difference. In case of addition and subtraction functions a certain order might bring you to the left of 0 in which case the game would "explode" unless you were using a number line for positive and negative numbers and 0.

At this point it would be a good idea to introduce multiplication functions. For example, let M be multiplication-by-2. Thus,

$$5 \xrightarrow{M} 10 \ , \ 12 \xrightarrow{M} 24 \ , \ 3 \xrightarrow{MM} 12 \ , \text{ etc.}$$

Notice, for example, that

$$4 \xrightarrow{MH} 18 \text{ and } 4 \xrightarrow{HM} 28.$$

This example shows that when you compose addition and multiplication functions, the order of application can make a difference in results. (If you compose addition-of-0 with any multiplication function, the order makes no difference. For what other pair of functions—an addition function and a multiplication function—is composition not affected by order?)

Professor David A. Page, presently affiliated with the University of Illinois and with Educational Services, Inc., has done much to popularize elementary school work with functions and to explore the possibilities of this topic.

Drill

Here is a suggested drill sheet:

Add.

23	62	26	40	37
49	21	32	27	29
3	8	39	23	20
—	—	—	—	—
18	53	6	59	50
26	26	41	15	9
13	9	28	4	12
—	—	—	—	—
31	37	50	9	40
28	25	30	27	13
8	17	20	33	19
—	—	—	—	—

Price List

Milk	7¢ a carton
Sandwiches	12¢ each
Pie	15¢ a piece

Mr. Barnes worked at a food counter where they used the price list shown on the left. He kept a record of each sale he made. He used M for milk, S for sandwiches, and P for pie.

No one bought more than 2 of any item.

Fill in the missing entries in his record:

0 M; 1 S; 1 P .... 27 ¢	2 M; 0 S; 1 P .... 29 ¢
2 M; 0 S; 0 P .... 14 ¢	1 M; 2 S; 1 P .... 46 ¢
1 M; 1 S; 1 P .... 34 ¢	0 M; 0 S; 1 P .... 15 ¢
1 M; 1 S; 0 P .... 19 ¢	1 M; 0 S; 1 P .... 22 ¢
2 M; 1 S; 0 P .... 26 ¢	1 M; 0 S; 2 P .... 37 ¢
0 M; 2 S; 2 P .... 54 ¢	2 M; 2 S; 2 P .... 68 ¢

WATCH THE SIGNS!

A

50	−	17	=	33
−		+		−
20	+	4	=	24
=		=		=
30	−	21	=	9

B

11	−	8	=	3
+		+		+
60	−	20	=	40
=		=		=
71	−	28	=	43

C

20	+	31	=	51
+		−		+
23	−	9	=	14
=		=		=
43	+	22	=	65

D

40	+	10	=	50
−		−		−
30	+	3	=	33
=		=		=
10	+	7	=	17

E

90	+	10	=	100
−		−		−
60	+	8	=	68
=		=		=
30	+	2	=	32

F

110	+	14	=	124
−		−		−
70	+	9	=	79
=		=		=
40	+	5	=	45

Tasks: Pupils complete the record of sales under the conditions of the price list and the fact that no one bought more than 2 of any item. Pupils supply the missing addends and sums for the Watch the Signs! *game.*

Purpose: (a) To provide pupils with practice in addition and subtraction; (b) to give pupils further insight into numeration systems.

Unifying Ideas: Addition and subtraction; numeration; structure.

Materials for Teacher: Chart described in the Notes on Pupil Page 27.

Materials for Pupils: None.

The Lesson:

It is not immediately obvious that the sale problem is basically one involving a numeration system. It would be obvious if the three items under consideration were priced at 1 cent, 10 cents, and 1 dollar respectively, and we were told that no one bought more than 2 of any item. Then, given the amount of a sale, we could tell at once precisely which items were purchased. For example, given that a sale amounted to $1.22, one knows at once that two of the 1-cent items, two of the 10-cent items, and one of the 1-dollar items were purchased. One could change the allowable-purchase condition (in the 1-cent, 10-cents, 1-dollar problem) to require that no one purchase more than 3 of any item. In that case, one could still tell precisely which items were purchased for a given sales total. What is the most liberal allowable-purchase condition for which one can still determine the number of each item purchased? (9 of any item)

In the problem at hand, we are given a sales total and asked to partition this number of cents into 15-groups, 12-groups, and 7-groups. Under the given allowable-purchase condition, each of the given sales totals can be so partitioned in just one way. If the allowable-purchase condition were liberalized to one under which no more than 5 of any item could be purchased, we could get a sales total for which the number of each item purchased could not be determined. For example, under this condition, for 60 cents one could purchase 5 sandwiches or he could purchase 4 pieces of pie.

Lexicographical Ordering

It is instructive to list all possible purchases and their costs. How can we make sure that we have listed all of the possible purchases? We need to use

more

Headline for the Day: Using whole numbers only, write all the true sentences that follow the pattern:

$$\square + \triangle + \bigcirc = 3$$

some kind of system which assures us that no purchase has been omitted from consideration.

We have three items to consider, and we can buy 0, 1, or 2 of each item. Let's use strings of 3 digits each to stand for the purchases. For example, the string 021 shall refer to a purchase consisting of 0 cartons of milk, 2 sandwiches, and 1 piece of pie.

Although the wife of one the authors insists that you have not made a purchase when you visit a store and buy nothing, we shall consider 000 to be a purchase, or at least a mathematical purchase. Let 000 be the first purchase in our list of all possible purchases. What purchase should we list next? We can get a clue by noticing how words are listed in a dictionary. Consider the words *abacist*, *aardvark*, *aarhus*, *aasvogel*, *abaft*, and *aardwolf*. A dictionary that lists these words does so in the following order:

aardvark
↓
aardwolf
↓
aarhus
↓
aasvogel
↓
abacist
↓
abaft

This type of ordering is known as *lexicographical ordering*. (Take a look at the first page of the A section in the telephone directory for a large city.)

Let's use lexicographical ordering in making our list of purchases:

000
↓
001
↓
002
↓
010
↓
011
↓
012

etc.

At the top of the next column is an almost completed table.

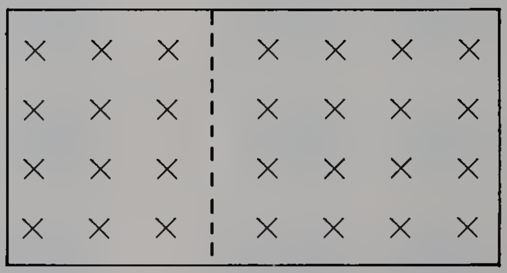
Cartons of milk	Sandwiches	Pieces of pie	Sales total in cents
0	0	0	0
0	0	1	15
0	0	2	30
0	1	0	12
0	1	1	27
0	1	2	42
0	2	0	24
0	2	1	39
0	2	2	54
1	0	0	7
1	0	1	22
1	0	2	37
1	1	0	19
1	1	1	34
1	1	2	49
1	2	0	31
1	2	1	46
1	2	2	61
2	0	0	14

How many possible purchases are there?

Drill

Here is a suggestion for a drill sheet:

Finish the sentences for this picture.



$12 + 16 = \_\_\_$ $4 \times 3 = \_\_\_$ $4 \times 4 = \_\_\_$

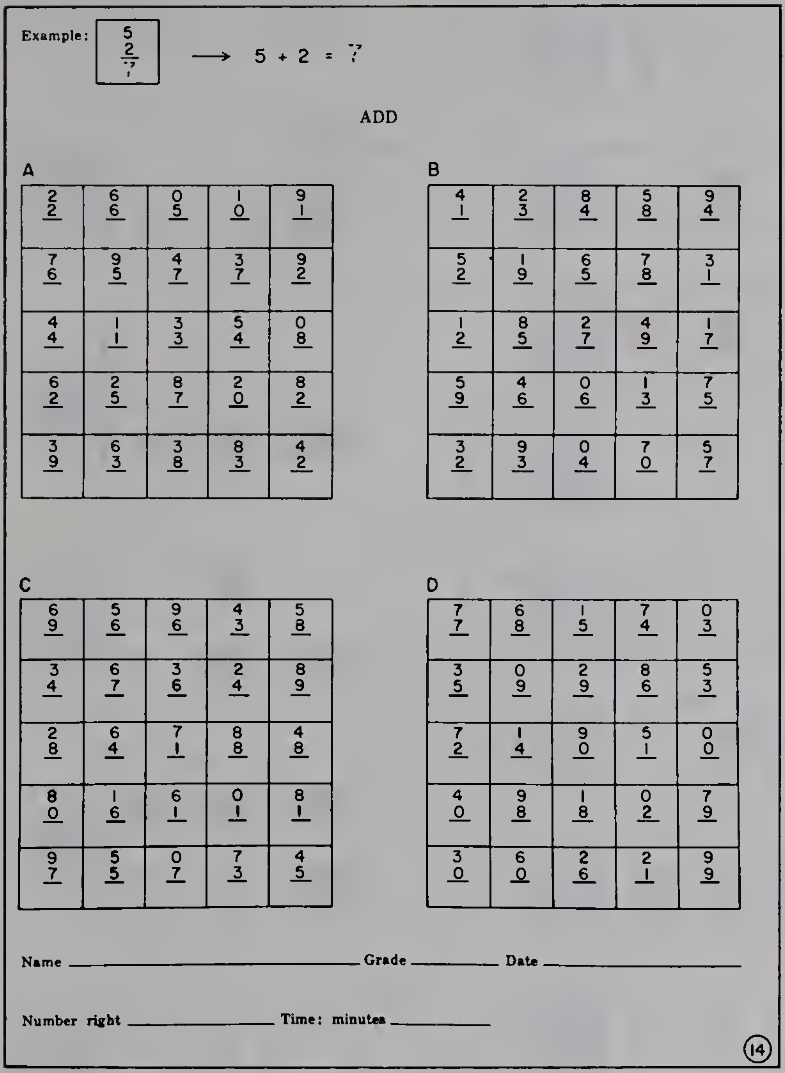
$(4 \times 3) + (4 \times 4) = 4 \times \_\_\_$

$(3 \times 4) + (\quad \times \quad) = \quad \times$

$(4 + 4 + 4) + (\qquad\qquad) = 28$

$(3 + 3 + 3 + 3) + (\qquad\qquad) = 28$

Consider using as tests pages 14 and 15 from MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL C, as shown on the following page.



Example:

5 2 7

→ 5 + 2 = 7

ADD

A

2 2	6 6	0 5	1 0	9 1
7 6	9 5	4 7	3 7	9 2
4 4	1 1	3 3	5 4	0 8
6 2	2 5	8 7	2 0	8 2
3 9	6 3	3 8	8 3	4 2

B

4 1	2 3	8 4	5 8	9 4
5 2	1 9	6 5	7 8	3 1
1 2	8 5	2 7	4 9	1 7
5 9	4 6	0 6	1 3	7 5
3 2	9 3	0 4	7 0	5 7

C

6 9	5 6	9 6	4 3	5 8
3 4	6 7	3 6	2 4	8 9
2 8	6 4	7 1	8 8	4 8
8 0	1 6	6 1	0 1	8 1
9 7	5 5	0 7	7 3	4 5

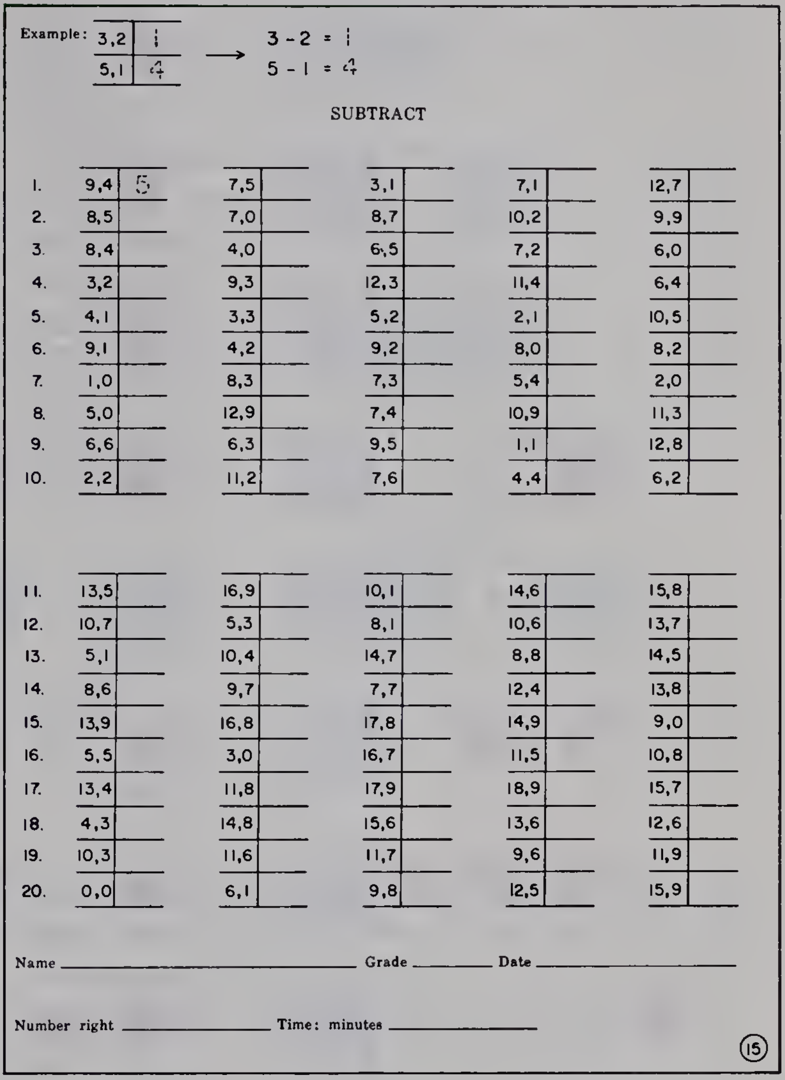
D

7 7	6 8	1 5	7 4	0 3
3 5	0 9	2 9	8 6	5 3
7 2	1 4	9 0	5 1	0 0
4 0	9 8	1 8	0 2	7 9
3 0	6 0	2 6	2 1	9 9

Name ______________________ Grade ________ Date ______________

Number right ____________ Time: minutes __________

⑭



Example:

3,2	1
5,1	4

→ 3 - 2 = 1
5 - 1 = 4

SUBTRACT

1.	9,4	5	7,5		3,1		7,1		12,7	
2.	8,5		7,0		8,7		10,2		9,9	
3.	8,4		4,0		6,5		7,2		6,0	
4.	3,2		9,3		12,3		11,4		6,4	
5.	4,1		3,3		5,2		2,1		10,5	
6.	9,1		4,2		9,2		8,0		8,2	
7.	1,0		8,3		7,3		5,4		2,0	
8.	5,0		12,9		7,4		10,9		11,3	
9.	6,6		6,3		9,5		1,1		12,8	
10.	2,2		11,2		7,6		4,4		6,2	

11.	13,5		16,9		10,1		14,6		15,8	
12.	10,7		5,3		8,1		10,6		13,7	
13.	5,1		10,4		14,7		8,8		14,5	
14.	8,6		9,7		7,7		12,4		13,8	
15.	13,9		16,8		17,8		14,9		9,0	
16.	5,5		3,0		16,7		11,5		10,8	
17.	13,4		11,8		17,9		18,9		15,7	
18.	4,3		14,8		15,6		13,6		12,6	
19.	10,3		11,6		11,7		9,6		11,9	
20.	0,0		6,1		9,8		12,5		15,9	

Name ______________________ Grade ________ Date ______________

Number right ____________ Time: minutes __________

⑮

WHAT'S MY RULE?

All you need is
a green crayon.

1. 1 | 1

2. 2 1 | 2

3. 2 1 | 3

4. 4 2 1 | 4

5. 4 2 1 | 5

A. 4 2 1 | 6

B. 4 2 1 | 7

C. 8 4 2 1 | 8

D. 8 4 2 1 | 9

E. 8 4 2 1 | 10

F. 8 4 2 1 | 11

G. 8 4 2 1 | 12

H. 8 4 2 1 | 13

I. 8 4 2 1 | 14

J. 8 4 2 1 | 15

K. 16 8 4 2 1 | 16

L. 16 8 4 2 1 | 17

M. 16 8 4 2 1 | 18

N. 16 8 4 2 1 | 19

O. 16 8 4 2 1 | 20

P. 16 8 4 2 1 | 21

Q. 16 8 4 2 1 | 22

R. 16 8 4 2 1 | 23

S. 16 8 4 2 1 | 24

T. 16 8 4 2 1 | 25

U. 16 8 4 2 1 | 26

V. 16 8 4 2 1 | 27

W. 16 8 4 2 1 | 28

X. 16 8 4 2 1 | 29

Y. 16 8 4 2 1 | 30

Z. 16 8 4 2 1 | 31

I
4 2 1
32 16 8 | 42

II
4 2 1
32 16 8 | 36

III
4 2 1
32 16 8 | 48

IV
4 2 1
32 16 8 | 50

V
4 2 1
32 16 8 | 39

VI
4 2 1
32 16 8 | 40

VII
4 2 1
32 16 8 | 30

VIII
4 2 1
32 16 8 | 58

IX
4 2 1
32 16 8 | 63

X
1
64 32 | 64

Notes on Pupil Page 120

Tasks: *Pupils shade with green crayon the appropriate boxes in each row so that the sum of the green-shaded numbers is the number in the blue box.*

Purpose: (a) To further extend pupils' understanding of numeration systems; (b) to provide pupils with practice in addition and subtraction.

Unifying Ideas: Numeration; addition and subtraction.

Materials for Teacher: Demonstration-size tagboard strips showing 16, 8, 4, 2, and 1 squares.

Materials for Pupils: Green crayons.

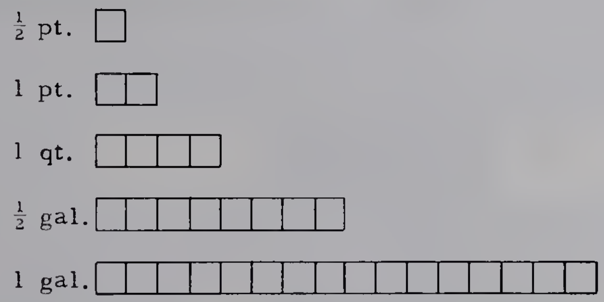
The Lesson:

You may wish to prepare pupils for this page by working through the following activity. Given the number of glasses of milk required at a party, determine the least number of containers (from among gallons, half-gallons, quarts, pints, and half-pints) that should be purchased to just meet the need.

Draw the following chart on the chalkboard:

Containers					
gal.	½ gal.	qt.	pt.	½ pt.	

Display the five tagboard strips:



Let each strip represent one of the containers; each little square represents 1 glass of milk. Write the name of the container on the back of the strip that represents the container.

Now, suppose that we need 6 glasses of milk for the party. Write a 6 in the cell at the right-hand end of the bottom row of your chart. Then point to the 16-card. Should we buy a gallon container of milk? No, that would be more than we need. Write a 0 in the left-hand cell of the top row of the chart. Point to the 8-card. Should we buy a half-gallon container? No, that's more than we need. Etc.

	Containers					
	gal.	½ gal.	qt.	pt.	½ pt.	
	0	0	1	1	0	
Glasses		4	2			6

Check: 4 + 2 = 6

Erase the entries in the chart and retain the check sentence. Work through several of the exercises on pupil page 120 in a manner similar to that used for the 6-glass exercise. For example:

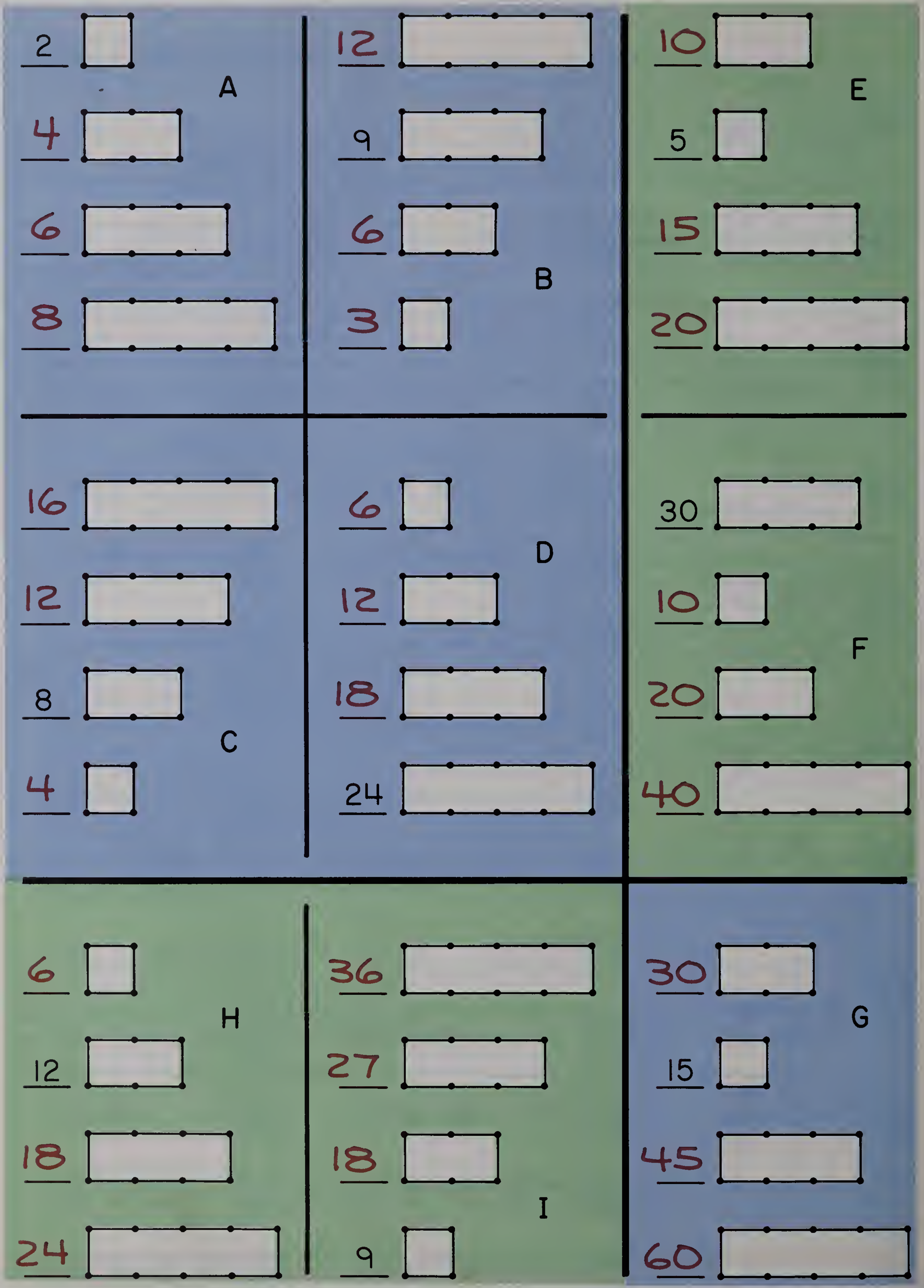
	Containers					
	gal.	½ gal.	qt.	pt.	½ pt.	
	1	0	1	0	1	
Glasses	16		4		1	21

Check: 4 + 2 = 6
16 + 4 + 1 = 21

As pupils work exercises 1-5 and A-Z in that order, they should develop a feeling for the exchanging procedure as manifested by shifts to the left.

Headline for the Day: Write a story that fits: 100 + 50 + ☐ = 275

CROSS NUMBER PUZZLE


A
2
4
6
8
B
12
9
6
3
E
10
5
15
20
C
16
12
8
4
D
6
12
18
24
F
30
10
20
40
H
6
12
18
24
I
36
27
18
9
G
30
15
45
60

Notes on Pupil Page 121

Tasks: Pupils record numbers suggested by the rectangular strips and the clues, and then complete the Cross Number Puzzle.

Purpose: To review and extend pupils' understanding of the four fundamental operations of elementary mathematics.

Unifying Ideas: Structure; addition and subtraction; multiplication and division.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

The activity on this page combines multiplication and division with the *Cross Number Puzzle.* As one might suspect, the distributive principle for multiplication over addition will play a role. Here's how. Note that if we restrict our attention to the smallest rectangles, we have a complete *Cross Number Puzzle.* Similarly, if we restrict our attention to the next larger rectangles, we again have a complete *Cross Number Puzzle*, etc.

2	3	5
4	6	10
6	9	15

4	6	10
8	12	20
12	18	30

6	9	15
12	18	30
18	27	45

8	12	20
16	24	40
24	36	60

In the case of the first *Cross Number Puzzle*, we note that

$$(2 + 3) + (4 + 6) = (2 + 4) + (3 + 6).$$

Now let's multiply $(2 + 3) + (4 + 6)$ by 2. In view of the above equation, this product must be the same as the one obtained by multiplying $(2 + 4) + (3 + 6)$ by 2:

$$\begin{aligned}&[(2 + 3) + (4 + 6)] \times 2\\ &= [(2 + 4) + (3 + 6)] \times 2\end{aligned}$$

By the distributive principle for multiplication over addition, this gives us:

$$\begin{aligned}&[(2 + 3) \times 2] + [(4 + 6) \times 2]\\ &= [(2 + 4) \times 2] + [(3 + 6) \times 2]\end{aligned}$$

Again by this principle, it follows that

$$\begin{aligned}&[(2 \times 2) + (3 \times 2)] + [(4 \times 2) + (6 \times 2)]\\ &= [(2 \times 2) + (4 \times 2)] + [(3 \times 2) + (6 \times 2)].\end{aligned}$$

This is precisely the situation for the second *Cross Number Puzzle.* Each of its rectangles is twice the size of the corresponding rectangle for the first *Cross Number Puzzle.* In other words, the second *Cross Number Puzzle* can be derived from the first just by using the distributive principle for multiplication over addition.

Drill

Here is a suggested drill sheet:

$2 \times \square = 8$

$7 \times \square = 0$

$6 \div 3 = \square$

$4 \times \square = 12$

$\square \times \triangle =$

$\square \times \triangle =$

$\square \times \triangle =$

$\square \times \triangle =$

15

$3\overline{)15}$ $\square$

$12 \div 3 = \square$

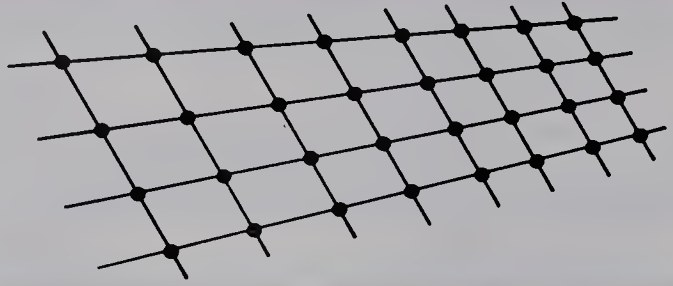
$$\begin{array}{r}5\\ \times 2\\ \hline \square\end{array} \qquad \begin{array}{r}6\\ \times \square\\ \hline 12\end{array} \qquad \begin{array}{r}\square\\ \times 3\\ \hline 9\end{array}$$

$12 = \square \times 6$

Headline for the Day: Together Bill and Andy have 13 cents. Each boy doubles his money. Now how much do they have together?

A

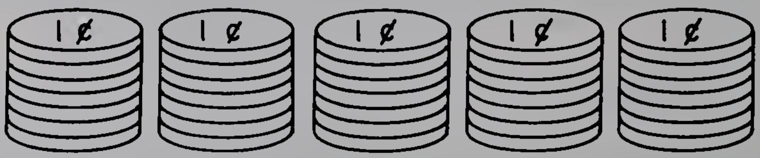
8 x 4 = 32



32 ÷ 4 = 8

B

5 x 7 = 35



35 ÷ 7 = 5

C

7 x 7 = 49

Harry put a penny in his bank each day for seven weeks. In that time he had saved 49 ¢.

49 ÷ 7 = 7

D

20 = 10 x 2

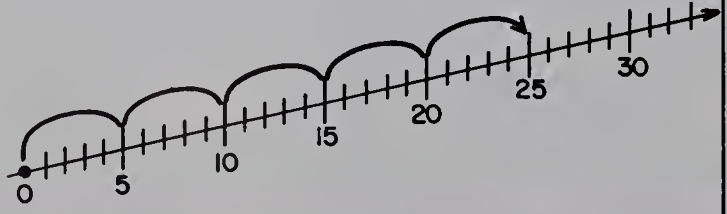
or 2 10

There were twenty children in the class. When they lined up by twos, there were 10 couples.

2 = 20 ÷ 10

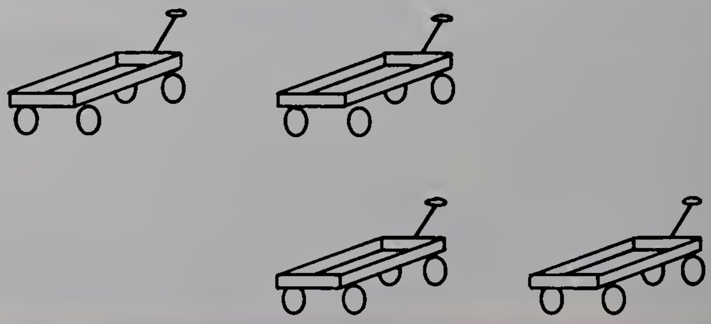
or 10 2

E



5 x 5 = 25

F



4 x 4 = 16

Notes on Pupil Page 122

Tasks: Pupils record the missing factors and products in the sentences suggested by the pictures.

Purpose: To provide pupils with practice in abstracting a multiplication or division sentence from a physical situation.

Unifying Ideas: Multiplication and division.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Pupils should need no prior explanation from you in order to do the exercises on pupil page 122.

Don't forget flash card drill on all four operations.

Drill

At the top of the next column is a suggested drill sheet which combines addition, multiplication, and the inequality relations:

Circle the one which is greater, and tell by how much it is greater.

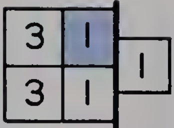
[✓ 3×4] $3+4$ greater by 5	5×2 $5+2$ greater by ___
$2+3$ 2×3 greater by ___	$4+4$ 4×4 greater by ___
3×5 $3+5$ greater by ___	$5+3$ 5×3 greater by ___
1×1 $1+1$ greater by ___	2×2 $2+2$ greater by ___

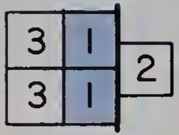
The lower right-hand box shows us a number whose sum with itself is the same as its product with itself—the number 2. Challenge pupils to find another example of such a number.

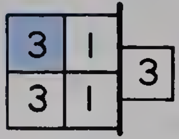
Headline for the Day: Write a number sentence that fits: There are 20 dimes in a roll of dimes.

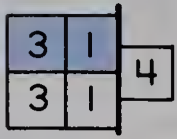
WHAT'S MY RULE?

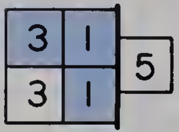
All you need is
a blue crayon.

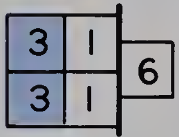


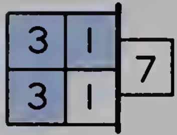


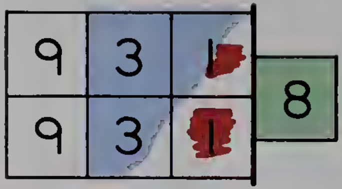


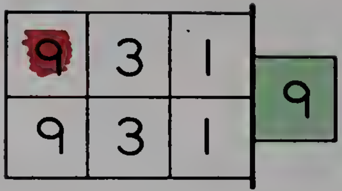


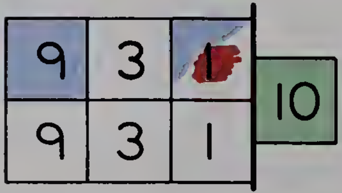


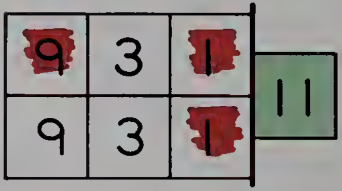


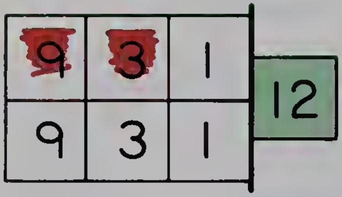


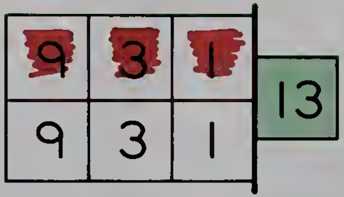


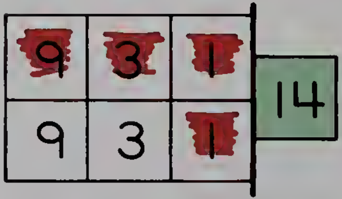


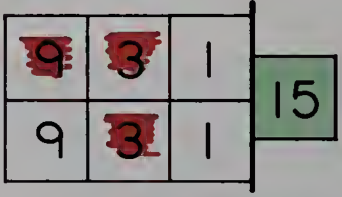


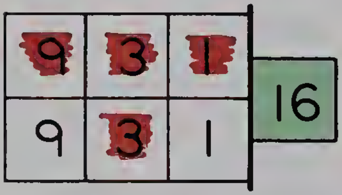


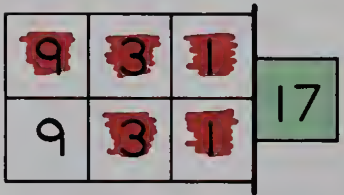


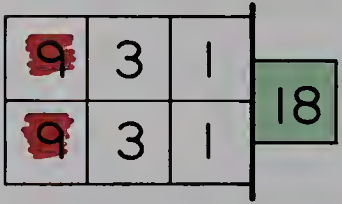


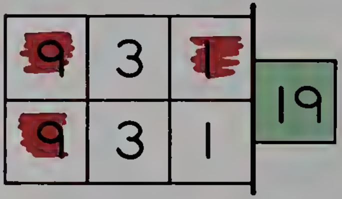




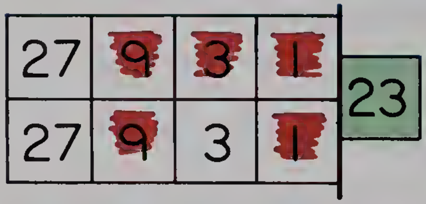




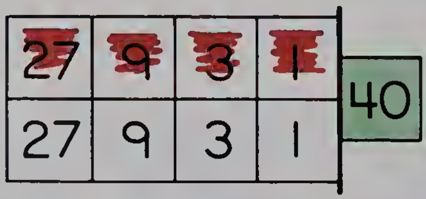




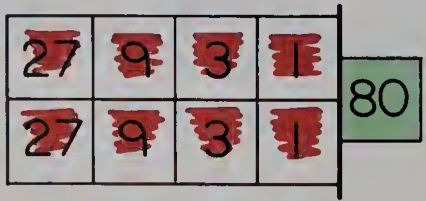
A


27 9 3 1
27 9 3 1
23

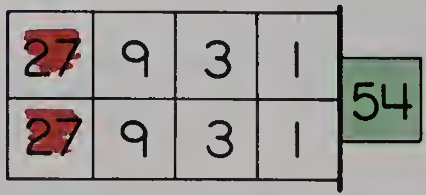
B


27 9 3 1
27 9 3 1
40

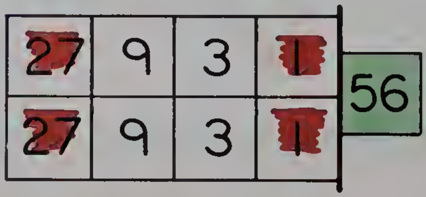
C


27 9 3 1
27 9 3 1
80

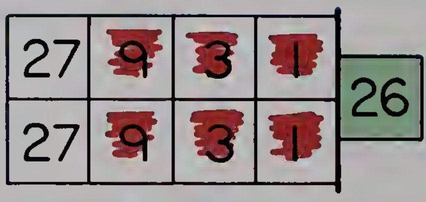
D


27 9 3 1
27 9 3 1
54

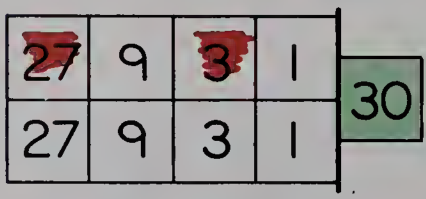
E


27 9 3 1
27 9 3 1
56

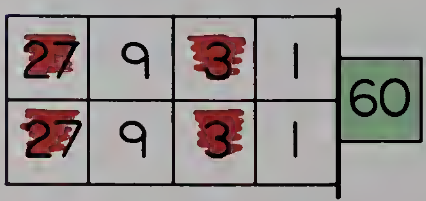
F


27 9 3 1
27 9 3 1
26

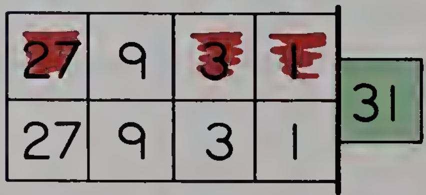
G


27 9 3 1
27 9 3 1
30

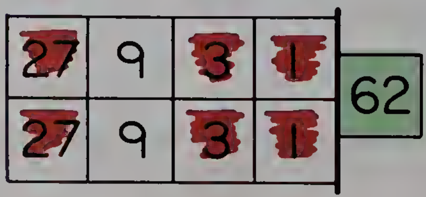
H


27 9 3 1
27 9 3 1
60

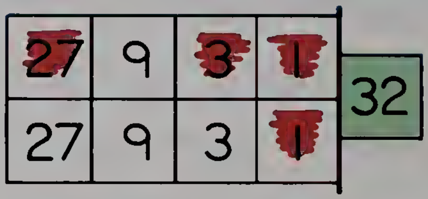
I


27 9 3 1
27 9 3 1
31

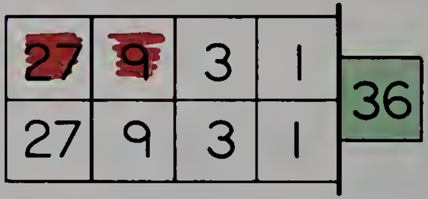
J


27 9 3 1
27 9 3 1
62

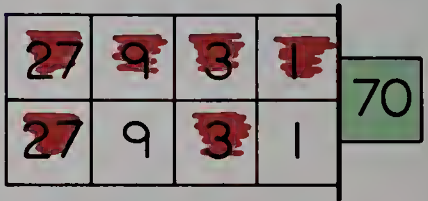
K


27 9 3 1
27 9 3 1
32

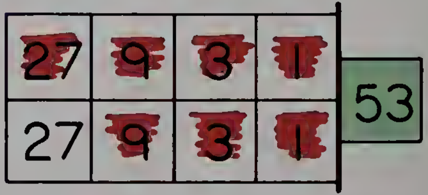
L


27 9 3 1
27 9 3 1
36

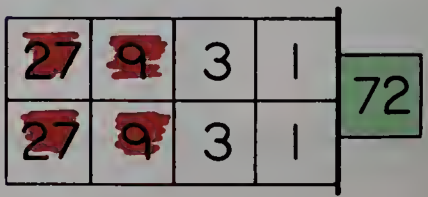
M


27 9 3 1
27 9 3 1
70

N


27 9 3 1
27 9 3 1
53

O


27 9 3 1
27 9 3 1
72

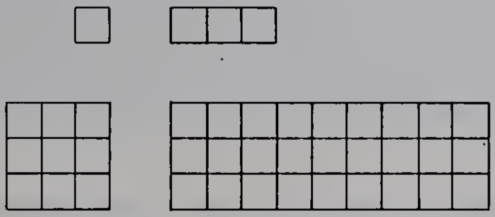
Notes on Pupil Page 123

Tasks: *Pupils shade with a blue crayon the appropriate boxes in each pair of rows so that the sum of the numbers in the blue-shaded boxes is the number in the green box.*

Purpose: (a) To further extend pupils' understanding of numeration systems; (b) to provide pupils with practice in addition and subtraction.

Unifying Ideas: Numeration; addition and subtraction.

Materials for Teacher: If possible, obtain two wooden cubes each 3″ on an edge and scored (or painted) on all six faces to show 9 squares on a face; two wooden blocks each 1″ by 3″ by 3″ and scored (or painted) to show 9 squares on each of the two larger faces and 3 squares on each of the other four faces; two wooden blocks each 1″ by 1″ by 3″ and scored (or painted) on all four of the long faces to show 3 squares on a face; two wooden cubes each 1″ on an edge. If these are not obtainable, you may wish to use two sets of cardboard strips like the following:



Materials for Pupils: Cuisenaire rods; blue crayons.

The Lesson:

This page deals with ternary (base 3) partitioning just as pupil page 120 dealt with binary partitioning. You may wish to prepare pupils for this page by carrying out the following activities.

Exhibit the eight blocks described above. How many of these little cubes would fit into this long block?(3) How many of these long blocks would fit into this flat block?(3) How many of these flat blocks would fit into this big cube?(3) How many of these little cubes would fit into this flat block?(9) How many of these long blocks would fit into this big cube?(9) How many of these little cubes would fit into this big cube?(27)

Let pupils combine their supplies of Cuisenaire rods to make such blocks (a white rod, a light green rod, a square floor of light green rods, a 3-floor stack of square floors of light green rods).

Try counting the pupils in the class with the blocks. When you get to the third pupil, make a big fuss about exchanging the pair of little cubes for the long block. Make a similar fuss when you get to the sixth pupil. Make a great big fuss when you get to the ninth pupil, for now you need to make a double exchange. Finally, for the first time in your career, consider yourself lucky if you have a twenty-seventh pupil. This makes it possible for you to make still another exchange.

Try some doubling problems with the blocks, such as going from a representation of 40 (exercise B) to a representation of 80 (exercise C).

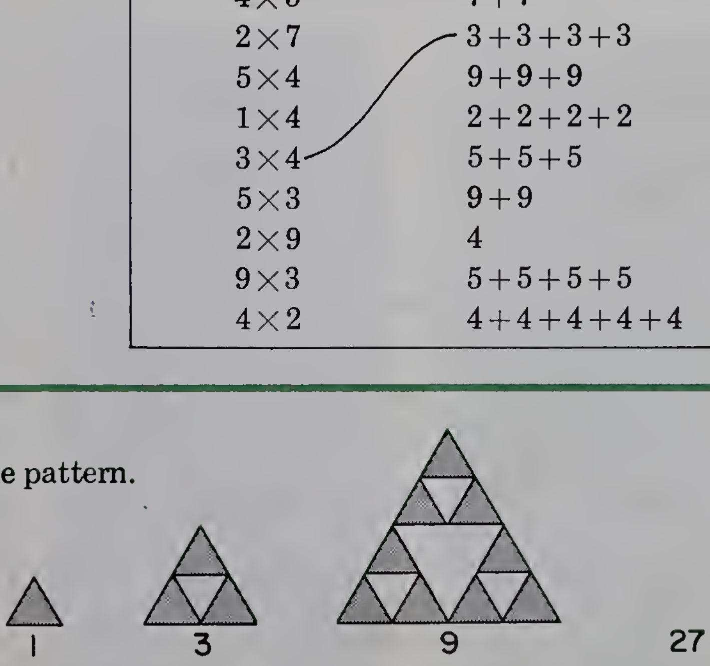
Drill

Try this worksheet:

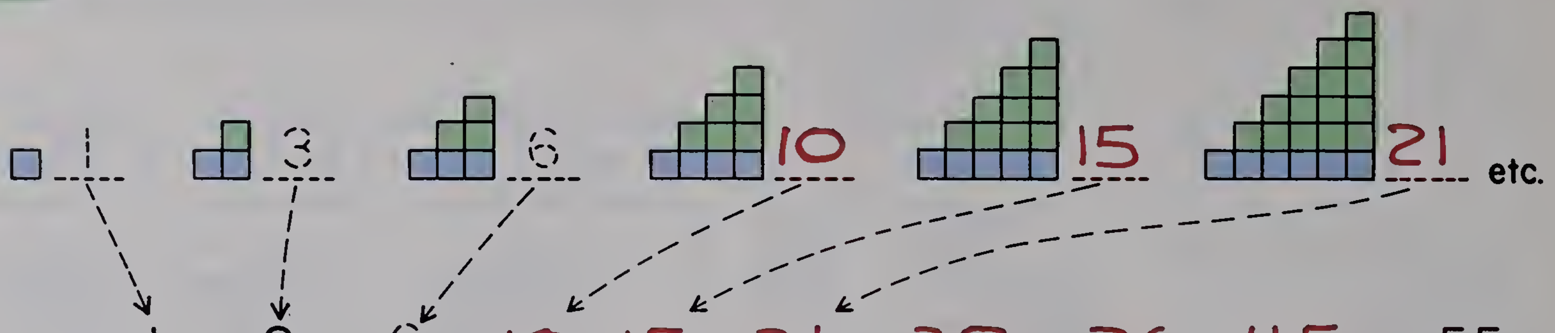
Matching

4×5	7+7
2×7	3+3+3+3
5×4	9+9+9
1×4	2+2+2+2
3×4	5+5+5
5×3	9+9
2×9	4
9×3	5+5+5+5
4×2	4+4+4+4+4

Headline for the Day: Draw the next picture in the pattern.



I A PATTERN OF GROWTH.



a. 1, 3, 6, 10, 15, 21, 28, 36, 45, 55

b. 2, 6, 12, 20, 30, 42, 56, 72, 90, 110

c. 3, 9, 18, 30, 45, 63, 84, 108, 135, 165

d. 5, 15, 30, 50, 75, 105, 140, 180, 225, 275

II What Are My Rules?

	A			B			C			D		
m	40	6	46	20	10	30	40	13	53	60	14	74
n	10	4	14	10	7	17	30	8	38	20	9	29
m − n	30	2	32	10	3	13	10	5	15	40	5	45
m + n	50	10	60	30	17	47	70	21	91	80	23	103

III Twice as much

30 minutes → 1 hour

8 inches → 1 ft., 4 in.

5 days → 1 wk., 3 da.

18 cents → 36 ¢

13 hours → 1 da., 2 hrs.

1 ft., 3 in. → 2 ft., 6 in.

3 quarts → 6 qts., 0 pt.

IV Three times as much

8 eggs → 2 dozen eggs

4 inches → 1 ft., 0 in.

8 hours → 1 day

5 days → 2 wks., 1 day

50 ¢ → $1.50

7 ounces → 1 lb., 5 oz.

$1.15 → $3.45

Notes on Pupil Page 124

Tasks: *In exercise Ia pupils compute the missing numbers in accordance with the rule of formation illustrated by the figures. In exercise Ib they use the given clue (1→2) and the derived clue (45→90) to guess that sequence b is the double of sequence a. In exercise II they apply their proposed rules in completing the table. In exercises III and IV they compute the missing quantities and record them in the notation specified.*

Purpose: General review.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; measurement; functions and relations.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

Prepare pupils for exercise I by putting the following on the board as a *What's My Rule?* game:

1	2	3	___	5	___	___

Ask what numbers the pupils think go in the blanks. Then introduce another row below it, like that shown at the top of the next column.

1	2	3	4	5	6	7
2	4	6	___	___	___	___
						↑

What number do you think will be written in the blank indicated by the arrow? Some may predict 14. To check, fill in each blank along the way. Then introduce another row:

1	2	3	4	5	6	7
2	4	6	8	10	12	14
3	6	9	___	___	___	___
				↑		↑

What numbers do you think will be written in the blanks with the arrows under them? Some may predict 15 and 21.

Instead of seeing the second and third sequences as the double and triple, respectively, of the first, a pupil will look at the columns in the example and see it this way:

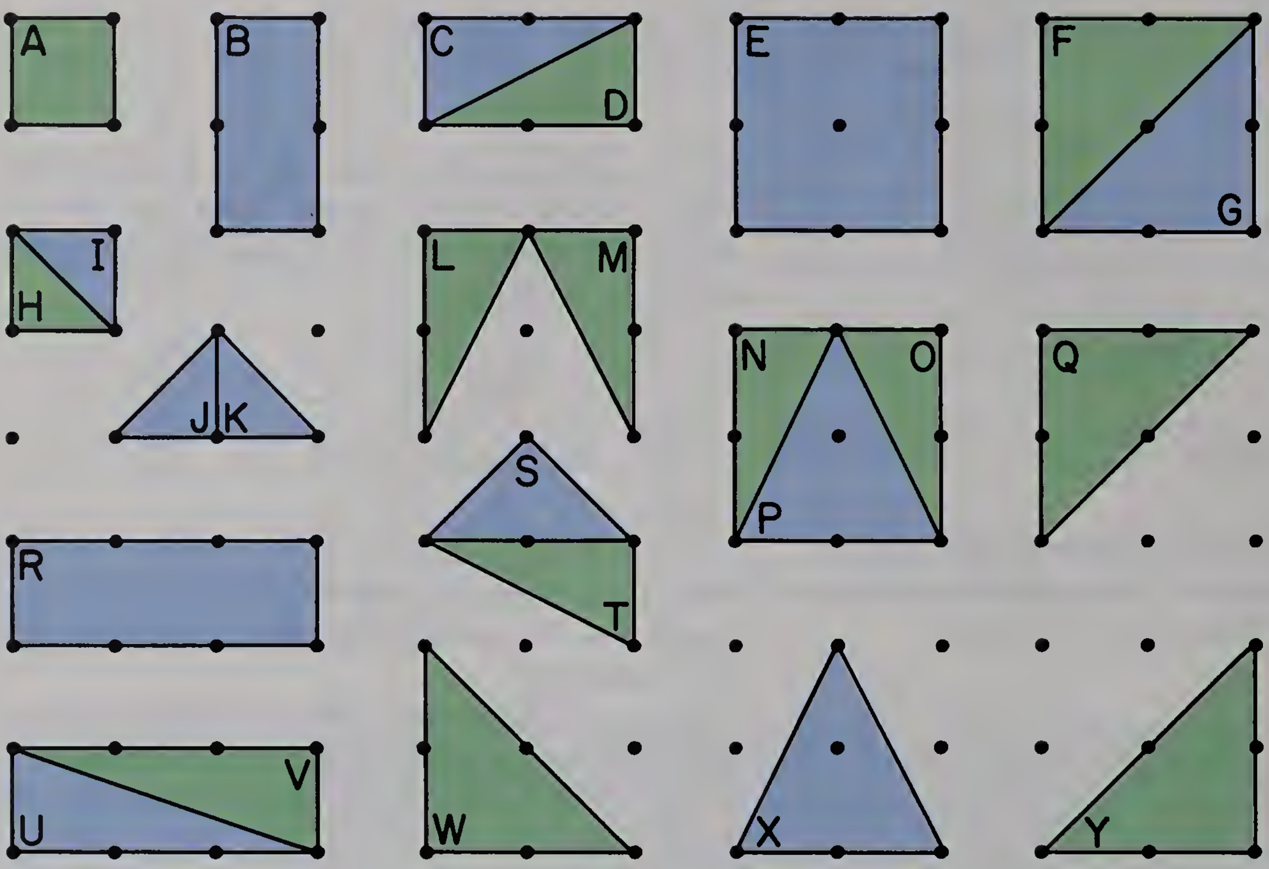
$$\begin{array}{rcrcrcrcr} 1 & & 2 & & 3 & & 5 & & 7 \\ +2 & & +4 & & +6 & \ldots & +10 & \ldots & +14 \\ \hline 3 & & 6 & & 9 & & 15 & & 21 \end{array}$$

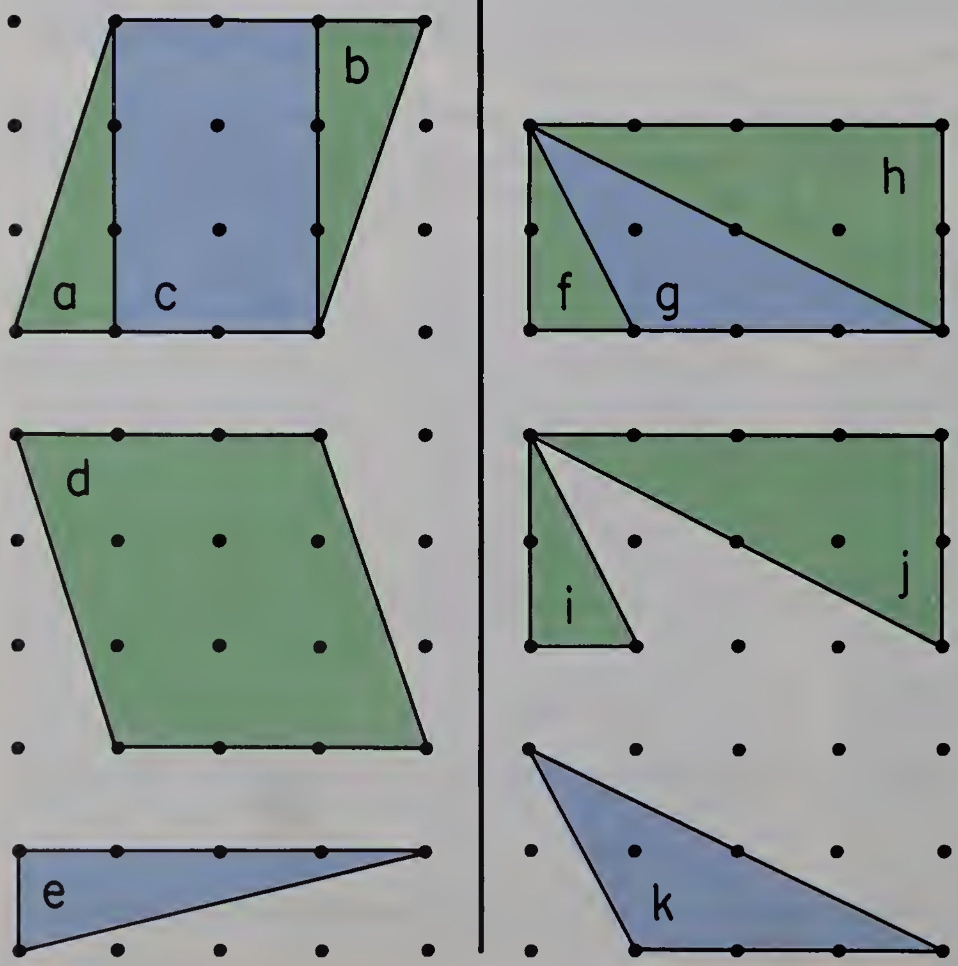
That both views lead to correct answers follows logically from the distributive principle for multiplication over addition.

Prepare for exercises III and IV by working with pupils on problems like those in the exercises until they understand how to proceed on their own.

Headline for the Day: Make a story for each pair of corresponding quantities in exercises III and IV on page 124.

FIND THE AREAS


A
B
C
D
E
F
G
I
H
L
M
J
K
N
O
Q
S
P
R
T
V
U
W
X
Y


b
a
c
h
f
g
d
j
i
e
k

a	$1\frac{1}{2}$
b	$1\frac{1}{2}$
c	6
d	9
e	2
f	1
g	3
h	4
i	1
j	4
k	3

A	1
B	2
C	1
D	1
E	4
F	2
G	2
H	$\frac{1}{2}$
I	$\frac{1}{2}$
J	$\frac{1}{2}$
K	$\frac{1}{2}$
L	1
M	1
N	1
O	1
P	2
Q	2
R	3
S	1
T	1
U	$1\frac{1}{2}$
V	$1\frac{1}{2}$
W	2
X	2
Y	2

Notes on Pupil Page 125

Tasks: *Pupils determine the number of units of area for each figure and record it in the chart.*

Purpose: (a) To review and extend pupils' understanding of area measurement; (b) to provide pupils with an opportunity for the use of fractions.

Unifying Ideas: Multiplication and division; addition and subtraction; geometry; measurement.

Materials for Teacher: A Counting Board described in the Notes on Pupil Page 33; rubber bands.

Materials for Pupils: None.

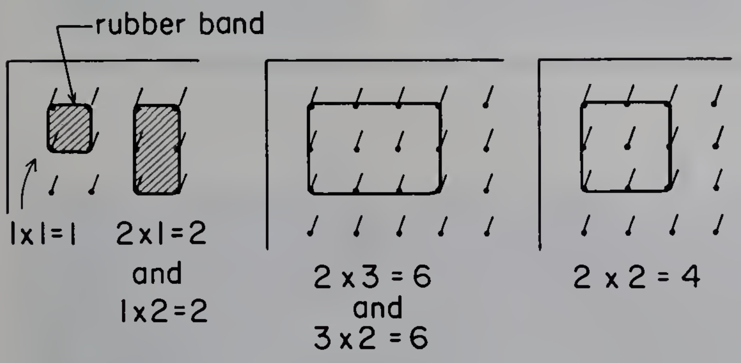
The Lesson:

The exercises on this page have been arranged in a developmental sequence. For example, pupils find the area of B and then find the areas of C and D by noting that each figure is half of B. This insight is then used to find the areas of F and G and of H and I. One finds the area of S by adding the areas of J and K, etc.

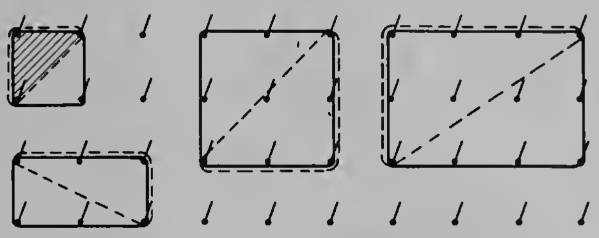
Note that one can find the area of *g* (or of *k*) by subtracting the sum of the areas of *f* and *h* from 2×4.

The Counting Board

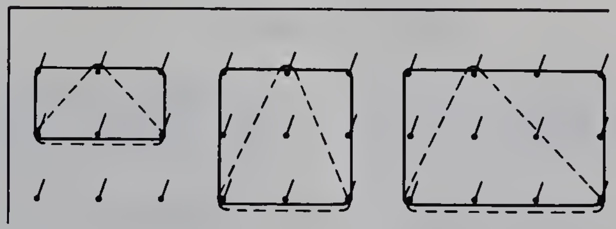
Use rubber bands to outline regions on the Counting Board:



A second rubber band in each figure can indicate a dissection of the region into regions with the same area:

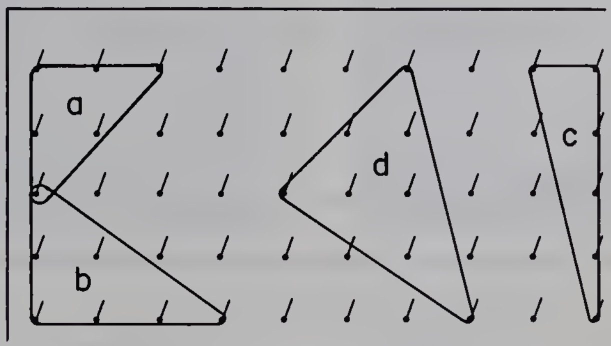


Other ways to create triangular shapes that are half of the enclosing rectangle are:



This anticipates an idea that will come later: the area of a triangle is half the product of its base and altitude. An expansion of this idea is illustrated below.

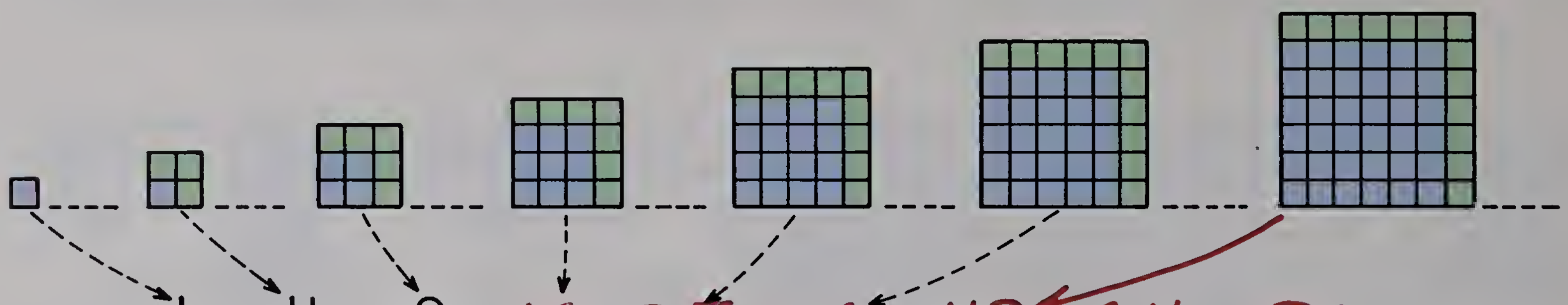
How many small squares (1×1=1) could be placed (if cut up properly) in spaces a, b, c, and d below?



This kind of argument in its entirety is beyond the reach of most beginners. However, most can follow in part and some will go surprisingly far. Again we are anticipating the mathematics to come.

Headline for the Day: Which number doesn't belong with the others? 3, 9, 6, 12, 10, 15

I A PATTERN OF GROWTH.



a. 1, 4, 9, 16, 25, 36, 49, 64, 81, 100

b. 2, 8, 18, 32, 50, 72, 98, 128, 162, 200

c. 3, 12, 27, 48, 75, 108, 147, 192, 243, 300

d. 4, 16, 36, 64, 100, 144, 196, 256, 324, 400

II Half as much

1 foot	→	6 inches
1 hour	→	30 minutes
50 ¢	→	25 ¢
1 wk., 5 days	→	6 days
1 hr., 20 min.	→	40 minutes
1 dozen eggs	→	6 eggs
$ 1.50	→	75 ¢
2000 miles	→	1000 miles

III One-third as much

30 ¢	→	10 ¢
1 dozen eggs	→	4 eggs
2 wks., 1 day	→	5 days
1 yard, 0 in.	→	1 ft., 0 in.
1 hour	→	20 minutes
1 foot	→	4 inches
39 miles	→	13 miles
$ 1.50	→	50 ¢

IV What Are My Rules?

	A			B			C			D		
q	10	3	13	20	5	25	30	1	31	9	20	29
r	10	8	18	30	4	34	10	7	17	8	30	38
s	10	7	17	10	5	15	10	6	16	1	10	11
t	10	2	12	10	6	16	20	3	23	2	10	12
q + r + s + t	40	20	60	70	20	90	70	17	87	20	70	90
q + s	20	10	30	30	10	40	40	7	47	10	30	40
r + t	20	10	30	40	10	50	30	10	40	10	40	50

Notes on Pupil Page 126

Tasks: *In exercise I pupils find the missing terms of the sequences. In exercises II and III they compute the missing quantities and record them in the notation specified. In exercise IV they apply their proposed rules in completing the table.*

Purpose: General review.

Unifying Ideas: Structure; number and counting; addition and subtraction; multiplication and division; measurement; functions and relations.

Materials for Teacher: A Counting Board.

Materials for Pupils: None.

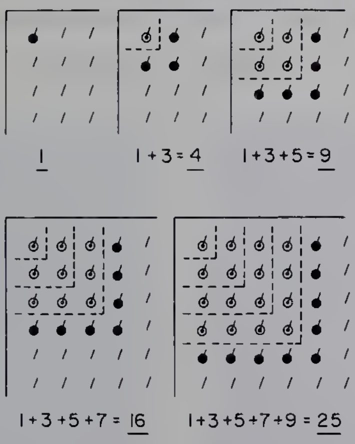
The Lesson:

See the Notes on Pupil Page 124 for suggestions relevant to pupil page 126.

The Counting Board

Another use for this device is to show the relationships among square numbers (1, 4, 9, 16, 25, etc.), odd numbers (1, 3, 5, 7, 9, etc.), and triangular numbers (1, 3, 6, 10, 15, etc.).

Starting with a washer in a corner, add successive rows and columns of washers to demonstrate square numbers:



Conclusion:

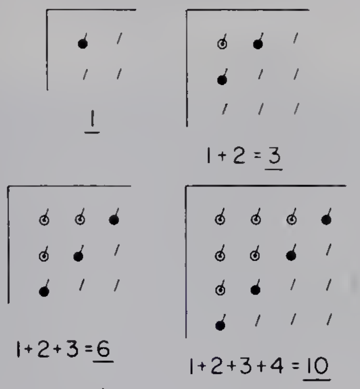
The sum of the first 2 consecutive odd numbers $=2\times2=4$.

The sum of the first 3 consecutive odd numbers $=3\times3=9$.

The sum of the first four $= 4 \times 4 = 16$.

The sum of the first five $= 5 \times 5 = 25$.

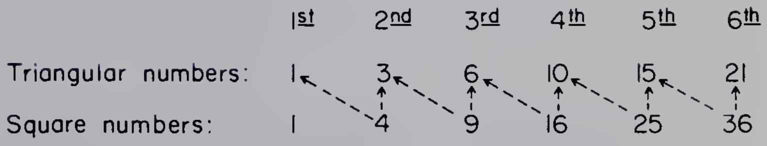
Build the triangular numbers by starting in one corner and, step by step, adding a diagonal row of washers:



and

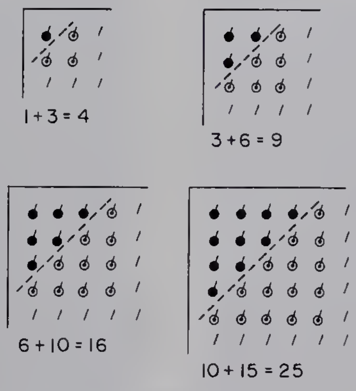
1+2+3+4+5 = 15 ,

1+2+3+4+5+6 = 21



1st
2nd
3rd
4th
5th
6th
Triangular numbers: 1 3 6 10 15 21
Square numbers: 1 4 9 16 25 36

Notice that each square number is the sum of the triangular number listed above it and the triangular number listed on its left. This relationship can be shown by putting washers on the nails as shown:

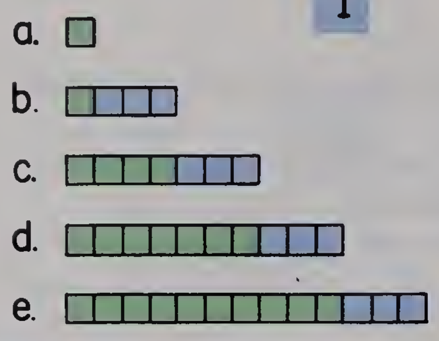


(See the Notes on Pupil Page 1.)

Headline for the Day: Which number doesn't belong with the others? 4, 12, 16, 8, 20, 6

What Are My Rules?

I



			a.	b.	c.	d.	e.	f.	g.	h.
If	1	then	4	7	10	13	16	19	22	
If	2	then	8	14	20	26	32	38	44	
If	3	then	12	21	30	39	48	57	66	

II

	A			B			C			D		
x	5	40	45	7	30	37	6	50	56	40	8	48
y	5	20	25	3	20	23	6	10	16	20	5	25
z	4	0	4	2	10	12	5	10	15	10	3	13
x + y + z	14	60	74	12	60	72	17	70	87	70	16	86
x − y	0	20	20	4	10	14	0	40	40	20	3	23
y − z	1	20	21	1	10	11	1	0	1	10	2	12
x − z	1	40	41	5	20	25	1	40	41	30	5	35

III

Twice as much

6 inches → 1 foot
8 inches → 1 ft., 4 in.
25 cents → 50 cents
11 inches → 1 foot, 10 inches
40 minutes → 1 hr., 20 min.
4 days → 1 wk., 1 da.
6 eggs → 1 dozen eggs
6 days → 1 week, 5 days
75 cents → 1 dollar, 50 cents

IV

Half as much

1 foot, 6 inches → 0 ft., 9 in.
3 hours → 1 hr., 30 min.
2 wks., 4 days → 1 wk., 2 da.
$1.20 → 60¢
1 day → 12 hours
1 quart → 1 pint
1 yard → 1 ft., 6 in.
1 year, 2 months → 7 months
1 gallon → 2 quarts

Notes on Pupil Page 127

Tasks: *Pupils record the missing entries in the* What Are My Rules? *games. They also record in the notation specified the missing entries in exercises III and IV.*

Purpose: General review.

Unifying Ideas: Structure; addition and subtraction; multiplication and division; functions and relations; measurement.

Materials for Teacher: None.

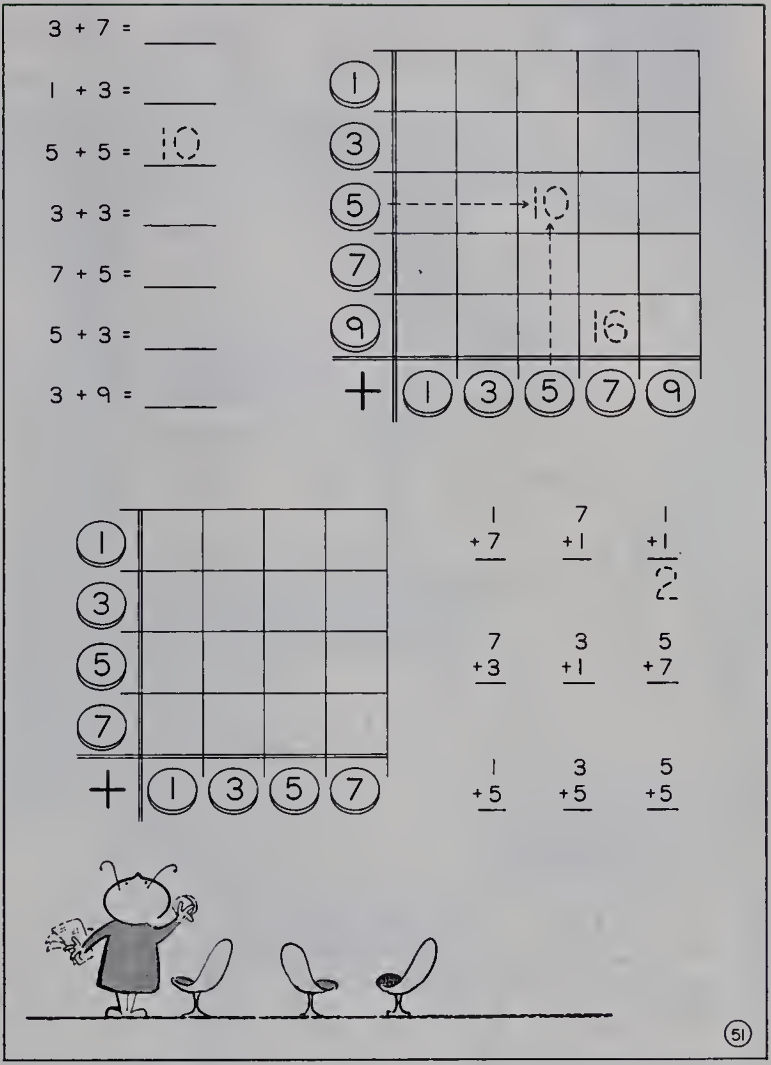
Materials for Pupils: None.

The Lesson:

Pupils should go right to work on this page.

Drill

Consider using as a very easy drill sheet page 51 of MATH WORKSHOP SUPPLEMENTARY ACTIVITIES, LEVEL B:



3 + 7 = ____

1 + 3 = ____

5 + 5 = 10

3 + 3 = ____

7 + 5 = ____

5 + 3 = ____

3 + 9 = ____

①					
③					
⑤			10		
⑦					
⑨				16	
+	①	③	⑤	⑦	⑨

①				
③				
⑤				
⑦				
+	①	③	⑤	⑦

$\begin{array}{r}1\\+7\\\hline\end{array}$ $\begin{array}{r}7\\+1\\\hline\end{array}$ $\begin{array}{r}1\\+1\\\hline 2\end{array}$

$\begin{array}{r}7\\+3\\\hline\end{array}$ $\begin{array}{r}3\\+1\\\hline\end{array}$ $\begin{array}{r}5\\+7\\\hline\end{array}$

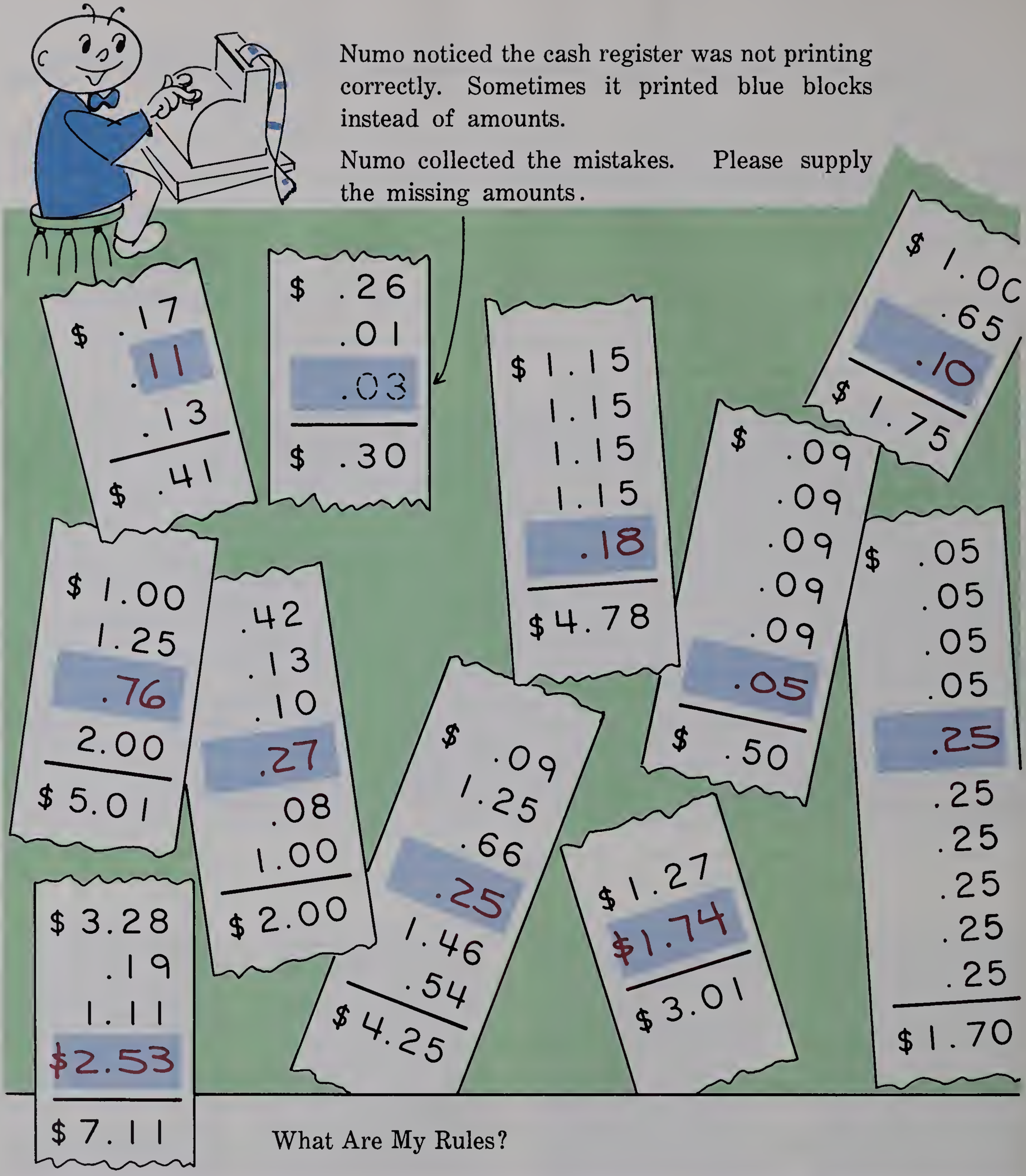
$\begin{array}{r}1\\+5\\\hline\end{array}$ $\begin{array}{r}3\\+5\\\hline\end{array}$ $\begin{array}{r}5\\+5\\\hline\end{array}$

(51)

Headline for the Day: What number doesn't belong with the others? 15, 45, 30, 40, 60, 75

Numo noticed the cash register was not printing correctly. Sometimes it printed blue blocks instead of amounts.

Numo collected the mistakes. Please supply the missing amounts.


$ .17 .11 .13 $ .41
$ .26 .01 .03 $ .30
$ 1.15 1.15 1.15 1.15 .18 $ 4.78
$ 1.00 .65 .10 $ 1.75
$ .09 .09 .09 .09 .09 .05 $ .50
$ .05 .05 .05 .05 .25 .25 .25 .25 .25 .25 $ 1.70
$ 1.00 1.25 .76 2.00 $ 5.01
.42 .13 .10 .27 .08 1.00 $ 2.00
$ .09 1.25 .66 .25 1.46 .54 $ 4.25
$ 1.27 $1.74 $ 3.01
$ 3.28 .19 1.11 $2.53 $ 7.11

What Are My Rules?

x	4	10	9	5	6	7	5	7	15	10	0	12	7
y	3	4	9	2	4	3	5	6	2	1	0	3	7
x + y	7	14	18	7	10	10	10	13	17	11	0	15	14
x − y	1	6	0	3	2	4	0	1	13	9	0	9	0
x · y	12	40	81	10	24	21	25	42	30	10	0	36	49

Notes on Pupil Page 128

Tasks: *Pupils record the missing addends or sums in each cash register record. They also record the missing entries in the* What Are My Rules? *table. (Notice that a raised dot is used as a multiplication sign in the last row.)*

Purpose: To review and extend pupils' understanding of addition and subtraction.

Unifying Ideas: Addition and subtraction; functions and relations.

Materials for Teacher: None.

Materials for Pupils: None.

The Lesson:

The check-out tapes with a missing item are not mysterious. They simply present long and involved computations. Even the most direct approach requires much figuring, and in any one of the steps involved it is possible to commit an error.

This activity demonstrates the fact that a small error anywhere along the way will result in an incorrect answer. So the most economical method is to have pupils work with an emphasis on accuracy. Speed is a secondary consideration.

Readiness Activities

You may wish to prepare for pupil page 128 with the following activities.

Activity 1

Helen bought a present for each of her three cousins. Altogether she spent $1.25. She remembered that the model for Bill cost 35¢ and the book for Martha cost 45¢, but she couldn't remember what she paid for the game for Roger. Can you help her out?

$$\begin{array}{r} \$\ .35 \\ .45 \\ .\ ? \\ \hline \$1.25 \end{array}$$

Was it more or less than the cost of the model? More or less than the book? (The game cost 45¢.)

Activity 2

Here is the scorepad for Roger's game. After each turn the four players recorded their total scores and added the totals. This was fortunate, because Helen's mischievous little brother erased all her scores. Who won each game?

	Games					
	1st	2nd	3rd	4th	5th	6th
Roger	15	18	30	28	25	32
Helen						
Martha	20	22	30	13	29	45
Bill	15	14	35	29	21	37
Totals	75	75	120	100	110	158

(Helen's scores were 25, 21, 25, 30, 35, and 44.)

In these introductory examples, do not suggest any rule such as: add the given numbers and subtract the sum from the total to get the missing number. Each pupil should be encouraged to tackle the examples any way he likes. It may well be by trial and error—a perfectly legitimate, time-wasting procedure. Or he may perform a series of subtractions: $75-15=60$, $60-20=40$, $40-15=25$; so Helen's score in the first game must have been 25.

There are many ways to work out the problem to a correct answer. It takes time for pupils to develop a sense which guides them toward the most economical method of reaching a correct answer. No set of rules is inclusive enough to meet all situations, and every set of rules is dangerous because there is no way to be sure that the pupil will select the right rule and apply it correctly. Instead of memorizing rules, a pupil must develop an insight into problems and learn to sense the shortest road to the goal. Deciding upon a method of attack is an art—it may never be mastered fully, but it can be hindered by rules.

Headline for the Day: There are 20 quarters in a roll of quarters. How many dollars is such a roll worth?

DATE DUE SLIP

EDUC [illegible] 1 '77	MAY 23 RETURN
MAR 28 RETURN	DUE EDUC AUG 13 '80
EDUC [illegible] 23 '77	AUG 11 RETURN
APR 13 RETURN	DUE EDUC. MAR 18 '82
[illegible] 9 '77	RETURN MAR 15 '82
MAY 9 RETURN	DUE EDUC OCT 1[illegible] '83
EDUC MR 2 '78	RETURN OCT 24 '83
RETURNED FEB 27 '78	DUE EDUC OCT 01 '84
MAR 8 RETURN	SEP 24 RETURN
DUE Educ MAY 31 '78	
RETURNED MAY 23 '78	
DUE Educ APR 30 '79	
JAN 2 RETURN	
DUE EDUC MAY 23 '80	

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